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# Fuzzy difference equations of volterra type

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**Abstract**

In this work we introduce the notion of fuzzy volterra difference equations and study the dynamical properties of some classes of this type of equations. We prove some comparison theorems for these equations in terms of ordinary volterra difference equations. Using these results the stability of the fuzzy nonlinear volterra difference equations is investigated.

*Keywords:* Volterra difference equations, Fuzzy, Attractivity, Stability.
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## 1. Introduction

Nonlinear difference equations are important in applications, such equations appear naturally as discrete analogues of differential and delay differential equations which model various diverse phenomena in biology, ecology, physics, engineering and economics.
Volterra difference equations are used to describe the processes whose future behavior depends on the whole previous history. These systems appear in the theory of viscoelasticity, models of propagation of perturbations in matter with memory, description of the motion of bodies with reference to hereditary, various problems of biomechanics, numerical solution of various type of equations with continuous time and also to solve optimal control problems, see [1,5,6,11] for furthere information.
In 1965, Zadeh initiated the development of the modified set theory known as fuzzy set theory. Using it, one can model the meaning of vague notions and also some kinds of human reasoning. The fuzzy set theory and its applications have been extensively developed since the seventies and the industrial interest in fuzzy control has increased since 1990.
Fuzzy difference equations have been studied to some extent, for example see [2-4,7-10]. We initiate in this paper, the notion of fuzzy volterra difference equations and prove some basic facts about this equations. We prove comparison theorems for some classes of this equations in terms of ordinary volterra difference equations. Using these results the stability of the fuzzy nonlinear volterra difference equations is investigated.

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## 2. Preliminaries

Let $P_k(\mathbb{R}^n)$ denote the family of all nonempty compact convex subsets of $\mathbb{R}^n$. Denote by $E^q = \{u : \mathbb{R}^n \to [0,1]\}$ such that $u$ satisfies the following conditions:

(1) $u$ is normal, that is, there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$.
(2) $u$ is fuzzy convex, $0 \leq \lambda \leq 1$:

$$u(\lambda x + (1-\lambda)y) \geq min[u(x), u(y)];$$

(3) $u$ is upper semi-continuous.
(4) $[u]^0 = cl - \{x \in \mathbb{R}^n : u(x) > 0\}$ is compact.
From (1)-(4) we have that $[u]^\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\} \in P_k(\mathbb{R}^n)$ for $\alpha \in (0,1]$.

Let $d_H(A,B)$ be the Hausdorf distance between the sets $A, B \in P_k(\mathbb{R}^n)$. Then we define:

$$d[u,v] = \sup_{0 \leq \alpha \leq 1} d_H([u]^\alpha, [v]^\alpha)$$

which is a metric in $E^q$ and $(E^q, d)$ is a complete metric space. We define $\hat{0} \in E^q$ as $\hat{0} = \chi_{\{0\}}$.
Let $\mathbb{N}$ denote the natural numbers. We denote by $\mathbb{N}^+_{n_0}$, the set:

$$\mathbb{N}^+_{n_0} = [n_0, n_0+1, ..., n_0+k, ...]$$

with $k, n_0 \in \mathbb{N}$.
We consider the fuzzy volterra difference equation given by:

$$x(n+1) = x(n) + f(n, x(n), \sum_{i=n_0}^{n-1} x(i)) \tag{2.1}$$

in which $f : \mathbb{N}^+_{n_0} \times E^q \times E^q \to E^q$ and $x(n) \in E^q$. For the study of properties of the solutions of this equation we use the following equation:

$$\Delta u(n) = h(n, u(n), \sum_{i=n_0}^{n-1} u(i)) \tag{2.2}$$

in which $h : \mathbb{N}^+_{n_0} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

## 3. Main results

First we obtain results about comparison of solutions of Eq.(2.1) and Eq.(2.2).

**Theorem 3.1.** *Assume that $h(n,u,v)$ is nondecreasing in $u$ and $v$ for each $n$ and*

$$d[f(n, x(n), \sum_{i=n_0}^{n-1} x(i)), \hat{0}] \leq h(n, d[x(n), \hat{0}], d[\sum_{i=n_0}^{n-1} x(i), \hat{0}]) \tag{3.1}$$

*where $h$ is given in Eq.(2.2). Then $d[x(n_0), \hat{0}] \leq u(n_0)$ implies $d[x(n), \hat{0}] \leq u(n)$ for $n \geq n_0$.*

**Proof .** Suppose that $d[x(i),\hat{0}]\leq u(i)$ for $n_0\leq i\leq k$, from this we have $\sup_{z\in[x(i)]_\alpha}\parallel z\parallel\leq u(i), n_0\leq i\leq k$, therefore,

$$\begin{aligned} d[\sum_{i=n_0}^{k} x(i),\hat{0}] &= \sup_{\alpha\in(0,1]} d_H([\sum_{i=n_0}^{k} x(i)]_\alpha,[\hat{0}]_\alpha) = \sup_{\alpha\in(0,1]}\ \sup_{z\in[\sum_{i=n_0}^{k} x(i)]_\alpha}\parallel z\parallel \\ &= \sup_{\alpha\in(0,1]}\ \sup_{z\in\sum_{i=n_0}^{k}[x(i)]_\alpha}\parallel z\parallel\leq \sum_{i=n_0}^{k}\sup_{\alpha\in(0,1]}\ \sup_{z_i\in[x(i)]_\alpha}\parallel z_i\parallel \\ &\leq \sum_{i=n_0}^{k} u(i) \end{aligned}$$

now from monotonicity of $h$ we have that:

$$\begin{aligned} d[x(k+1),\hat{0}] &= d[x(k)+f(k,x(k),\sum_{i=n_0}^{k-1} x(i)),\hat{0}] \\ &\leq d[x(k),\hat{0}]+d[f(k,x(k),\sum_{i=n_0}^{k-1} x(i)),\hat{0}] \\ &\leq u(k)+h(k,d[x(k),\hat{0}],d[\sum_{i=n_0}^{k-1} x(i),\hat{0}] \\ &\leq u(k)+h(k,u(k),\sum_{i=n_0}^{k-1} u(k)) = u(k+1) \end{aligned}$$

which completes the proof. □

Now we consider a more general form of Eq.(2.1)

$$x(n+1)=x(n)+f(n,x(n),\sum_{i=n_0}^{n-1} G(n,i,x(i))) \tag{3.2}$$

in which $f:\mathbb{N}^+_{n_0}\times E^q\times E^q\to E^q$ and $G:\mathbb{N}^+_{n_0}\times\mathbb{N}^+_{n_0}\times E^q\to E^q$. We use a function of the form $V:\mathbb{N}^+_{n_0}\times E^q\to\mathbb{R}^+$ and consider $\Delta V(n,x(n))=V(n+1,x(n+1))-V(n,x(n))$.

**Theorem 3.2.** *Assume that $h(n,u,v)$ is nondecreasing in $u$ and $v$ for each $n\in\mathbb{N}^+_{n_0}$ and*

$$\Delta V(n,x(n))\leq h(n,V(n,x(n)),\sum_{s=n_0}^{n-1} V(s,x(s))) \tag{3.3}$$

*then $V(n_0,x(n_0))\leq u(n_0)$ implies $V(n,x(n))\leq u(n)$ for each $n\geq n_0$ where $x(n)$ is the solution of Eq.(3.2) and $u(n)$ is the solution of Eq.(2.2).*

**Proof .** Suppose that the claim is not true. Then there exists a $k\in\mathbb{N}^+_{n_0}$ such that $V(m,x(m))\leq u(m)$ for $n_0\leq m\leq k$ and $V(k+1,x(k+1))>u(k+1)$. Using the monotone character of $h(n,r,s)$ we have:

$$V(k,x(k)) \quad + \quad h(k,V(k,x(k)),\sum_{s=n_0}^{k-1} V(s,x(s)))\geq V(k+1,x(k+1))$$

$$
\begin{aligned}
&> \quad u(k+1) = u(k) + h(k, u(k), \sum_{s=n_0}^{k-1} u(s)) \\
&\geq \quad u(k) + h(k, V(k, x(k)), \sum_{s=n_0}^{k-1} V(s, x(s)))
\end{aligned}
$$

and therefore $V(k, x(k)) > u(k)$ which is a contradiction. □

**Corollary 3.3.** *Suppose that $L : \mathbb{N}^+_{n_0} \times E^q \to E^q$, $h(n, u, v)$ is nondecreasing in $u$ and $v$ for each $n \in \mathbb{N}^+_{n_0}$ and:*

$$
\begin{aligned}
d[L(n+1, x(n+1)), \hat{0}] \quad &\leq \quad d[L(n, x(n)), \hat{0}] \\
&+ \quad h(n, d[L(n, x(n)), \hat{0}], \sum_{s=n_0}^{n-1} d[L(s, x(s)), \hat{0}])
\end{aligned}
$$

*then $d[L(n_0, x(n_0)), \hat{0}] \leq u(n_0)$ implies that $d[L(n, x(n)), \hat{0}] \leq u(n)$ for all $n \geq n_0$.*

**Proof .** Set $V(n, x(n)) = d[L(n, x(n)), \hat{0}]$ in the above theorem. □

Using the above results we obtain the following criterion for comparison of dynamics of volterra difference equations and its fuzzy version.

**Theorem 3.4.** *Suppose that there exists functions $V(n, x)$ and $h(n, u, v)$ satisfying the conditions:*
*(i) $h : \mathbb{N}^+_{n_0} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$, $h(n, 0, 0) = 0$ and $h(n, u, v)$ is nondecreasing in $u$ and $v$ for each $n \in \mathbb{N}^+_{n_0}$,*
*(ii) $V : \mathbb{N}^+_{n_0} \times E^q \to \mathbb{R}^+$ and*

$$
\Delta V(n, x(n)) \leq h(n, V(n, x(n)), \sum_{s=n_0}^{n-1} V(s, x(s)))
$$

*(iii) $b(d[x(n), \hat{0}]) \leq V(n, x(n)) \leq a(d[x(n), \hat{0}])$ where $a, b \in \Gamma$ and $n \in \mathbb{N}^+_{n_0}$.*
*Then the asymptotic stability of the trivial solution of Eq.(2.2) imply the asymptotic stability of Eq.(3.2).*

**Proof .** Assume that the trivial solution of Eq.(2.2) is stable and $\epsilon > 0$. Then there exists $\delta_1 = \delta_1(n_0, \epsilon) > 0$ such that

$$
u(n_0) < \delta_1 \text{ implies } u(n) < b(\epsilon), \; n \geq n_0
$$

Choose $\delta = \delta(n_0, \epsilon) > 0$ such that $a(\delta) < \delta_1$ and $u(n_0) = V(n_0, x(n_0))$. Then theorem gives $V(n, x(n)) \leq u(n), n \geq n_0$ and we have:

$$
b(d[x(n), \hat{0}]) \leq V(n, x(n)) \leq u(n), n \geq n_0
$$

And

$$
u(n_0) = V(n_0, x(n_0)) \leq a(d[x(n_0), \hat{0}]) \leq a(\delta) < \delta_1
$$

Therefore

$$
b(d[x(n), \hat{0}]) \leq V(n, x(n)) \leq u(n) < b(\epsilon)
$$

Which implies the stability of the trivial solution of Eq.(3.2). Furthere more the relation $b(d[x(n), \hat{0}]) \leq u(n)$ implies the attractivity if the trivial solution of Eq.(2.2) is attractive. □

**Remark:** We have been studying the class of fuzzy volterra difference equations (2.1) which correspond to volterra difference equations (2.2). A general volterra difference equation is of the following form:

$$x(n+1) = f(n, x(n), x(n-1), ..., x(n_0)), n \geq n_0 \tag{3.4}$$

The definition and study of the dynamical properties of a general fuzzy volterra difference equation corresponding to (3.4) needs furthere work and is an idea for a useful research work in future.

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