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A comparative study on numerical, non-Bayes and Bayes estimation for the shape parameter of Kumaraswamy distribution

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Abstract

This paper is considered with Kumaraswamy distribution. Numerical, non-Bayes and Bayes methods of estimation were used to estimate the unknown shape parameter. The maximum likelihood is obtained as a non-Bayes estimator. As well as, Bayes estimators under a symmetric loss function (De-groot and NLINEX) by using four types of informative priors three double priors and one single prior. In addition, numerical estimators are obtained by using Newton's method and the false position method. Simulation research is conducted for the comparison of the effectiveness of the proposed estimators. Matlab 2015 will be used to obtain the numerical results.

Keywords: Kumaraswamy distribution, Bayes, non-Bayes, Numerical estimator.

1. Introduction

Kumaraswamy (1976,1978) [16, 17] has showed that the will know probability distribution function such as the normal, log-normal, beta and empirical distribution such as Johson's and polynomial-transformed-normal, etc. do not fit well hydrological data, such as daily rainfall, daily stream flow, etc. and developed a new probability density function known as the since power probability density function. Bantan et al. (2019) [6] introduced truncated inverted KD, and Ghosh (2019) [12] introduced bivariate and multivariate weighted KD.

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Kumaraswamy distribution using different methods of estimation and introduced by many authors some of whom are AL Noor N. H. and Ibraheem S.k. (2016) [3] used the maximum likelihood, Bayes and empirical Bayes methods of estimation for obtaining the estimate of the unknown shape parameter of KD under complete samples assuming that the other shape parameter is known. Abraheem S.k.et al (2020) [1] used classical maximum likelihood and Bayes methods estimator for obtaining the estimate of the unknown shape parameter of KD with a symmetric loss function via three types of informative priors two single prior and one double priors. As well as using expansion method. Al-obedy, N.J. et al. (2020) [4] used maximum Likelihood; Bayes methods estimation are used to estimate the unknown shape parameter of basic Gompertz distribution. The failure rate (hazad) function with the least loss was found using different priors under symmetric loss function, also numerical solution of failure rate function by using expansion method (Bernstein polynomial and power function).

In this work, deriving some estimators of the unknown shape parameter of KD based on a complete data assuming that the other shape parameter is known using: "maximum likelihood estimator" as the classical method in addition to Bayes estimator by assuming joint informative priors represented by (gamma-exponential, gamma-chi-squared, chi-squared-exponential and gamma priors) under asymmetric loss functions (De-groot and NLENEX loss functions) and using numerical methods to estimate the unknown shape parameter using (Newton's method and false position method). Compare the efficiency of classical, Bayesian and numerical estimators, using Mont-Carlo simulation method in terms of mean square error (MSE).

The probability density function of KD random variable is given by [11].

$$f(t;\varnothing,\ \ v)=\varnothing\ \ v\ v\ t^{v-1}(1-t^{v})^{\varnothing-1};\qquad 0<t\ <1\ ,\ \ \varnothing,v>0 \tag{1.1}$$

where $\varnothing$ and v are shape parameters respectively. Here we assume that v is the known shape parameter. The corresponding cumulative distribution function (cdf) is given by:

$$F(t;\varnothing,v)=1-(1-t^{v})^{\varnothing};\qquad 0<t<1;\varnothing,\ \ v>0 \tag{1.2}$$

The reliability and failure rate functions of KD are given, respectively by:

$$R(t)=1-F(t;\varnothing,v)=(1-t^{v})^{\varnothing};\qquad 0<t\ <1;\ \varnothing,\ v>0 \tag{1.3}$$

$$h(t)=\frac{f(t)}{R(t)}=\frac{\varnothing\ v\ t^{v-1}}{1-t^{v}};\qquad 0<t\ <1\ \varnothing,\ v>0 \tag{1.4}$$

2. Numerical Estimator of shape parameter

In this section, we will illustrate how to estimate the shape parameter $\varnothing$ numerically by using numerical methods (Newton method (NM) and false position method (FP) [7]). The fundamental idea in NM, starting with the initial estimate t_0, new estimate t_1 is the x-intercept of the tangent line to the function Y at (t_0 , $Y(t_0)$) the next estimate t_2 is the x-intercept of the tangent tine to the function Y at (t_1, $Y(t_1)$) and so on [10, 19]. From this process we get,

$$t_{n+1}=t_n-\frac{Y(t_n)}{\acute{Y}(t_n)}\qquad n=0,1,\ldots \tag{2.1}$$

The method of FP proceeds can be presented as follow [7, 13]:
Choose two initial estimate t_0, t_1 and define a_1=t_0 and b_1=t_1 . Then,

$$t_n=a_n-Y(a_n)\frac{a_{n-b_n}}{Y\ (a_n)-Y(b_n)}\qquad for\ n=1,2,\ldots \tag{2.2}$$

If $Y(t_n)$. $Y(a_n) > 0$ then $a_{n+1}=t_n$, $b_{n+1}=b_n$ else $a_{n+1}=a_n$, $b_{n+1}=t_n$.
In which must we find $Y(\varnothing) = 0$, from (1.3) we have:

$$Y(\varnothing) = R(t) - (1 - t^v)^{\varnothing} = 0 \ldots \tag{2.3}$$

In NM, let one initial value $\varnothing_0$ and find $\acute{Y}(\varnothing)$ as:

$$\acute{Y}(\varnothing) = -(1 - \ t^v)^{\varnothing} \ln(1 - t^v) \tag{2.4}$$

Now find $Y(\varnothing_0)$ and $\acute{Y}(\varnothing_0)$ then by Newton's iteration formula as in (2.1), we get:

$$\varnothing_1 = \varnothing_0 - \frac{Y(\varnothing_0)}{\acute{Y}(\varnothing_0)}$$

By repeat this proses, we have numerical estimate of $\varnothing$ called $\widehat{\varnothing}_{\mathrm{NM}}$.
In FP, let two initial values $a_1 = \varnothing_0$ and $b_1 = \varnothing_1$ and apply (2.2) to find $\varnothing_2$ from 1st iteration. Additional iterations can be performing to get the numerical estimate of $\varnothing$ called $\widehat{\varnothing}_{\mathrm{FP}}$. In above methods, we stop and find the estimate value of $\varnothing$ if :

$$|\varnothing_n - \varnothing_{n-1}| < \ \epsilon \qquad \textit{where } \epsilon \textit{ is very small}$$

3. Maximum Likelihood Estimator Method (MLEM)

The maximum likelihood method is attributed to Fisher. However, the method can be traced back to the works the 18-century scientists Lambert and Bernoulli. Fisher introduced the method of maximum likelihood in his first statistical publications in (1912) [9], and he developed it in (1920) [25]. Let t = $(t_1, t_2, \ldots . t_n)$ be the life time of random sample of size n drawn independently from KD defined by (1.1). Then the likelihood function for the given sample observations is:

$$L(\varnothing, v\,|t) = \prod_{i=1}^{n} f(t_i\,|\varnothing, v) = \varnothing^v v^n \prod_{i=1}^{n} \left[(t_i)^{v-1} (1 - t_i^v)^{\varnothing-1}\right]$$

Then

$$L(\varnothing, v\,|t\ ) = \varnothing^n v^n e^{(v-1)\sum_{i=1}^{n} \ln(t_i)} e^{(\varnothing-1)\sum_{i=1}^{n} \ln(1-t_i^v)} \tag{3.1}$$

We take the natural logarithm for the likelihood function so we get the function:

$$\ln L(\varnothing, v\,|t) = n \ln \varnothing + n \ln v (v-1) \sum_{i=1}^{n} \ln(t_i) + (\varnothing - 1) \sum_{i=1}^{n} \ln(1 - t_i^v) \tag{3.2}$$

The partial derivative for log-likelihood function with respect to unknown parameter $\varnothing$ is:

$$\frac{\partial \ln L}{\partial \varnothing} = \frac{n}{\varnothing} + \sum_{i=1}^{n} \ln(1 - t_i^v) \tag{3.3}$$

Then we equate the partial derivate force to zero, and get the following formula:

$$\begin{gathered} \frac{n}{\varnothing} + \sum_{i=1}^{n} \ln(1 - t_i^v) = 0 \\ \widehat{\varnothing}_{\mathrm{ML}} = \frac{-n}{\sum_{i=1}^{n} \ln(1 - t_i^v)} = \frac{-n}{T} \end{gathered} \tag{3.4}$$

where

$$T = \sum_{i=1}^{n} \ln(1 - t_i^v) \tag{3.5}$$

4. Standard Bayes Estimator Method (SBEM)

The standard Bayesian estimator method (SBEM) assumes that the random sample $t_1, t_2, \dots t_n$ taken from population with pdf $f(t;\varnothing)$. However the unknown parameter $\varnothing$ is considered to be as random variable in some real situation. There is often additional information available about $\varnothing$ (that means there is a prior knowledge exit about the parameter $\varnothing$. This method is based on the notation g($\varnothing$) that represent the prior distribution for parameter $\varnothing$ which come from prior knowledge, additional information and past experience. The Bayes estimator depend on the probability density function (posterior pdf) which includes information from previous knowledge and sample information [24]. The steps of standard Bayes estimator method are as the followings:

1. Finding conditional density function for parameter $\varnothing$ of the random variable $t_1, t_2, \dots t_n$

$$\pi\left(\varnothing \,|t\ \right) = \frac{L\left(\varnothing \,|t\ \right) g(\varnothing)}{\int_{\varnothing} L\left(\varnothing \,|t\ \right) g\left(\varnothing\right) d\varnothing} \tag{4.1}$$

   is called posterior density function there are two variables terms in Eq.(4.1), one term is the likelihood function $L(\varnothing\,|t)$, and the second is the prior probability of the parameter, $g(\varnothing)$ [5].

2. Using loss functions L($\widehat{\varnothing}, \varnothing$) which is defined to be real function satisfying :

   a- L($\widehat{\varnothing}, \varnothing$) ≥ 0 for all possible estimator $\widehat{\varnothing}$ and all parameter $\varnothing$.
   b- $L(\widehat{\varnothing}, \varnothing) = 0 \ for \ \widehat{\varnothing} = \varnothing$ [23].

   There are two function which as the following:

   - the De-groot loss function (weighted balance loss function) [8, 2].
   - Non-linear exponential (NLINEX) loos function [15].

3. Finding the risk function for parameter $\widehat{\varnothing}$.

$$Risk\ \left(\widehat{\varnothing}\right) = E\left[\ \ L(\widehat{\varnothing}, \varnothing)\right] = \int_{\varnothing} L\left(\widehat{\varnothing}, \varnothing\right) \pi\left(\varnothing|t\right) d\varnothing \tag{4.2}$$

   where $\widehat{\varnothing}$ is an estimate of $\varnothing$. If $\widehat{\varnothing} = \varnothing$ there is no loss if $\widehat{\varnothing} < \varnothing$ we call it under estimation, on the other hand if $\widehat{\varnothing} > \varnothing$ then we call it overestimation [26]. The value of $\widehat{\varnothing}$ which minimize the risk function is called the standard Bayes estimator.

4.1. Joint Posterior Density Function Using Gamma and Exponential Priors

The most widely used prior distribution of the parameter $\varnothing$ is the gamma distribution with hyper-parameters $\alpha \ and \ \beta$ with probability density function given by [21]:

$$g_1\left(\varnothing\right) = \frac{\beta^{\alpha}}{\Gamma\left(\alpha\right)} \varnothing^{\alpha-1} e^{-\varnothing\beta}; \qquad \varnothing > 0 \ \ , \alpha, \beta > 0 \tag{4.3}$$

The first posterior density function of the unknown parameter $\varnothing$ of KD have been obtained by combining the likelihood function (3.1) with the density function of gamma prior (4.3) and using (4.1) as:

$$\begin{aligned}\pi_1\left(\varnothing\,|t\ \right) &= \frac{\frac{\beta^{\alpha}}{\Gamma(\alpha)} \varnothing^{\alpha-1} e^{-\varnothing\beta} \varnothing^{n} \nu^{n} * e^{(\nu-1)\sum_{i=1}^{n} \ln t_i + (\varnothing-1)\sum_{i=1}^{n} \ln(1-t_i)}}{\int_0^{\infty} \frac{\beta^{\alpha}}{\Gamma(\alpha)} \varnothing^{n-1} e^{-\varnothing\beta} \varnothing^{n-1} e^{-\varnothing\beta} * \varnothing^{n} \nu^{n} e^{(\nu-1)\sum_{i=1}^{n} \ln t_i + (\varnothing-1)\sum_{i=1}^{n} \ln(1-t_i)} d\varnothing} \\ &= \frac{\varnothing^{n+\alpha-1} e^{-\varnothing(\beta-T)} (\beta-T)^{n+\alpha}}{\Gamma\left(n+\alpha\right)}\end{aligned} \tag{4.4}$$

The second prior distributing is exponential distribution with hyper parameter 'c' with probability density function given by [21].

$$g_2(\varnothing) = ce^{-\varnothing c}; \qquad \varnothing > 0, \qquad c > 0 \tag{4.5}$$

Can combining the likelihood (3.1) with the density function of exponential prior (4.5) via (4.1), results the second posterior density function of $\varnothing$.

$$\begin{aligned} \pi_2(\varnothing\,|t\,) &= \frac{\varnothing^n v^n * e^{(v-1)\sum_{i=1}^n \ln t_i + (\varnothing-1)\sum_{i=1}^n \ln\left(1-t_i^v\right)} \;*\; ce^{-\varnothing c}}{\int_0^\infty \varnothing^n v^n * e^{(v-1)\sum_{i=1}^n \ln t_i + (\varnothing-1)\sum_{i=1}^n \ln\left(1-t_i^v\right)} d\varnothing} \\ &= \frac{\varnothing^n e^{-\varnothing(c-T)}(c-T)^{n+1}}{\Gamma(n+\alpha)} \end{aligned} \tag{4.6}$$

By combining (4.3) and (4.5), obtain the double prior distribution (gamma-exponential) for $\varnothing$ as [22]:

$$g_3(\varnothing) = g_1(\varnothing)\, g_2(\varnothing) = \frac{c\beta^\alpha}{\Gamma(\alpha)} \varnothing^{\alpha-1} e^{-\varnothing(\beta+c)}; \qquad \varnothing > 0, \quad \alpha,\beta,c > 0 \tag{4.7}$$

Hence, the posterior distribution based on this double prior distribution of $\varnothing$ for given data t can be obtained, using (4.5), (4.6) and via (4.1) as :

$$\begin{aligned} \pi_3(\varnothing\,|t\,) &= \frac{\varnothing^n v^n * e^{(v-1)\sum_{i=1}^n \ln t_i + (\varnothing-1)\sum_{i=1}^n \ln\left(1-t_i^v\right)} \;*\; \frac{c\beta^\alpha}{\Gamma(\alpha)}\varnothing^{\alpha-1}e^{-\varnothing(\beta+c)}}{\int_0^\infty \varnothing^n v^n * e^{(v-1)\sum_{i=1}^n \ln t_i + (\varnothing-1)\sum_{i=1}^n \ln\left(1-t_i^v\right)} * \frac{c\beta^\alpha}{\Gamma(\alpha)}\varnothing^{\alpha-1}e^{-\varnothing(\beta+c)} d\varnothing} \\ &= \frac{\varnothing^{n+\alpha-1} e^{-\varnothing(\beta+c-T)}(\beta+c-T)^{n+\alpha}}{\Gamma(n+\alpha)} \end{aligned} \tag{4.8}$$

The probability density function (4.8) is similar to gamma distribution G (α_1, β_1) where $\alpha_1 = (n+\alpha)$ and $\beta_1 = (\beta + c - T)$

4.2. Joint Posterior Density Function Using Gamma and Chi-Squared Priors

Chi-squared priors of the parameter $\varnothing$ with hyper- parameterC_2 defined by the following density [5]:

$$g_4(\varnothing) = \frac{e^{-\frac{\varnothing}{2}} \;\; \varnothing^{\frac{c_2}{2}-1}}{2^{\frac{c_2}{2}}\Gamma\left(\frac{c_2}{2}\right)}; \qquad \varnothing > 0, \quad c_2 > 0 \tag{4.9}$$

Combining the likelihood function (3.1) with the density function of chi-squared prior (4.9) and using (4.1), results the fourth posterior density function of $\varnothing$.

$$\begin{aligned} \pi_4(\varnothing\,|t\,) &= \frac{\varnothing^n v^n * e^{(v-1)\sum_{i=1}^n \ln t_i + (\varnothing-1)\sum_{i=1}^n \ln\left(1-t_i^v\right)} \;*\; \frac{e^{\frac{-\varnothing}{2}} \;\varnothing^{\frac{c_2}{2}-1}}{2^{\frac{c_2}{2}}\Gamma\left(\frac{c_2}{2}\right)}}{\int_0^\infty \varnothing^n v^n * e^{(v-1)\sum_{i=1}^n \ln t_i + (\varnothing-1)\sum_{i=1}^n \ln\left(1-t_i^v\right)} * \frac{e^{\frac{-\varnothing}{2}} \;\varnothing^{\frac{c_2}{2}-1}}{2^{\frac{c_2}{2}}\Gamma\left(\frac{c_2}{2}\right)} d\varnothing} \\ &= \frac{\varnothing^{n+\frac{c_2}{2}-1} e^{-\varnothing\left(\frac{1}{2}-T\right)}\left(\frac{1}{2}-T\right)^{n+\frac{c_2}{2}}}{\Gamma\left(n+\frac{c_2}{2}\right)} \end{aligned} \tag{4.10}$$

By combining (4.3) and (4.9), obtain the double prior distribution (gamma-chi- squared) for $\varnothing$ as [20]:

$$g_5(\varnothing) = g_1(\varnothing) \;.\; g_4(\varnothing) = \frac{\beta^\alpha \varnothing^{\alpha+\frac{c_2}{2}-2} e^{-\varnothing\left(\beta+\frac{1}{2}\right)}}{\Gamma\left(\alpha+\frac{c_2}{2}\right)2^{\frac{c_2}{2}}}; \qquad \varnothing > 0, \quad \alpha,\beta,c_2 > 0 \tag{4.11}$$

Hence, the posterior distribution of ∅ based on this double prior distribution of ∅ for given data t can be obtained, using (3.1) and (4.11), as:

$$\pi_5(\varnothing|t) = \frac{\varnothing^n v^n * e^{(v-1)\sum_{i=1}^n \ln t_i + (\varnothing-1)\sum_{i=1}^n \ln\left(1-t_i^v\right)} * \frac{\beta^\alpha \varnothing^{\alpha+\frac{c_2}{2}-2} e^{-\varnothing(\beta-\frac{1}{2})}}{\Gamma\left(\alpha+\frac{c_2}{2}\right) 2^{\frac{c_2}{2}}}}{\int_0^\infty \varnothing^n v^n * e^{(v-1)\sum_{i=1}^n \ln t_i + (\varnothing-1)\sum_{i=1}^n \ln\left(1-t_i^v\right)} * \frac{\beta^\alpha \varnothing^{\alpha+\frac{c_2}{2}-2} e^{-\varnothing(\beta-\frac{1}{2})}}{\Gamma\left(\alpha+\frac{c_2}{2}\right) 2^{\frac{c_2}{2}}} d\varnothing}$$
$$= \frac{\varnothing^{n+\alpha+\frac{c_2}{2}-2} e^{-\varnothing(\beta+\frac{1}{2}-T)} \left(\beta+\frac{1}{2}-T\right)^{n+\alpha+\frac{c_2}{2}-1}}{\Gamma\left(n+\alpha+\frac{c_2}{2}-1\right)} \tag{4.12}$$

The probability density in (4.12) is similar to gamma distribution G (α_2, β_2) where $\alpha_2 = \left(n+\alpha+\frac{c_2}{2}-1\right)$ and $\beta_2 = \left(\beta+\frac{1}{2}-T\right)$.

4.3. Joint Posterior Density Function Using Chi-Squared and Exponential Priors

In a similar manner, assume both the prior distributions have pdfs given by (4.5) and (4.9). Hence, the double prior distribution ∅ becomes [20]

$$g_6(\varnothing) = g_2(\varnothing)\, g_4(\varnothing) = \frac{c\varnothing^{\frac{c_2}{2}-1} e^{-\varnothing(c+\frac{1}{2})}}{2^{\frac{c_2}{2}}\Gamma\left(\frac{c_2}{2}\right)}; \qquad \varnothing > 0, \quad c, c_2 > 0 \tag{4.13}$$

and the poster for distribution of ∅ given the data t, based on this double prior distribution, comes out to be (3.1) and (4.13), as:

$$\pi_6(\varnothing|t) = \frac{\varnothing^n v^n * e^{(v-1)\sum_{i=1}^n \ln t_i + (\varnothing-1)\sum_{i=1}^n \ln\left(1-t_i^v\right)} \left[\frac{c\varnothing^{\frac{c_2}{2}-1} e^{-\varnothing(c+\frac{1}{2})}}{2^{\frac{c_2}{2}}\Gamma\left(\frac{c_2}{2}\right)}\right]}{\int_0^\infty \varnothing^n v^n * e^{(v-1)\sum_{i=1}^n \ln t_i + (\varnothing-1)\sum_{i=1}^n \ln\left(1-t_i^v\right)} \left[\frac{c\varnothing^{\frac{c_2}{2}-1} e^{-\varnothing(c+\frac{1}{2})}}{2^{\frac{c_2}{2}}\Gamma\left(\frac{c_2}{2}\right)}\right] d\varnothing}$$
$$= \frac{\varnothing^{n+\frac{c_2}{2}-1} e^{-\varnothing(c+\frac{1}{2}-T)} \left(c+\frac{1}{2}-T\right)^{n+\frac{c_2}{2}}}{\Gamma\left(n+\frac{c_2}{2}\right)} \tag{4.14}$$

The probability density function in (4.14) is similar to gamma distribution G (α_3, β_3), where $\alpha_3 = \left(n+\frac{c_2}{2}\right)$ and $\beta_3 = \left(c+\frac{1}{2}-T\right)$.

4.4. Posterior Density Function Using Gamma Distribution

Here, we consider only a single gamma prior distribution for ∅ , given by (4.3) [20, 14], and corresponding posterior distribution for ∅ as (4.4). Which is also a gamma distribution $G(\alpha_4, \beta_4)$ with parameter $\alpha_4 = (n+\alpha)$ and $\beta_4 = (\beta - T)$, thus in all the cases of the different types of double Prior distribution and in the case of a single prior distribution, the posterior distribution of ∅ given the data t becomes a gamma distribution.

5. Loss Functions under Study

Some Bayesian estimators are obtained based on two loss function which are: De-groot loss function (weighted balance loss function) and Non-Linear exponential (NLINEX) loss function as asymmetric loss functions.

5.1. De-groot Loss Function (Weighted Balance Loss Function)

In Bayesian estimation, we consider a type of loss function which is classified as asymmetric function, was introduced by De-groot (2005) [8], which is widely used in most estimation problems. It can be defined as [2, 18].

$$L(\widehat{\varnothing},\varnothing)=\frac{\left(\varnothing-\widehat{\varnothing}\right)^{2}}{\widehat{\varnothing}^{2}} \tag{5.1}$$

According to (4.2) by taking the derivative of loss function (5.1) with respect to $\widehat{\varnothing}$ and setting it equal to zero, the Bayes estimator of $\varnothing$ based on De-groot loss function, denoted by $\widehat{\varnothing}_d$, can obtained as:

$$\widehat{\varnothing}_d=\frac{E_{\pi(\varnothing^2|t\ )}}{E_{\pi(\varnothing|t\ )}} \tag{5.2}$$

5.2. Non- Liner Exponential (NLINEX) Loss Function

In this section we have proposed a new loss function that is asymmetric in nature and non-linear function of the error called non-linear exponential (NLINEX) loss function was proposed by Islam et al. (2004) is linear combination of LINEX loss function and squared error loss function [15].
For NLINEX loss function, the Bayes estimator a parameter $\varnothing$ is,

$$\widehat{\varnothing}_{\mathrm{NL}}=\frac{-\left[\ln E_{\pi}\left(e^{-\ c_1\varnothing}\right)-2E_{\pi}\left(\varnothing\right)\right]}{(c_1+2)} \tag{5.3}$$

where E_π stands for posterior expectation.

5.3. Bayes Estimators under the De-groot Loss Function (Weighted Balance Loss Function)

In this subsections, we obtain Bayes estimators of $\varnothing$ for KD corresponding to different posterior distributions .

***Corresponding to $\pi_3(\varnothing\,|t\ )$**

Under the gamma- exponential prior distribution, using (4.8) and (5.1), the Bayes estimator of $\varnothing$ based on De-groot loss function corresponding to $\pi_3(\varnothing\,|t\ )$ and using (4.1) can be found as:

$$E_{\pi_3}\left(\ \varnothing|\,t\right)=\frac{\Gamma\left(n+\alpha+1\right)}{\Gamma\left(n+\alpha\right)\left(\beta+c-T\right)} \tag{5.4}$$

And

$$E_{\pi_3}\left(\ \varnothing^2\middle|\,t\right)=\frac{\Gamma\left(n+\alpha+2\right)}{\Gamma\left(n+\alpha\right)\left(\beta+c-T\right)^2} \tag{5.5}$$

From (5.4) and (5.5), the Bayes estimator of $\varnothing$ based on De-groot loss function under the assumption of gamma-exponential prior information is given by:

$$\widehat{\varnothing}_{\mathrm{Bdge}}=\frac{(n+\alpha+2)}{(\beta+c-T)} \tag{5.6}$$

***Corresponding to $\pi_5\left(\ \varnothing|\,t\right)$**

Under the gamma-chi-squared prior distribution, using (4.12) and (5.2), the Bayes estimator of $\varnothing$ based on De-groot loss function corresponding to $\pi_5\left(\ \varnothing|\,t\right)$ and by (4.2) can be found as :

$$E_{\pi_5}\left(\ \varnothing|\,t\right)=\frac{\Gamma\left(n+\alpha+\frac{c_2}{2}\right)}{\Gamma\left(n+\alpha+\frac{c_2}{2}\ \ -1\right)\left(\beta+\frac{1}{2}-T\right)} \tag{5.7}$$

And

$$E_{\pi_5}\left(\varnothing^2\,|t\;\right)=\frac{\Gamma\left(n+\alpha+\frac{c_2}{2}\;\;+1\right)}{\Gamma\left(n+\alpha+\frac{c_2}{2}\;\;-1\right)\left(\beta+\frac{1}{2}\;\;-T\right)^2} \tag{5.8}$$

From (5.7) and (5.8), the Bayes estimator of $\varnothing$ based on De-groot loss function under the assumption of gamma-chi-squared prior information is given by :

$$\widehat{\varnothing}_{\text{Bdgch}}=\frac{\left(n+\alpha+\frac{c_2}{2}+1\right)}{\left(\beta+\frac{1}{2}-T\right)} \tag{5.9}$$

***Corresponding to** $\pi_{\mathbf{6}}\left(\;\varnothing|\,t\right)$

Under the chi-squared-exponential prior distribution, using (4.14) and (5.2), the Bayes estimator of $\varnothing$ based on De-groot loss function corresponding to $\pi_{\mathbf{6}}\left(\;\varnothing|\,t\right)$, by the same way can be found as :

$$E_{\pi_6}\left(\;\varnothing|\,t\right)=\frac{\Gamma\left(n+\frac{c_2}{2}+1\right)}{\Gamma\left(n+\frac{c_2}{2}\right)\left(c+\frac{1}{2}-T\right)} \tag{5.10}$$

And

$$E_{\pi_6}\left(\varnothing^2\,|t\;\right)=\frac{\Gamma\left(n+\frac{c_2}{2}+2\right)}{\Gamma\left(n+\frac{c_2}{2}\right)\left(c+\frac{1}{2}\;\;-T\right)^2} \tag{5.11}$$

From (5.10) and (5.11), the Bayes estimator of $\varnothing$ based on De-groot loss function under the assumption of chi-squared-exponential prior information is given by :

$$\widehat{\varnothing}_{\text{Bdche}}=\frac{\left(n+\frac{c_2}{2}+2\right)}{\left(c+\frac{1}{2}-T\right)} \tag{5.12}$$

***Corresponding to** $\pi_{\mathbf{1}}\left(\varnothing\,|\mathbf{t}\;\right)$

From (4.4) and (5.2), the Bayes estimator of $\varnothing$ based on De-groot loss function corresponding to $\pi_1\left(\varnothing\,|t\;\right)$ can be found as:

$$E_{\pi_1}\left(\varnothing\,|t\;\right)=\frac{\Gamma\left(n+\alpha+1\right)}{\Gamma\left(n+\alpha\right)\left(\beta-T\right)} \tag{5.13}$$

And

$$E_{\pi_1}\left(\varnothing^2\,|t\;\right)=\frac{\Gamma\left(n+\alpha+2\right)}{\Gamma\left(n+\alpha\right)\left(\beta-T\right)^2} \tag{5.14}$$

From (5.13) and (5.14), the Bayes estimator of $\varnothing$ based on De-groot loss function under the assumption of gamma prior information is given by :

$$\widehat{\varnothing}_{\text{Bdg}}=\frac{\left(n+\alpha+2\right)}{\left(\beta-T\right)} \tag{5.15}$$

5.4. Bayes Estimators under the Non – Linear Exponential (NLINEX) Loss Function

Here, based on NLINEX loss function, we obtain Bayes estimators of $\varnothing$ and R(t) for KD corresponding to different posterior distributions.

***Corresponding to** $\pi_{\mathbf{3}}\left(\varnothing\,|t\;\right)$

From (4.8) and (5.2), the Bayes estimator of $\varnothing$ based on NLINEX loss function corresponding to $\pi_3(\varnothing|t)$ can be found as:

$$\begin{aligned} E_{\pi_3}\left(e^{-c_1\varnothing}|t\right) &= \left(\frac{\beta+c-T}{\beta+c-T+c_1}\right)^{n+\alpha} \\ \ln E_{\pi_3}\left(e^{-c_1\varnothing}|t\right) &= (n+\alpha)\ln\left(\frac{\beta+c-T}{\beta+c-T+c_1}\right) \end{aligned} \tag{5.16}$$

From (5.4) and (5.16), the Bayes estimator of $\varnothing$ based on NLINEX loss function under the assumption of gamma-exponential prior information is given by :

$$\widehat{\varnothing}_{\text{BNLge}} = \frac{-\left[(n+\alpha)\ln\left(\frac{\beta+c-T}{\beta+c-T+c_1}\right) - 2\frac{(n+\alpha+1)}{(\beta+c-T)}\right]}{(c_1+2)} \tag{5.17}$$

***Corresponding to** $\pi_5(\varnothing|t)$

From (4.12) and (5.2), the Bayes estimator of $\varnothing$ based on NLINEX loss function corresponding to $\pi_5(\varnothing|t)$ can be found as:

$$\begin{aligned} E_{\pi_5}\left(e^{-c_1\varnothing}|t\right) &= \left(\frac{\beta+\frac{1}{2}-T}{\beta+\frac{1}{2}-T+c_1}\right)^{n+\alpha+\frac{c_2}{2}-1} \\ \ln\ \mathrm{E}_{\pi_5}\left(e^{-c_1\varnothing}|t\right) &= \left(n+\alpha+\frac{c_2}{2}-1\right)\ln\left(\frac{\beta+\frac{1}{2}-T}{\beta+\frac{1}{2}-T+c_1}\right) \end{aligned} \tag{5.18}$$

From (5.7) and (5.18), the Bayes estimator of $\varnothing$ based on NLINEX loss function under the assumption of gamma -chi-squared prior information is given by:

$$\widehat{\varnothing}_{\text{BNLgch}} = \frac{-\left[\left(n+\alpha+\frac{c_2}{2}-1\right)\ln\left(\frac{\beta+\frac{1}{2}-T}{\beta+\frac{1}{2}-T+c_1}\right) - 2\frac{\left(n+\alpha+\frac{c_2}{2}\right)}{\left(\beta+\frac{1}{2}-T\right)}\right]}{(c_1+2)} \tag{5.19}$$

***Corresponding to** $\pi_6(\varnothing|t)$

From (4.14) and (5.2), the Bayes estimator of $\varnothing$ based on NLINEX loss function corresponding to $\pi_6(\varnothing|t)$ can be found as:

$$\begin{aligned} E_{\pi_6}\left(e^{-c_1\varnothing}|t\right) &= \left(\frac{c+\frac{1}{2}-T}{c+\frac{1}{2}-T+c_1}\right)^{n+\frac{c_2}{2}} \\ \ln E_{\pi_6}\left(e^{-c_1\varnothing}|t\right) &= \left(n+\frac{c_2}{2}\right)\ln\left(\frac{c+\frac{1}{2}-T}{c+\frac{1}{2}-T+c_1}\right) \end{aligned} \tag{5.20}$$

From (5.10) and (5.20), the Bayes estimator of $\varnothing$ based on NLINEX loss function under the assumption of chi-squared-exponential prior information is given by:

$$\widehat{\varnothing}_{\text{BNLche}} = \frac{-\left[\left(n+\frac{c_2}{2}\right)\ln\left(\frac{c+\frac{1}{2}-T}{c+\frac{1}{2}-T+c_1}\right) - 2\frac{\left(n+\frac{c_2}{2}+1\right)}{\left(c+\frac{1}{2}-T\right)}\right]}{(c_1+2)} \tag{5.21}$$

***Corresponding to** $\pi_1(\varnothing|t)$

From (4.4) and (5.2), the Bayes estimator of $\varnothing$ based on NLINEX loss function corresponding to $\pi_1(\varnothing\,|t\,)$ can be found as:

$$
\begin{aligned}
E_{\pi_1}\left(e^{-c_1\varnothing}\,|t\,\right) &= \left(\frac{\beta-T}{\beta-T+c_1}\right)^{n+\alpha} \\
\ln E_{\pi_1}\left(e^{-c_1\varnothing}\,|t\,\right) &= (n+\alpha)\ln\left(\frac{\beta-T}{\beta-T+c_1}\right)
\end{aligned}
\tag{5.22}
$$

From (5.13) and (5.22), the Bayes estimator of $\varnothing$ based on NLINEX loss function under the assumption of gamma prior information is given by:

$$
\widehat{\varnothing}_{\mathrm{BNLg}} = \frac{-\left[(n+\alpha)\ln\left(\frac{\beta-T}{\beta-T+c_1}\right)-2\left(\frac{n+\alpha+1}{\beta-T}\right)\right]}{(c_1+2)} \tag{5.23}
$$

6. Simulation Study

The simulation study state that used to estimate $\varnothing$ of KD can be summarized by the following steps:

Step (1):

- In this step, it has been set default values of parameters and constants for simulation experiments summarized in the following table.

Table 1: Default Values of Parameters and Constants that have been used in Simulation Experiments

Simple size	n	***10, 15, 25, 50, 100***
Shape parameter	$\varnothing$	***1.5, 2, 2.5***
Hyper-Parameter-Gamma-Exponential	α	***3***
	β	***2***
	C	***1.5***
Hyper-Parameter-Gamma-Chi-Squared	α	***3***
	β	***2***
	C_2	***2***
Hyper-Parameter-Chi-Squared- Exponential	C	***1.5***
	C_2	***2***
Hyper-Parameter-Gamma	α	***3***
	β	***2***
Number of Sample Replicate	L	***1000***

- The shape parameters of KD which are varied into nine cases to observe their effect on the estimates when $v > \varnothing$, $v = \varnothing$ and $v < \varnothing$.
- The values of NLINEX loss function constant (C_1) used are different values which are indicated in the tables.
- The simulation study process is replicate 1000 times to get independent samples from different sizes.

Step (2): At this step, we are generating random samples as follows:
Suppose that U is a random variable with uniform distribution in (0, 1), then the data of this distribution can be created using the inverse transformation method of the cdf where:

$$U = F(t) \tag{6.1}$$

$$t = F^{-1}(U) \tag{6.2}$$

Now, substituting equation (1.2) in equation (6.1), we get

$$U_i = F\,(t_i) = \ \ 1 - \ (1 - t_i^v)^{\varnothing} \ ; \qquad t \ > 0 \ ; \ \ \varnothing, \ v \ > 0$$

Simplify this equation, we have

$$t_i = [\ 1 - \ (1 - U_i)^{\frac{1}{\varnothing}}]^{\frac{1}{v}} \ ; \qquad i = 1, \ \ldots, \ n \tag{6.3}$$

Step (3): Calculate the non- Bayes, Bayes and numerical estimators of the unknown shape parameter $\varnothing$ of KD according to the formulas that we have obtained in the previous section.

Step (4): Compare the different estimation methods for the shape parameter according to mean squared error (MSE).
The best estimator is the estimator that gives the smallest value of MSE where MSE are given as:

$$MSE\ (\widehat{\varnothing}) = \frac{\sum_{j=1}^{L} (\widehat{\varnothing}_j - \ \varnothing)^2}{L} \tag{6.4}$$

where,
L : is the number of sample replicated.
$\widehat{\varnothing}_j$: is the estimate of $\varnothing$ at the j^{th} replicate.

7. Simulation Results for Estimating the Shape Parameter

The table (2), include nine different cases which contains the MSE values for non-Bayes, Bayes and numerical estimators, it appears that:

- From case (I) – case (VI), with case (VIIII) when n=100, the best prior of Bayes estimators based on De-groot loss function is gamma–exponential while case (VII) to case (VIIII) is gamma–chi–squared for all samples sizes.
- From case (I) to case (VIIII), the best prior of Bayes estimator based on NLINEX loss function is gamma for all sample sizse.
- From case (I) and case (II), the best loss is De-groot loss function with gamma–exponential and chi–squared–exponential priors while the best loss NLINEX loss function with gamma–chi–squared and gamma priors for all sample sizes except n=100.
- From case (III): $\varnothing = 1.5 and v = 2$, the best loss is NLINEX loss function for all priors and all sample sizes except $n = 10, 25$ the De-groot loss function is best when prior is gamma–exponential.
- From case (IV): $\varnothing = 2 and v = 1$, the best loss is De-groot loss function for all priors and all sample sizes except n= 10, 50 NLINEX loss function is best when prior is gamma.

- From case (V) – case (VIIII), the best loss is De-groot loss function for all priors but in case (V) and case(VI) the best loss is NLINEX loss function when the prior is gamma.
- The MSE values associated with numerical estimators are the better than non-Bayes and Bayes estimates with different cases and all sample sizes.
- Newton's method is the best estimator than false position method for all different cases.
- From all cases the best prior is double prior function with De-groot while the single prior is the best with NLINEX.
- From case (I) to case (III) the Bayes estimation methods with all priors are the best from non-Bayes estimation method for all sample sizes while n= 15, 25, 50, 100 the non-Bayes method is the best estimation from Bayes method with gamma prior of De-groot loss function.
- From case (IV) – case (VIIII) the Bayes estimation methods for all priors of De-groot are the best from non-Bayes estimation method.

Table (2): MSE Values for Non-Bayes, Numerical and Bayes Estimators of the Shape Parameter ($\varnothing$) of Kumaraswamy Distribution with Different cases

Case (I): $\varnothing = 1.5, v = 1$

n	Non-Bayes Method ML $\widehat{\Theta}(t)_{ML}$	Numerical Methods		Bayes Methods			Best Loss
		$\widehat{\Theta}(t)_{PF}$	$\widehat{\Theta}(t)_{BP}$	Perior	Degroot $\widehat{\Theta}(t)_{Bd}$	NLINEX $\widehat{\Theta}(t)_{BNL}$ C1=6.7	
10	0.3862	5.2982e-14	2.2627e-04	Gamma-Exponential	0.1035	0.1876	Degroot
				Gamma-Chi-squared	0.0064	0.0048	NLINEX
				Chi-squared- Exponential	0.1677	0.2149	Degroot
				Gamma	0.3304	0.1136	NLINEX
Best Prior					Gamma-Exponential	Gamma	
15	0.1763	8.4342e-12	4.1279e-05	Gamma-Exponential	0.0776	0.1334	Degroot
				Gamma-Chi-squared	0.1332	0.0980	NLINEX
				Chi-squared- Exponential	0.1081	0.1475	Degroot
				Gamma	0.1842	0.0861	NLINEX
Best Prior					Gamma-Exponential	Gamma	
25	0.0760	3.1967e-16	4.0645e-08	Gamma-Exponential	0.0456	0.0648	Degroot
				Gamma-Chi-squared	0.0722	0.0484	NLINEX
				Chi-squared- Exponential	0.0575	0.0688	Degroot
				Gamma	0.0936	0.0434	NLINEX
Best Prior					Gamma-Exponential	Gamma	
50	0.0449	3.5596e-12	3.0876e-05	Gamma-Exponential	0.0354	0.0428	Degroot
				Gamma-Chi-squared	0.0425	0.0374	NLINEX
				Chi-squared- Exponential	0.0394	0.0444	Degroot
				Gamma	0.0479	0.0357	NLINEX
Best Prior					Gamma-Exponential	Gamma	
100	.0221 0	1.5433e-14	1.7097e-04	Gamma-Exponential	0.0196	0.0202	Degroot
				Gamma-Chi-squared	0.0220	0.0190	NLINEX
				Chi-squared- Exponential	0.0208	0.0207	NLINEX
				Gamma	0.0236	0.0187	NLINEX
Best Prior					Gamma-Exponential	Gamma	

Case (II): $\varnothing = 1.5, \upsilon = 1.5$

n	*Non-Bayes Method ML* $\widehat{\Theta}(t)_{ML}$	*Numerical Methods*		*Bayes Methods*			*Best Loss*
		$\widehat{\Theta}(t)_{PF}$	$\widehat{\Theta}(t)_{BP}$	*Perior*	*Degroot* $\widehat{\Theta}(t)_{Bd}$	*NLINEX* $\widehat{\Theta}(t)_{BNL}$ *C1=6.5*	
10	*0.3065*	*2.0207e-13*	*1.2154e-04*	*Gamma-Exponential*	*0.0884*	*0.1788*	*Degroot*
				Gamma-Chi-squared	*0.1887*	*0.1220*	*NLINEX*
				Chi-squared- Exponential	*0.1412*	*0.2045*	*Degroot*
				Gamma	*0.2911*	*0.1025*	*NLINEX*
Best Prior					*Gamma-Exponential*	*Gamma*	
15	*0.1847*	*2.1918e-17*	*7.3362e-06*	*Gamma-Exponential*	*0.0767*	*0.1121*	*Degroot*
				Gamma-Chi-squared	*0.1439*	*0.0804*	*NLINEX*
				Chi-squared- Exponential	*0.1105*	*0.1239*	*Degroot*
				Gamma	*0.2032*	*0.0712*	*NLINEX*
Best Prior					*Gamma-Exponential*	*Gamma*	
25	*0.1068*	*4.1989e-11*	*1.4220e-05*	*Gamma-Exponential*	*0.0653*	*0.0829*	*Degroot*
				Gamma-Chi-squared	*0.0928*	*0.0678*	*NLINEX*
				Chi-squared- Exponential	*0.0807*	*0.0890*	*Degroot*
				Gamma	*0.1150*	*0.0636*	*NLINEX*
Best Prior					*Gamma-Exponential*	*Gamma*	
50	*0.0471*	*6.3263e-18*	*1.0337e-04*	*Gamma-Exponential*	*0.0371*	*0.0423*	*Degroot*
				Gamma-Chi-squared	*0.0449*	*0.0375*	*NLINEX*
				Chi-squared- Exponential	*0.0414*	*0.0440*	*Degroot*
				Gamma	*0.0506*	*0.0362*	*NLINEX*
Best Prior					*Gamma-Exponential*	*Gamma*	
100	*.0241 0*	*1.5641e-10*	*4.1088e-07*	*Gamma-Exponential*	*0.0215*	*0.0219*	*Degroot*
				Gamma-Chi-squared	*0.0238*	*0.0208*	*NLINEX*
				Chi-squared- Exponential	*0.0227*	*0.0225*	*NLINEX*
				Gamma	*0.0254*	*0.0206*	*NLINEX*
Best Prior					*Gamma-Exponential*	*Gamma*	

Case (III): $\varnothing = 1.5, \upsilon = 2$

n	*Non-Bayes Method ML* $\widehat{\Theta}(t)_{ML}$	*Numerical Methods*		*Bayes Methods*			*Best Loss*
		$\widehat{\Theta}(t)_{PF}$	$\widehat{\Theta}(t)_{BP}$	*Perior*	*Degroot* $\widehat{\Theta}(t)_{Bd}$	*NLINEX* $\widehat{\Theta}(t)_{BNL}$ *C1=6.3*	
10	*0.4143*	*2.3027e-14*	*5.6722e-06*	*Gamma-Exponential*	*0.1015*	*0.1432*	*Degroot*
				Gamma-Chi-squared	*0.2435*	*0.0945*	*NLINEX*
				Chi-squared- Exponential	*0.1741*	*0.1637*	*NLINEX*
				Gamma	*0.3806*	*0.0824*	*NLINEX*
Best Prior					*Gamma-Exponential*	*Gamma*	
15	*0.2281*	*8.4342e-12*	*4.1279e-05*	*Gamma-Exponential*	*0.0883*	*0.0853*	*NLINEX*
				Gamma-Chi-squared	*0.1806*	*0.0612*	*NLINEX*
				Chi-squared- Exponential	*0.1323*	*0.0945*	*NLINEX*
				Gamma	*0.2580*	*0.0578*	*NLINEX*
Best Prior					*Gamma-Exponential*	*Gamma*	
25	*0.1066*	*7.3468e-24*	*2.1538e-11*	*Gamma-Exponential*	*0.0628*	*0.0672*	*Degroot*
				Gamma-Chi-squared	*0.0963*	*0.0551*	*NLINEX*
				Chi-squared- Exponential	*0.0794*	*0.0724*	*NLINEX*
				Gamma	*0.1221*	*0.0528*	*NLINEX*
Best Prior					*Gamma-Exponential*	*Gamma*	
50	*0.0505*	*5.2982e-14*	*2.2627e-04*	*Gamma-Exponential*	*0.0391*	*0.0381*	*NLINEX*
				Gamma-Chi-squared	*0.0492*	*0.0350*	*NLINEX*
				Chi-squared- Exponential	*0.0441*	*0.0400*	*NLINEX*
				Gamma	*0.1221*	*0.0528*	*NLINEX*
Best Prior					*Gamma-Exponential*	*Gamma*	
100	*0.0233*	*2.9129e-11*	*4.4305e-04*	*Gamma-Exponential*	*0.0206*	*0.0206*	
				Gamma-Chi-squared	*0.0230*	*0.0197*	*NLINEX*
				Chi-squared- Exponential	*0.0218*	*0.0212*	*NLINEX*
				Gamma	*0.0247*	*0.0195*	*NLINEX*
Best Prior					*Gamma-Exponential*	*Gamma*	

Case (IV): $\varnothing = 2, v = 1$

n	Non-Bayes Method ML $\widehat{\Theta}(t)_{ML}$	Numerical Methods		Bayes Methods			Best Loss
		$\widehat{\Theta}(t)_{PF}$	$\widehat{\Theta}(t)_{BP}$	Perior	Degroot $\widehat{\Theta}(t)_{Bd}$	NLINEX $\widehat{\Theta}(t)_{BNL}$ C1=6.7	
10	0.4143	2.3027e-14	5.6722e-06	Gamma-Exponential	0.1015	0.14327	Degroot
				Gamma-Chi-squared	0.2435	0.0945	NLINEX
				Chi-squared- Exponential	0.1741	0.1637	NLINEX
				Gamma	0.3806	0.0824	NLINEX
Best Prior					Gamma-Exponential	Gamma	
15	0.2281	8.4342e-12	4.1279e-05	Gamma-Exponential	0.0883	0.0853	NLINEX
				Gamma-Chi-squared	0.1806	0.0612	NLINEX
				Chi-squared- Exponential	0.1323	0.0945	NLINEX
				Gamma	0.2580	0.0578	NLINEX
Best Prior					Gamma-Exponential	Gamma	
25	0.1066	7.3468e-24	2.1538e-11	Gamma-Exponential 0.0628	0.0672	Degroot	
				Gamma-Chi-squared	0.0963	0.0551	NLINEX
				Chi-squared- Exponential	0.0794	0.0724	NLINEX
				Gamma	0.1221	0.0528	NLINEX
Best Prior					Gamma-Exponential	Gamma	
50	0.0505	5.2982e-14	2.2627e-04	Gamma-Exponential	0.0391	0.0381	NLINEX
				Gamma-Chi-squared	0.0492	0.0350	NLINEX
				Chi-squared- Exponential	0.0441	0.0400	NLINEX
				Gamma	0.0562	0.0346	NLINEX
Best Prior					Gamma-Exponential	Gamma	
100	0.0233	2.9129e-11	4.4305e-04	Gamma-Exponential	0.0206	0.0206	
				Gamma-Chi-squared	0.0230	0.0197	NLINEX
				Chi-squared- Exponential	0.0218	0.0212	NLINEX
				Gamma	0.0247	0.0195	NLINEX
Best Prior					Gamma-Exponential	Gamma	

Case (V): $\varnothing = 2, v = 2$

n	Non-Bayes Method ML $\widehat{\Theta}(t)_{ML}$	Numerical Methods		Bayes Methods			Best Loss
		$\widehat{\Theta}(t)_{PF}$	$\widehat{\Theta}(t)_{BP}$	Perior	Degroot $\widehat{\Theta}(t)_{Bd}$	NLINEX $\widehat{\Theta}(t)_{BNL}$ C1=6.7	
10	0.6848	1.0433e-21	1.7172e-06	Gamma-Exponential	0.1376	0.5566	Degroot
				Gamma-Chi-squared	0.1942	0.3857	Degroot
				Chi-squared- Exponential	0.1930	0.5605	Degroot
				Gamma	0.3190	0.3082	NLINEX
Best Prior					Gamma-Exponential	Gamma	
15	0.2855	1.4994e-11	2.7734e-05	Gamma-Exponential	0.1104	0.3945	Degroot
				Gamma-Chi-squared	0.1376	0.2776	Degroot
				Chi-squared- Exponential	0.1388	0.3847	Degroot
				Gamma	0.1963	0.2263	Degroot
Best Prior					Gamma-Exponential	Gamma	
25	0.1699	6.3281e-12	1.9610e-05	Gamma-Exponential	0.0876	0.2276	Degroot
				Gamma-Chi-squared	0.1087	0.1658	Degroot
				Chi-squared- Exponential	0.1075	0.2159	Degroot
				Gamma	0.1385	0.1407	Degroot
Best Prior					Gamma-Exponential	Gamma	
50	0.0814	7.9126e-13	8.5065e-06	Gamma-Exponential	0.0576	0.1031	Degroot
				Gamma-Chi-squared	0.0659	0.0815	Degroot
				Chi-squared- Exponential	0.0650	0.0972	Degroot
				Gamma	0.0751	0.0733	NLINEX
Best Prior					Gamma-Exponential	Gamma	
100	0.0358	4.9757e-17	6.0661e-04	Gamma-Exponential	0.0304	0.0452	Degroot
				Gamma-Chi-squared	0.0323	0.0382	Degroot
				Chi-squared- Exponential	0.0322	0.0429	Degroot
				Gamma	0.0344	0.0356	Degroot
Best Prior					Gamma-Exponential	Gamma	

Case (VI): $\varnothing = 2, v = 2.5$

n	*Non-Bayes Method ML* $\widehat{\Theta}(t)_{ML}$	*Numerical Methods* $\widehat{\Theta}(t)_{PF}$	$\widehat{\Theta}(t)_{BP}$	*Bayes Methods* *Perior*	*Degroot* $\widehat{\Theta}(t)_{Bd}$	*NLINEX* $\widehat{\Theta}(t)_{BNL}$ *C1=6.2*	*Best Loss*
10	*0.6239*	*3.6789e-14*	*2.0796e-05*	*Gamma-Exponential*	*0.1446*	*0.5388*	*Degroot*
				Gamma-Chi-squared	*0.1912*	*0.3709*	*Degroot*
				Chi-squared- Exponential	*0.1943*	*0.5418*	*Degroot*
				Gamma	*0.3061*	*0.2962*	*NLINEX*
Best Prior					*Gamma-Exponential*	*Gamma*	
15	*0.3134*	*1.4994e-11*	*2.7734e-05*	*Gamma-Exponential*	*0.1133*	*0.3652*	*Degroot*
				Gamma-Chi-squared	*0.1483*	*0.2531*	*Degroot*
				Chi-squared- Exponential	*0.1472*	*0.3541*	*Degroot*
				Gamma	*0.2135*	*0.2061*	*NLINEX*
Best Prior					*Gamma-Exponential*	*Gamma*	
25	*0.1862*	*4.8309e-18*	*2.0896e-08*	*Gamma-Exponential*	*0.0978*	*0.2228*	*Degroot*
				Gamma-Chi-squared	*0.1198*	*0.1645*	*Degroot*
				Chi-squared- Exponential	*0.1195*	*0.2123*	*Degroot*
				Gamma	*0.1502*	*0.1417*	*NLINEX*
Best Prior					*Gamma-Exponential*	*Gamma*	
50	*0.0960*	*1.0234e-24*	*2.8807e-11*	*Gamma-Exponential*	*0.0667*	*0.1028*	*Degroot*
				Gamma-Chi-squared	*0.0772*	*0.0839*	*Degroot*
				Chi-squared- Exponential	*0.0761*	*0.0983*	*Degroot*
				Gamma	*0.0878*	*0.0772*	*NLINEX*
Best Prior					*Gamma-Exponential*	*Gamma*	
100	*0.0392*	*2.8181e-11*	*4.3141e-07*	*Gamma-Exponential*	*0.0328*	*0.0445*	*Degroot*
				Gamma-Chi-squared	*0.0353*	*0.0384*	*Degroot*
				Chi-squared- Exponential	*0.0350*	*0.0426*	*Degroot*
				Gamma	*0.0378*	*0.0363*	*NLINEX*
Best Prior					*Gamma-Exponential*	*Gamma*	

Case (VII): $\varnothing = 2.5, v = 1$

n	*Non-Bayes Method ML* $\widehat{\Theta}(t)_{ML}$	*Numerical Methods* $\widehat{\Theta}(t)_{PF}$	$\widehat{\Theta}(t)_{BP}$	*Bayes Methods* *Perior*	*Degroot* $\widehat{\Theta}(t)_{Bd}$	*NLINEX* $\widehat{\Theta}(t)_{BNL}$ *C1=6.7*	*Best Loss*
10	*0.8694*	*5.0276e-15*	*7.5770e-07*	*Gamma-Exponential*	*0.3218*	*1.2449*	*Degroot*
				Gamma-Chi-squared	*0.2136*	*0.9032*	*Degroot*
				Chi-squared- Exponential	*0.2776*	*1.1856*	*Degroot*
				Gamma	*0.2833*	*0.7306*	*Degroot*
Best Prior					*Gamma-Exponential*	*Gamma*	
15	*0.5037*	*7.1685e-18*	*4.5819e-08*	*Gamma-Exponential*	*0.2322*	*0.8739*	*Degroot*
				Gamma-Chi-squared	*0.1834*	*0.6266*	*Degroot*
				Chi-squared- Exponential	*0.2183*	*0.7983*	*Degroot*
				Gamma	*0.2357*	*0.5092*	*Degroot*
Best Prior					*Gamma-Exponential*	*Gamma*	
25	*0.2783*	*4.9625e-10*	*4.4992e-05*	*Gamma-Exponential*	*0.1617*	*0.5177*	*Degroot*
				Gamma-Chi-squared	*0.1497*	*0.3739*	*Degroot*
				Chi-squared- Exponential	*0.1647*	*0.4567*	*Degroot*
				Gamma	*0.1795*	*0.3100*	*Degroot*
Best Prior					*Gamma-Exponential*	*Gamma*	
50	*0.1208*	*1.0025e-13*	*2.9523e-08*	*Gamma-Exponential*	*0.0910*	*0.2266*	*Degroot*
				Gamma-Chi-squared	*0.0886*	*0.1695*	*Degroot*
				Chi-squared- Exponential	*0.0932*	*0.1965*	*Degroot*
				Gamma	*0.0977*	*0.1456*	*Degroot*
Best Prior					*Gamma-Exponential*	*Gamma*	
100	*0.0605*	*6.8525e-22*	*8.6986e-10*	*Gamma-Exponential*	*0.0521*	*0.0950*	*Degroot*
				Gamma-Chi-squared	*0.0519*	*0.0763*	*Degroot*
				Chi-squared- Exponential	*0.0532*	*0.0842*	*Degroot*
				Gamma	*0.0546*	*0.0688*	*Degroot*
Best Prior					*Gamma-Exponential*	*Gamma*	

Case (VIII): $\varnothing = 2.5, v = 2.5$

n	*Non-Bayes Method ML* $\widehat{\Theta}(t)_{ML}$	*Numerical Methods* $\widehat{\Theta}(t)_{PF}$	$\widehat{\Theta}(t)_{BP}$	*Bayes Methods* *Perior*	*Degroot* $\widehat{\Theta}(t)_{Bd}$	*NLINEX* $\widehat{\Theta}(t)_{BNL}$ *C1=6.2*	*Best Loss*
10	*0.8497*	*5.3441e-10*	*4.6580e-05*	*Gamma-Exponential*	*0.2987*	*1.1594*	*Degroot*
				Gamma-Chi-squared	*0.1977*	*0.8160*	*Degroot*
				Chi-squared- Exponential	*0.2546*	*1.0883*	*Degroot*
				Gamma	*0.2750*	*0.6455*	*Degroot*
Best Prior					*Gamma-Exponential*	*Gamma*	
15	*0.4959*	*1.2582e-19*	*5.9588e-07*	*Gamma-Exponential*	*0.2359*	*0.8302*	*Degroot*
				Gamma-Chi-squared	*0.1870*	*0.5862*	*Degroot*
				Chi-squared- Exponential	*0.2214*	*0.7519*	*Degroot*
				Gamma	*0.2387*	*0.4722*	*Degroot*
Best Prior					*Gamma-Exponential*	*Gamma*	
25	*0.2884*	*6.3579e-23*	*1.6080e-06*	*Gamma-Exponential*	*0.1744*	*0.5025*	*Degroot*
				Gamma-Chi-squared	*0.1603*	*0.3625*	*Degroot*
				Chi-squared- Exponential	*0.1769*	*0.4428*	*Degroot*
				Gamma	*0.1886*	*0.3015*	*Degroot*
Best Prior					*Gamma-Exponential*	*Gamma*	
50	*0.1322*	*2.7505e-14*	*4.0295e-06*	*Gamma-Exponential*	*0.0935*	*0.2088*	*Degroot*
				Gamma-Chi-squared	*0.0950*	*0.1562*	*Degroot*
				Chi-squared- Exponential	*0.0988*	*0.1809*	*Degroot*
				Gamma	*0.1066*	*0.1350*	*Degroot*
Best Prior					*Gamma-Exponential*	*Gamma*	
100	*0.0548*	*3.6357e-21*	*4.7092e-06*	*Gamma-Exponential*	*0.0452*	*0.0807*	*Degroot*
				Gamma-Chi-squared	*0.0461*	*0.0638*	*Degroot*
				Chi-squared- Exponential	*0.0468*	*0.0707*	*Degroot*
				Gamma	*0.0493*	*0.0573*	*Degroot*
Best Prior					*Gamma-Exponential*	*Gamma*	

Case (VIIII): $\varnothing = 2.5, v = 3$

n	*Non-Bayes Method ML* $\widehat{\Theta}(t)_{ML}$	*Numerical Methods* $\widehat{\Theta}(t)_{PF}$	$\widehat{\Theta}(t)_{BP}$	*Bayes Methods* *Perior*	*Degroot* $\widehat{\Theta}(t)_{Bd}$	*NLINEX* $\widehat{\Theta}(t)_{BNL}$ *C1=6.2*	*Best Loss*
10	*0.8716*	*2.3637e-16*	*7.3094e-06*	*Gamma-Exponential*	*0.3095*	*1.1722*	*Degroot*
				Gamma-Chi-squared	*0.2048*	*0.8298*	*Degroot*
				Chi-squared- Exponential	*0.2657*	*1.1038*	*Degroot*
				Gamma	*0.2785*	*0.6597*	*Degroot*
Best Prior					*Gamma-Exponential*	*Gamma*	
15	*0.4890*	*2.3428e-11*	*1.0770e-05*	*Gamma-Exponential*	*0.2084*	*0.7961*	*Degroot*
				Gamma-Chi-squared	*0.1680*	*0.5512*	*Degroot*
				Chi-squared- Exponential	*0.1966*	*0.7135*	*Degroot*
				Gamma	*0.2277*	*0.4375*	*Degroot*
Best Prior					*Gamma-Exponential*	*Gamma*	
25	*0.3053*	*1.1973e-20*	*2.9622e-09*	*Gamma-Exponential*	*0.1653*	*0.4835*	*Degroot*
				Gamma-Chi-squared	*0.1596*	*0.3454*	*Degroot*
				Chi-squared- Exponential	*0.1741*	*0.4237*	*Degroot*
				Gamma	*0.1942*	*0.2860*	*Degroot*
Best Prior					*Gamma-Exponential*	*Gamma*	
50	*0.1277*	*4.8082e-19*	*2.5081e-05*	*Gamma-Exponential*	*0.0973*	*0.2204*	*Degroot*
				Gamma-Chi-squared	*0.0946*	*0.1658*	*Degroot*
				Chi-squared- Exponential	*0.0997*	*0.1917*	*Degroot*
				Gamma	*0.1035*	*0.1434*	*Degroot*
Best Prior					*Gamma-Exponential*	*Gamma*	
100	*0.0613*	*2.9837e-27*	*4.6045e-12*	*Gamma-Exponential*	*0.0517*	*0.0883*	*Degroot*
				Gamma-Chi-squared	*0.0523*	*0.0712*	*Degroot*
				Chi-squared- Exponential	*0.0533*	*0.0784*	*Degroot*
				Gamma	*0.0553*	*0.0646*	*Degroot*
Best Prior					*Gamma-Exponential*	*Gamma*	

8. Conclusions

Depending on simulation results based on complete data, the most essential conclusions are summarized by:

1. For all sample sizes and for all cases the MSE values associated with numerical estimators are better than non-Bayes and Bayes estimates.

2. For all cases the MSE values of double priors distributions based on De-groot loss function are better than single prior distribution.

3. All MSE values and for all cases single prior distribution based on NLINEX are better than double prior distributions.

4. All Bayes methods for all priors the MSE values are best from non-Bayes method except for some special cases it was mentioned in the previous section.

5. Newton-Raphson method is the best from false position method to estimate MSE values for all cases and all sample sizes.

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