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Some common fixed point theorems for strict contractions

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Abstract

The extension of Aamri and El Moutawakil's property [1] to set-valued mappings arena is given. Also, some common fixed point theorems for strict contractions are established. These theorems extend results in [1,8].

Keywords: Property (M. V.), Set-valued, Single-valued, Weakly compatible.
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1. Introduction

In the last few decades, fixed point theory and its applications have attracted the attention of many authors. For examples, some fixed point theorems have been applied to show the existence of the solutions of differential equations, integral equations and many other applied mathematics (see, e.g., [13, 14, 16]). Also, many results on common fixed point theorems for compatible and weakly compatible mappings in set-valued setting have appeared (see, for instance, [2-7, 11, 12, 15]). Recently, in 2002, the authors [1] gave new common fixed point theorems under strict contractive conditions for single-valued mappings satisfying certain property.

In the sequel, let (X, d) be a metric space. Suppose that $B(X)$ is the set of all nonempty bounded subsets of X. As in [9], it is defined that

$$\delta(A, B) = \sup\{d(a, b), a \in A, b \in B\} \text{ and } D(A, B) = \inf\{d(a, b), a \in A, b \in B\}$$

for all A, B in $B(X)$. If $A = \{a\}$, we denote $\delta(a, B)$ and $D(a, B)$ for $\delta(A, B)$ and $D(A, B)$, respectively. Also, if B consists of a single point b, one can deduce that $\delta(A, B) = D(A, B) = d(a, b)$.

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It follows immediately from the definition of $\delta(A, B)$ that, for all $A, B, C \in B(X)$,

$$\delta(A, B) = \delta(B, A) \geq 0; \delta(A, B) \leq \delta(A, C) + \delta(C, B);$$

$$\delta(A, B) = 0 \text{ iff } A = B = \{a\}; \delta(A, B) = diam A.$$

Definition 1.1. [9] A sequence (A_n) of nonempty subsets of X is said to be convergent to $A \subseteq X$ if

(i) each point a in A is the limit of a convergent sequence (a_n), where a_n is in A_n for $n \in N$ ($N :=$ the set of all positive integers);

(ii) for arbitrary $\epsilon > 0$, there exists an integer m such that $A_n \subseteq A_\epsilon$ for all $n \geq m$, where A_ϵ denotes the set of all points x in X for which there exists a point a in A, depending on x, such that $d(x, a) < \epsilon$. A is then said to be the limit of the sequence (A_n).

Lemma 1.1 [9]. If (A_n) and (B_n) are sequences in $B(X)$ converging to A and B in $B(X)$, respectively, then the sequence $(\delta(An, Bn))$ converges to $\delta(A, B)$.

Lemma 1.2 [9]. Let (A_n) be a sequence in $B(X)$ and y be a point in X such that $\delta(A_n, y) \to 0$. Then the sequence (A_n) converges to the set $\{y\}$ in $B(X)$.

Definition 1.2 [11]. The mappings $I : X \to X$ and $F : X \to B(X)$ are δ-compatible if $\lim_{n\to\infty} \delta(FIx_n, IFx_n) = 0$ whenever (x_n) is a sequence in X such that $IFx_n \in B(X), Fx_n \to t$ and $Ix_n \to \{t\}$ for some $t \in X$.

Definition 1.3 [11]. The mappings $I : X \to X$ and $F : X \to B(X)$ are weakly compatible if they commute at coincidence points, i.e., for each point $u \in X$ such that $Fu = \{Iu\}$, we have $FIu = IFu$ (Note that the equation $Fu = \{Iu\}$ implies that Fu is a singleton).

If F is a single-valued mapping, then Definition 1.2 (resp. Definition 1.3) reduces to the definition of compatible (resp. weakly compatible) single-valued mappings of Jungck [10].

It can be seen that any δ-compatible pair $\{F, I\}$ is weakly compatible. Examples of weakly compatible pairs which are not δ-compatible are given in [11].

Definition 1.4 [1]. Two self-mappings I and J of a metric space (X, d) satisfy Property $(E.A.)$ if there exists a sequence (x_n) in X such that $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Jx_n = t$ for some $t \in X$.

The aim of this paper is to extend Property $(E.A.)$ to set-valued mappings setting. Also, we establish two common fixed point theorems for strict contractions. These theorems extend Theorems 1 and 2 in [1] and Theorem 1[8]. Finally, we conclude some remarks on our results.

2. Main Results

In this section, we introduce an extension of Property $(E.A)$ [1] to set-valued mappings arena. This extension is supported by some examples. Also, we prove two common fixed point theorems for strict contractions in metric spaces. Finally, we list some remarks that exploit the importance of our results.

First we introduce the extension of Property $(E.A)$ to set-valued mappings setting as follows.

Definition 2.1. Let (X, d) be a metric space, and $I : X \to X$ and $F : X \to B(X)$ be two mappings. The mappings I and F satisfy Property $(M.V.)$ if there exists a sequence (x_n) in X such that $Fx_n \to \{t\}$ and $Ix_n \to t$ for some $t \to X$.

Examples 2.1. (1) Let $X = [0, 1]$. Define I, F by $Ix = \frac{x}{4}$ and $Fx = [0, \frac{3x}{4}]$, for all $x \in X$. Consider the sequence $(x_n) = (\frac{1}{n})$. Clearly $\lim_{n\to\infty} Fx_n = \{0\}$ and $\lim_{n\to\infty} Ix_n = 0$. Then F and I satisfy the property $(M.V.)$.

(2) Let $X = [2, \infty)$. Define I, F by $Ix = x + 1$ and $Fx = [2, 2x + 1]$, for all $x \in X$. Suppose that Property $(M.V.)$ holds, then there exists a sequence (x_n) in X satisfying $\lim_{n\to\infty} Ix_n = t$ and

$\lim_{n\to\infty} Fx_n = \{t\}$ for some $t \in X$. Therefore, $\lim_{n\to\infty} x_n = t-1$ and $\lim_{n\to\infty} x_n = \frac{t-1}{2}$. Then $t = 1$, which is a contradiction since $1 \not\subseteq X$. Hence I and F don't satisfy the property $(M.V.)$.

Remark 2.1. Assume that there exists at least one sequence (x_n) in X such that $IFx_n \in B(X)$, $Fx_n \to \{t\}, Ixn \to t$ for some t in X, but $\lim_{n\to\infty} \delta(FIx_n, IFx_n)$ is either non-zero or non-existent. In this case, two mappings $I : X \to X$ and $F : X \to B(X)$ are not δ-compatible. Therefore, two maps I and F which are not δ-compatible satisfy the property $(M.V.)$.

Let Φ be the set of all functions $\phi : [0,\infty)^5 \to [0,\infty)$ satisfying the following conditions:

(ϕ_1) ϕ is nondecreasing in each coordinate variables and upper semi-continuous from the right;

(ϕ_2) $\phi(t) = \max \phi(0,t,0,0,t), \phi(0,0,t,t,0), \phi(t,0,0,t,t) < t$ for each $t \in (0,1)$.

Now, we state and prove the main theorem in the following way.

Theorem 2.1. Let I, J be self-mappings of a metric space (X,d) and $F, G : X \to B(X)$ be two set-valued mappings with $\cup F(X) \subseteq J(X)$ and $\cup G(X) \subseteq I(X)$. Furthermore, suppose that the pairs $\{F,I\}, \{G,J\}$ are weakly compatible and one of them satisfy the property $(M.V.)$. Also, suppose that one of $I(X)$, $J(X)$, $\cup F(X)$ and $\cup G(X)$ is a closed subset of X. If there exists $\phi \in \Phi$ such that, for all $x, y \in X$,

$$\delta(Fx, Gy) < \phi(d(Ix,Jy), \delta(Ix,Fx), \delta(Jy,Gy), D(Ix,Gy), D(Jy,Fx)) \tag{2.1}$$

whenever the right hand side of (2.1) is positive, then F, G, I and J have a unique common fixed point.

Proof. Suppose that the pair $\{G,J\}$ satisfies the property $(M.V.)$. Then there exists a sequence (x_n) in X such that $\lim_{n\to\infty} Gx_n = \{t\}$ and $\lim_{n\to\infty} Jx_n = t$ for some $t \in X$. Since $\cup G(X) \subseteq I(X)$ and $\lim_{n\to\infty} Gx_n = \{t\}$, then there exists a sequence (a_n) in $\cup G(X)$ such that $a_n \in Gx_n$ $\forall n \in N$. Hence there exists a sequence (y_n) in X such that $Iy_n = a_n$ and $\lim_{n\to\infty} Iy_n = \lim_{n\to\infty} a_n = t$. Now, we want to show that $\lim_{n\to\infty} Fy_n$ exists. From the inequality (2.1), we have that

$$\begin{aligned}\delta(Fy_n, Gx_n) \quad &< \phi(d(Iy_n,Jx_n), \delta(Iy_n,Fy_n), \delta(Jx_n,Gx_n), D(Iy_n,Gx_n), D(Jx_n,Fy_n)) \\ &\le \phi(\delta(Gx_n,Jx_n), \delta(Gx_n,Fy_n), \delta(Gx_n,Jx_n), 0, \delta(Jx_n,Gx_n) + \delta(Gx_n,Fy_n)).\end{aligned}$$

Letting $n \to \infty$ and suppose that $\lim_{n\to\infty} \delta(Fy_n, Gx_n) \neq 0$, we obtain from Lemma 1.1 that $\lim_{n\to\infty} \delta(Fy_n,Gx_n) \le \phi(0, \lim_{n\to\infty} \delta(Fy_n,Gx_n), 0, 0, \lim_{n\to\infty} \delta(Fy_n,Gx_n)) < \lim_{n\to\infty} \delta(Fy_n,Gx_n)$. This contradiction demands that $\lim_{n\to\infty} \delta(Fy_n,Gx_n) = 0$. Since $\delta(Fy_n,t) \le \delta(Fy_n,Gx_n) + \delta(Gx_n,t)$, then $\lim_{n\to\infty} \delta(Fy_n,t) = 0$. Lemma 1.2 gives that $\lim_{n\to\infty} \delta Fy_n$ exists and $\lim_{n\to\infty} \delta Fy_n = \{t\}$. Suppose that $I(X)$ is a closed subset of X. Since (Iy_n) is a sequence in $I(X)$ such that $\lim_{n\to\infty} Iy_n = t$, then $t \in I(X)$. Hence, there exists $u \in X$ such that $t = Iu$. Subsequently, it yields that $\lim_{n\to\infty} \delta Fy_n = \lim_{n\to\infty} \delta Gx_n = Iu$ and $\lim_{n\to\infty} \delta Iy_n = \lim_{n\to\infty} \delta Jx_n = Iu$.

From inequality (2.1), one can estimate that

$$\begin{aligned}\delta(Fu, Gx_n) \quad &< \phi(d(Iu,Jx_n), \delta(Iu,Fu), \delta(Jx_n,Gx_n), D(Iu,Gx_n), D(Jx_n,Fu)) \\ &\le \phi(d(Iu,Jx_n), \delta(Iu,Fu), \delta(Jx_n,Gx_n), D(Iu,Gx_n), D(Jx_n,Fu)).\end{aligned}$$

As $n \to \infty$ and suppose that $\delta(Fu, Iu) \neq 0$, it follows from Lemma 1.1 that

$$\delta(Fu, Iu) \leq \phi(0, \delta(Iu, Fu), 0, 0, \delta(Iu, Fu)) < \delta(Iu, Fu).$$

This contradiction implies that $Fu = \{Iu\}$. The weak compatibility of F and I leads to $FFu = FIu = IFu = \{IIu\}$. On the other hand, since $\cup F(X) \subseteq J(X)$, then there is an element $v \in X$ such that $Fu = \{Jv\}$. We claim that $Gv = \{Jv\}$. Using inequality (2.11) and suppose that $\delta(Fu, Gv) \neq 0$, we have that

$$\begin{aligned}\delta(Fu, Gv) \quad &< \phi(d(Iu, Jv), \delta(Iu, Fu), \delta(Jv, Gv), D(Iu, Gv), D(Jv, Fu)) \\ &\leq \phi(0, 0, \delta(Fu, Gv), \delta(Fu, Gv), 0) < \delta(Fu, Gv).\end{aligned}$$

This contradiction leads to $\delta(Fu, Gv) = 0 \Rightarrow Fu = Gv = \{Iu\} = \{Jv\}$. From the weak compatibility of G and J, we find that $GGv = GJv = JGv = \{JJv\}$. Now, we show that Fu is a common fixed point of F, G, I and J. In view of the inequality (2.1), it follows that

$$\begin{aligned}\delta(Fu, F^2u) = \delta(F^2u, Gv) \quad &< \phi(d(IFu, Jv), \delta(IFu, F^2u), \delta(Jv, Gv), D(IFu, Gv), D(Jv, F^2u)) \\ &\leq \phi(\delta(Fu, F^2u), 0, 0, \delta(Fu, F^2u), 0, \delta(Fu, F^2u)) < \delta(Fu, F^2u).\end{aligned}$$

This contradiction implies that $F^2u = Fu = \{I2u\}$. Hence, $\{Iu\} = Fu = F^2u = FIu = \{I2u\}$. So, Iu is a common fixed point of F and I. Similarly, one can prove that Jv is a common fixed point of G and J. Since $z = Iu = Jv$, then we conclude that z is a common fixed point of F, G, I and J. The proof, assuming the closedness of $J(X)$, is similar to the above. It is given that $\cup F(X) \subseteq J(X)$ and $\cup G(X) \subseteq I(X)$. So, the proof, assuming the closedness of $\cup F(X)$ (resp. $\cup G(X)$), is similar to the case in which $J(X)$ (resp. $I(X)$) is closed. Finally, one can deduce from the inequality (1) that z is a unique common fixed point of F, G, I and J. The proof, assuming that the pair $\{F, I\}$ satisfies the property $(M.V.)$, is similar to the above.

Remark 2.2. Let $f : [0, \infty) \to [0, \infty)$ be a function satisfying the following conditions:

$(F1)$ f is nondecreasing in $[0, \infty)$ and continuous from the right at zero,

$(F2)$ $0 < f(t) < t$ for each $t \to (0, \infty))$.

In Theorem 2.1, if we put $\phi(t_1, t_2, t_3, t_4, t_5) = f(\max t_1, t_2, t_3, t_4, t_5)$, then we obtain the multivalued version of Theorem 2[1].

Remark 2.3. Let $\mathfrak{R}$ be the set of all continuous mappings $g : [0, \infty)^5 \to [0, \infty)$ satisfying the following conditions:

(g_1) g is nondecreasing in each coordinate variables,

(g_2) $g(r, r, r, 2r, 0) \leq r$ and $g(r, r, r, 0, 2r) \leq r$ for every $r \geq 0$.

Suppose that there is a function $g \in \mathfrak{R}$ such that, for all $(x, y) \in X \times X - \{(x, x) : Fx = Gx\}$,

$$\delta(Fx, Gy) < g(d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy), D(Ix, Gy), D(Jy, Fx)) \tag{2.2}$$

whenever the right hand side of (2.2) is positive. If we replace the inequality (2.1) by (2.2), we find that the conclusion of Theorem 2.1 remains valid. This result is a multivalued version of Theorem 1[1] with $g(t_1, t_2, t_3, t_4, t_5) = \max\{t_1, \frac{t_2+t_3}{2}, \frac{t_4+t_5}{2}\}), F = GandI = J$. Also, this result is a multivalued version of Theorem 1[8].

In Theorem 2.1, if the mapping F (resp. G) is replaced by F_α (resp. G_α), $\alpha \in \Lambda$ where Λ is an index set, we obtain the following theorem.

Theorem 2.2. Let I, J be two self-mappings of a metric space X and for each$\alpha \in \Lambda$, $F_\alpha, G_\alpha : X \to B(X)$ be set-valued functions with $\cup[\cup_{\alpha\in\Lambda} F_\alpha(X)] \subseteq J(X)$ and $\cup[\cup_{\alpha\in\Lambda} G_\alpha(X)] \subseteq I(X)$. Furthermore, suppose that the pairs $\{F_\alpha, I\}, \{G_\alpha, J\}$ are weakly compatible for every $\alpha \in \Lambda$ and one of them satisfy the property $(M.V.)$. Also, suppose that one of $I(X), J(X)$, $\cup[\cup_{\alpha\in\Lambda} F_\alpha(X)]$ and $\cup[\cup_{\alpha\in\Lambda} G_\alpha(X)]$ is a closed subset of X. If there exists $\phi \in \Phi$ such that, for all $x, y \in X$, the following inequality

$$\delta(F_\alpha x, G_\alpha y) < \phi(d(Ix, Jy), \delta(Ix, F_\alpha x), \delta(Jy, G_\alpha y), D(Ix, G_\alpha y), D(Jy, F_\alpha x)) \tag{2.3}$$

holds whenever the right hand side of (2.3) is positive, then F_α, G_α, I and J have a unique common fixed point for all $\alpha \in \Lambda$.

Proof. Using Theorem 2.1, we obtain that there is a unique point $z_\alpha X$ such that $Iz_\alpha = Jz_\alpha = z_\alpha$ and $F_\alpha z_\alpha = G_\alpha z_\alpha = z_\alpha$ for any $\alpha \in \Lambda$. For all $\alpha, \beta \in \Lambda$, suppose that $d(z_\alpha, z_\beta) \neq 0$, we obtain that

$$\begin{aligned} d(z_\alpha, z_\beta) \leq \quad & \delta(F_\alpha z_\alpha, G_\beta z_\beta) \\ < \quad & \phi(d(Iz_\alpha, Jz_\beta), \delta(Iz_\alpha, F_\alpha z_\alpha), \delta(Jz_\beta, G_\beta z_\beta), D(Iz_\alpha, G_\beta z_\beta), D(Jz_\beta, F_\alpha z_\alpha)) \\ \leq \quad & \phi(d(z_\alpha, z_\beta), 0, 0, d(z_\alpha, z_\beta), d(z_\beta, z_\alpha)) \\ < \quad & d(z_\beta, z_\alpha) \end{aligned}$$

This contradiction yields that $d(z_\alpha, z_\beta) = 0 \Rightarrow z_\alpha = z_\beta$.

Remark 2.4. Assume that there is a $g \in \mathfrak{R}$ such that, for all $(x, y) \in X \times X - \{(x, x) : Fx = Gx\}$ and for each $\alpha \in \Lambda$,

$$\delta(F_\alpha x, G_\alpha y) < g(d(Ix, Jy), \delta(Ix, F_\alpha x), \delta(Jy, G_\alpha y), D(Ix, G_\alpha y), D(Jy, F_\alpha x)) \tag{2.4}$$

whenever the right hand side of (2.4) is positive. So, the conclusion of Theorem 2.2 remains valid if we replace the inequality (2.3) by (2.4).

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References

[1] M. Aamri and D. El Moutawakil, *Some new common fixed point theorems under strict contractive conditions*, J. Math. Anal. Appl. 270 (2002) 181–188.

[2] M.A. Ahmed, *Common fixed point theorems for set-valued and single-valued mappings*, Demonstratio Math. 36(2) (2003) 471–481.

[3] M.A. Ahmed, *Common fixed point theorems for weakly compatible mappings*, Rocky Mountain J. Math. 33(4) (2003) 1189–1203.

[4] M.A. Ahmed, *Common fixed point theorems under contractive conditions of Skof type*, PU. M. A. 15(1) (2004) 17–27.

[5] M.A. Ahmed and B.E. Rhoades, *Some common fixed point theorems for compatible mappings*, Indian J. Pure Appl. Math. 32(8) (2001) 1247–1254.

[6] T.-H. Chang, *Fixed point theorems for contractive type set-valued mappings*, Math. Japonica 38(4) (1993), 675-790.

[7] Lj.B. Ciric and J.S. Ume, *Some common fixed point theorems for weakly compatible mappings*, J. Math. Anal. Appl. 314 (2006) 488–499.

[8] A. Constantin, *A unified approach for some fixed point theorems*, Indian J. Math. 36(2) (1994) 91–101.

[9] B. Fisher, *Common fixed points of mapppings and set-valued mappings*, Rostick. Math. Kollaq. 18 (1981) 69–77.

[10] G. Jungck, *compatible mappings and common fixed points*, Int. J. Math. Math. Sci. 9 (1986) 771–779.

[11] G. Jungck and B. E. Rhoades, *Fixed points for set valued functions without continuity*, Indian J. Pure Appl. Math. 16(3) (1998) 227–238.

[12] A.R. Khan, A.A. Domlo and N. Hussain, *Coincidences of Lipchitz-type hybrid maps and invariant approximation*, Numer. Funct. Anal. Optim. 28(9-10) (2007) 1165–1177.

[13] H.K. Pathak and B. Fisher, *Common fixed point theorems with applications in dynamic programming*, Glas. Math. 31(51) (1996) 321–328.

[14] H.K. Pathak, S.N. Mishra and A.K. Kalinde, *Some Gregus type common fixed point theorems with applications*, Demonstratio Math. 36(2) (2003) 413–426.

[15] B.E. Rhoades, *Common fixed points of compatible set-valued mappings*, Publ. Math. Debrecen 48(3-4) (1996) 237–240.

[16] M. Zima, *A general fixed point theorem and its applications to integral-functional equations*, Bull. Austral. Math. Soc. 46 (1992) 179–186.