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Escape criteria for one parameter family of complex functions $\mathbf{f_k}\left(\mathbf{z}\right)$=kcsc$\left(\mathbf{z}\right)$ via non-standard iterations

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Abstract

In this research we stated and proved the some escape criteria theorems of the one parameter family of the transcendental meromorphic-functions $F=\{f_k\left(z\right)=k\ csc\left(z\right) : k\in\mathbb{C}$ and z$\in\mathbb{C}\}$. Furthermore, we used non-standard iterations: Mann, Ishikawa and Noor iterations in the complex plane. This research can be considered as an extension of [11].

Keywords: Escape criteria, meromorphic functions, transcendental functions, Mann iteration, Ishikawa iteration, Noor iteration.

1. Introduction

A very important method of generating fractal in the complex plane is the repeated iteration of a complex function. The most common example of such fractals is the Julia sets. Julia sets were studied much earlier, namely in the early twentieth century by French mathematicians Pierre Fatou [6] and Gaston Julia [7].
The Julia sets is not only interesting from a mathematical point of view. It has applications in other fields also, e.g. physics [1], biology [2] and robotics [14]. One of the most natural applications of the Julia sets – because of her beauty – was her use in computer graphics.
So many researchers have studied different properties of the Julia sets and proposed accordingly

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various generalizations. The first was the use of the z^p+c function instead of the quadratic one used by Mandelbrot [4, 9]. Further, additional types of functions were studied: rational [10], transcendental [5], elliptic [8] and anti-polynomials [3] etc. They applied results that can be explained in fixed point theory. In this theory, we can described methods of locating fixed points that change the feedback process $z_{n+1} = f_k(z_n)$ with other kinds of iteration processes. The obtain of different iteration processes began in 2004 in the works of Rani and Kumar [12, 13]. To obtain how one can visualize Julia sets, one will describe the well-known method called "The escape criterion method" which construct the filled in Julia sets do not have interiors, however, this method often ends up plotting the interior along with the Julia sets. [11]displayed the Mann iteration and proved new escape criteria for the generation of the Julia sets using this kind of iteration. Further studied different iteration as Ishikawa, Noor, SP and CR processes.

In this research we established the some escape criterion which performed an important job to generate the Julia sets. We take $F=\{f_k(z) = k\ csc(z) : k\in \mathbb{C}$ and z$\in \mathbb{C}\}$ then we use the Mann, Ishikawa and Noor iteration scheme.

2. Preliminaries

In this section we shall obtain some definitions from [15, 11]; for the sake of convenience those definitions will be put here.

Definition 2.1 (Julia sets[11]). *Suppose that $f : \mathbb{C} \to \mathbb{C}$ be a complex function and $k\in \mathbb{C}$ is a parameter. Hence the set of points*

$$J_f = \{z \in \mathbb{C} : \{|f^n(z)|\}_{n=0}^{\infty} \text{ is bounded}\} \tag{2.1}$$

Such that $f^n(z)$ is the n-th iterate of zis called the filled Julia set. The set boundary points of J_f is called Julia set.

Definition 2.2 (Picard-iteration [15]). *Suppose thatf $: \mathbb{C} \to \mathbb{C}$ be a complex function. Then for any $z_0\in \mathbb{C}$ the Picard-iteration in complex plane is defined as:*

$$z_{n+1} = f(z_n) \tag{2.2}$$

for all $n = 0, 1, 2, \dots$.

Definition 2.3 (Mann-iteration [15]). *Suppose that $f : \mathbb{C} \to \mathbb{C}$ be a complex function. Then for any $z_0\in \mathbb{C}$ the Mann-iteration in complex plane is defined as:*

$$z_{n+1} = (1 - a_1) z_n + a_1 f(z_n) \tag{2.3}$$

such that $a_1 \in (0, 1]$ and $n = 0, 1, 2, \dots$.

Definition 2.4 (Ishikawa-iteration [15]). *Suppose that $f : \mathbb{C} \to \mathbb{C}$ be a complex function. Then for any $z_0\in \mathbb{C}$ the Ishikawa-iteration in complex plane is defined as:*

$$\begin{aligned} z_{n+1} &= (1 - a_1) z_n + a_1 f(y_n) \\ y_n &= (1 - a_2) z_n + a_2 f(z_n), \end{aligned} \tag{2.4}$$

such that $a_1, a_2 \in (0, 1]$ and $n = 0, 1, 2, \dots$.

Definition 2.5 (Noor-iteration [15]). *Suppose that $f : \mathbb{C} \to \mathbb{C}$ be a complex function. Then for any $z_0 \in \mathbb{C}$ the Noor-iteration in complex plane is defined as:*

$$\begin{aligned} z_{n+1} &= (1-a_1)\, z_n + a_1 f\,(y_n) \\ y_n &= (1-a_2)\, z_n + a_2 f(x_n) \\ x_n &= (1-a_3)\, z_n + a_3 f(z_n), \end{aligned} \tag{2.5}$$

such that $a_1, a_2, a_3 \in (0,1]$ and $n = 0, 1, 2, \ldots$.

There are many fixed point iterations process in mathematics, but note that, all one step iterations have same escape criteria and same argument for two and three steps iterations, then we describe our results in Mann, Ishikawa, Noor iterations.

3. Main Results

In this section, we will prove some escape criteria as mentioned for the functions in $F = \{f_k\,(z) = \text{k } \csc(z) : k \in \mathbb{C} \text{ and z} \in \mathbb{C}\}$

Theorem 3.1. *Let $f_k\,(z)$ be a meromorphic transcendental functions in $F$with $|z| \geq |k| > \frac{1}{|a|}$ where $|a| \in (0,1]$. If the sequence of the iterates $\{z_n\}_{n\in\mathbb{N}}$ for Picard iteration is defined as follows:*

$$z_{n+1} = f_k(z_n)$$

where $n = 0, 1, 2, \ldots$, then $|z_n| \to \infty$ when $n \to \infty$.
Proof .*Let $f_k\,(z) = k \csc z$, then Picard iteration is*

$$|z_{n+1}| = |f_k(z_n)|$$

For $n = 1$, let $z_0 = z$, then we have

$$\begin{aligned} |z_1| = |f_k(z)| = |k\ csc\ z| &= \left| k \left(\frac{1}{z} + \frac{1}{3!} z + \left(\frac{1}{3!3!} - \frac{1}{5!} \right) z^3 + \ldots \right) \right| \\ &= |k|\,|z| \left| \frac{1}{z^2} + \frac{1}{3!} + \left(\frac{1}{3!3!} - \frac{1}{5!} \right) z^2 + \ldots \right|. \end{aligned}$$

Let $\left| \frac{1}{z^2} + \frac{1}{3!} + \left(\frac{1}{3!3!} - \frac{1}{5!} \right) z^2 + \ldots \right| > |a|$ where $|a| \in (0,1]$ and $z \in \mathbb{C}$, except those values of z wherefore $|a| = 0$. So, we have $|z_1| \geq |k|\,|a|\,|z|$.

For $n = 2$, we get

$$\begin{aligned} |z_2| = |f_k(z_1)| = |k\ csc\ z_1| &= |k|\,|z_1| \left| \frac{1}{{z_1}^2} + \frac{1}{3!} + \left(\frac{1}{3!3!} - \frac{1}{5!} \right) {z_1}^2 + \ldots \right| \\ &\geq |k|\,|a|\,|z_1| \geq |k|\,|a|\,(|k|\,|a|\,|z|) = |k|^2\,|a|^2\,|z|. \end{aligned}$$

Now, we continue to n^{th} term , we conclude that

$$|z_n| \geq |k|^n\,|a|^n\,|z|.$$

Since $|z| > |k| > \frac{1}{|a|}$, then $|k|\,|a| = |ka| > 1$. Hence $|z_n| \to \infty$ when $n \to \infty$. $\square$

Corollary 3.2. *Let* $|z_m| > \max\left\{|k|, \dfrac{1}{|a|}\right\}$ *for some* $m \geq 0$.
Since $|ak| > 1$, *then* $|z_{m+n}| > |z|\,|ak|^{m+n}$. *Therefore* $|z_n| \to \infty$ *as* $n \to \infty$.

Theorem 3.3. *Let* $f_k(z)$ *be a meromorphic transcendental functions in* F *with* $|z| \geq |k| > \frac{1}{a_1|a|}$ *where* $|a| \in (0,1]$. *If the sequence of the iterates* $\{z_n\}_{n\in\mathbb{N}}$ *for Mann iteration is defined as follows:*

$$z_{n+1} = (1 - a_1)\, z_n + a_1 f_k(z_n)$$

where $a_1 \in (0,1]$ *and* $n = 0, 1, 2, \ldots$, *then* $|z_n| \to \infty$ *when* $n \to \infty$.
Proof . *Let* $f_k(z) = k \csc z$, *then Mann iteration is*

$$|z_{n+1}| = |(1 - a_1)\, z_n + a_1 f_k(z_n)|$$

For $n = 1$, *let* $z_0 = z$, *then we have*

$$\begin{aligned}
|z_1| &= |(1 - a_1)\, z + a_1 f_k(z)| = |a_1\,(k\ csc\ z) + (1 - a_1)\, z| \\
&\geq a_1\,|(k\ csc\ z)| - (1 - a_1)\,|z| = a_1 \left| k \left(\frac{1}{z} + \frac{1}{3!} z + \left(\frac{1}{3!3!} - \frac{1}{5!} \right) z^3 + \ldots \right) \right| - (1 - a_1)\,|z| \\
&\geq |z|\,(a_1\,|k|\,|a| - 1)
\end{aligned}$$

Where $\left| \frac{1}{z^2} + \frac{1}{3!} + \left(\frac{1}{3!3!} - \frac{1}{5!} \right) z^2 + \ldots \right| > |a|$ *such that* $|a| \in (0,1]$ *and* $z \in \mathbb{C}$ *except those values of* z *wherefore* $|a| = 0$. *So, we get* $|z_1| \geq |z|\,(a_1\,|k|\,|a| - 1)$.
Now, for $n = 2$, *we have*

$$\begin{aligned}
|z_2| &= |(1 - a_1)\, z_1 + a_1 f_k(z_1)| = |a_1\,(k\ csc\ z_1) + (1 - a_1)\, z_1| \geq a_1\,|(k\ csc\ z_1)| - (1 - a_1)\,|z_1| \\
&= a_1 \left| k \left(\frac{1}{z_1} + \frac{1}{3!} z_1 + \left(\frac{1}{3!3!} - \frac{1}{5!} \right) {z_1}^3 + \ldots \right) \right| - (1 - a_1)\,|z_1| \geq |z_1|\,(a_1\,|k|\,|a| - 1) \geq |z|\,(a_1\,|k|\,|a| - 1)^2.
\end{aligned}$$

Since $|z_1| \geq |z|\,(a_1\,|k|\,|a| - 1)$ *and* $|z_1| \geq |z| \geq |k| > \frac{1}{a_1|a|}$, *this yields*

$$|z_1|\,(a_1\,|k|\,|a| - 1) \geq |z|\,(a_1\,|k|\,|a| - 1).$$

Then iterating up to n^{th} *term, we obtain*

$$|z_n| \geq |z|\,(a_1\,|k|\,|a| - 1)^n$$

This condition, $|z| > \frac{2}{a_1|a|}$, *yields to* $(a_1\,|k|\,|a| - 1) > 1$ *and therefore* $|z_n| \to \infty$ *when* $n \to \infty$. □

Corollary 3.4. *Let* $|z_m| > max\{|k|, \dfrac{2}{a_1\,|a|}\}$ *for some* $m \geq 0$. *Since* $a_1\,|ak| > 2$, *then* $|z_{m+n}| \geq |z|\,(a_1\,|k|\,|a| - 1)^{m+n}$. *Therefore* $|z_n| \to \infty$ *as* $n \to \infty$.

Theorem 3.5. *Let* $f_k(z)$ *be a meromorphic transcendental functions in* F *with* $|z| \geq |k| > \frac{1}{a_1|b|}$ *and* $|z| > |k| > \frac{2}{a_2|a|}$ *where* $|a|, |b| \in (0,1]$. *If the sequence of the iterates* $\{z_n\}_{n\in\mathbb{N}}$ *for Ishikawa iteration is defined as follows:*

$$\begin{aligned}
z_{n+1} &= (1 - a_1)\, z_n + a_1 f_k(y_n) \\
y_n &= (1 - a_2)\, z_n + a_2 f_k(z_n)
\end{aligned}$$

where $a_1, a_2 \in (0,1]$*and* $n = 0,1,2,\ldots,$ *then* $|z_n| \to \infty$ *when* $n \to \infty$.

Proof .*Let* $f_k(z) = k \csc z$, *then the first step of Ishikawa iteration is*

$$|y_n| = |(1-a_2)\, z_n + a_2 f_k(z_n)|$$

For $n = 0$, *let* $z_0 = z$ *and* $y_0 = y$, *then we have*

$$\begin{aligned} |y_0| &= |(1-a_2)\, z + a_2 f_k(z)| = |a_2\,(k\ csc\ z) + (1-a_2)\, z| \\ &\geq a_2 \,|(k\ csc\ z)| - (1-a_2)\,|z| = a_2 \left| k \left(\frac{1}{z} + \frac{1}{3!} z + \left(\frac{1}{3!3!} - \frac{1}{5!} \right) z^3 + \ldots \right) \right| - (1-a_2)\,|z| \\ &\geq a_2\,|k|\,|a|\,|z| - |z| \end{aligned}$$

where $\left| \frac{1}{z^2} + \frac{1}{3!} + \left(\frac{1}{3!3!} - \frac{1}{5!} \right) z^2 + \ldots \right| > |a|$ *such that* $|a| \in (0,1]$ *and* $z \in \mathbb{C}$ *except those values of* z *wherefore* $|a| = 0$.

$$|y_0| \geq |z|\,(a_2\,|k|\,|a| - 1).$$

The second step of Ishikawa iteration is

$$|z_{n+1}| = |(1-a_1)\, z_n + a_1 f_k(y_n)|\,.$$

When $n = 0$, *we consider*

$$\begin{aligned} |z_1| &= |(1-a_1)\, z + a_1 f_k(y_0)| = |a_1\,(k\ csc\ y_0) + (1-a_1)\, z| \\ &\geq a_1 \,|(k\ csc\ y_0)| - (1-a_1)\,|z| = a_1 \left| k \left(\frac{1}{y_0} + \frac{1}{3!} y_0 + \left(\frac{1}{3!3!} - \frac{1}{5!} \right) {y_0}^3 + \ldots \right) \right| - (1-a_1)\,|z| \\ &\geq a_1\,|k|\,|b|\,|y_0| - |z| \geq a_1\,|k|\,|b|\,|z|\,(a_2\,|k|\,|a| - 1) - |z|\,, \end{aligned}$$

where $\left| \frac{1}{{y_0}^2} + \frac{1}{3!} + \left(\frac{1}{3!3!} - \frac{1}{5!} \right) {y_0}^2 + \ldots \right| > |b|$*and* $(a_2\,|k|\,|a| - 1) = 1$. *Hence,* $|z_1| \geq |z|\,(a_1\,|k|\,|b| - 1)$. *Iterating up to* n^{th} *term, we have*

$$|z_n| \geq |z|\,(a_1\,|k|\,|b| - 1)^n.$$

Since $|z| > |k| > \frac{2}{a_1|b|}$ *and* $|z| > |k| > \frac{2}{a_2|a|}$, $|z_n| \to \infty$ *when* $n \to \infty$. $\square$

Corollary 3.6. *Let* $|z_m| > \max\left\{ \left| \frac{2}{a_2\,|a|} \right|, \frac{2}{a_1\,|b|} \right\}$ *for some* $m \geq 0$. *Since* $a_1\,|bk| > 2$ *and* $a_2\,|ak| > 2$, *then* $|z_{m+n}| > a_1\,|bk|^{m+n}\,|z|$. *Therefore* $|z_n| \to \infty$ *as* $n \to \infty$.

Theorem 3.7. *Let* $f_k(z)$ *be a meromorphic transcendental functions in* F *with* $|z| \geq |k| > \frac{1}{a_1|b|}$, $|z| \geq |k| > \frac{2}{a_2|c|}$ *and* $|z| \geq |k| > \frac{2}{a_3|a|}$*where* $|a|, |b|, |c| \in (0,1]$. *If the sequence of the iterates* $\{z_n\}_{n\in\mathbb{N}}$ *for Noor iteration is defined as follows:*

$$\begin{aligned} z_{n+1} &= (1-a_1)\, z_n + a_1 f_k(y_n) \\ y_n &= (1-a_2)\, z_n + a_2 f_k(x_n) \\ x_n &= (1-a_3)\, z_n + a_3 f_k(z_n), \end{aligned}$$

where $a_1, a_2, a_2 \in (0,1]$*and* $n = 0,1,2,\ldots,$ *then* $|z_n| \to \infty$ *when* $n \to \infty$.

Proof . *Let $f_k(z) = k \csc z$, then the first step of Noor iteration is*

$$|x_n| = |(1-a_3) z_n + a_3 f_k(z_n)|$$

For $n = 0$, let $z_0 = z$, $x_0 = x$ and $y_0 = y$, then we have

$$\begin{aligned} |x| &= |(1-a_3) z + a_3 f_k(z)| = |a_3 (k\ csc\ z) + (1-a_3) z| \\ &\geq a_3 |(k\ csc\ z)| - (1-a_3)|z| = a_3 \left| k \left(\frac{1}{z} + \frac{1}{3!} z + \left(\frac{1}{3!3!} - \frac{1}{5!} \right) z^3 + \ldots \right) \right| - (1-a_1)|z| \\ &\geq |z| (a_3 |k| |a| - 1), \end{aligned}$$

where $\left| \frac{1}{z^2} + \frac{1}{3!} + \left(\frac{1}{3!3!} - \frac{1}{5!} \right) z^2 + \ldots \right| > |a|$ *such that* $|a| \in (0, 1]$ *and* $z \in \mathbb{C}$ *except those values of* z *wherefore*$|a| = 0$.

$$|x| \geq |z| (a_3 |k| |a| - 1).$$

Because $|z| > \frac{2}{a_3 |a|}$, *this yields* $a_3 |k| |a| - 1) > 1$. *So, we conclude*

$$|x| \geq |z|.$$

Now the second step of Noor iteration is

$$|y_n| = |(1-a_2) z_n + a_2 f_k(x_n)|.$$

For $n = 0$, we have

$$\begin{aligned} |y| &= |(1-a_2) z + a_2 f_k(x)| = |a_2 (k\ csc\ x) + (1-a_2) z| \\ &\geq a_2 |(k\ csc\ x)| - (1-a_2)|z| = a_2 \left| k \left(\frac{1}{x} + \frac{1}{3!} x + \left(\frac{1}{3!3!} - \frac{1}{5!} \right) x^3 + \ldots \right) \right| - (1-a_z)|z| \\ &\geq a_2 |k| |c| |z| - |z| \end{aligned}$$

Since $\left| k \left(\frac{1}{x} + \frac{1}{3!} x + \left(\frac{1}{3!3!} - \frac{1}{5!} \right) x^3 + \ldots \right) \right| \geq |c|$ *where* $|c| \in (0, 1]$ *and* $x \in \mathbb{C}$ *except those values of* x *wherefore* $|c| = 0$.

$$|y| \geq |z| (a_2 |k| |c| - 1).$$

Since $|z| > \frac{2}{a_2 |c|}$, *this yields* $|y| \geq |z|$. *Now for the last step of Noor iteration we have*

$$|z_{n+1}| = |(1-a_1) z_n + a_1 f_k(y_n)|.$$

For $n = 1$, we have

$$\begin{aligned} |z_1| &= |(1-a_1) z + a_1 f_k(y)| = |a_1 (k\ csc\ y) + (1-a_1) z| \\ &\geq a_1 |(k\ csc\ y)| - (1-a_1)|z| = a_1 \left| k \left(\frac{1}{y} + \frac{1}{3!} y + \left(\frac{1}{3!3!} - \frac{1}{5!} \right) y^3 + \ldots \right) \right| - (1-a_1)|z| \\ &\geq a_1 |k| |b| |y| - (1-a_1)|z| \geq a_1 |k| |b| |y| - |z| \geq a_1 |k| |b| |z| - |z| \end{aligned}$$

Iterating up to nth term, we get

$$|z_n| \geq |z| (a_1 |k| |b| - 1)^n.$$

Since $|z| > |k| > \frac{2}{a_1 |b|}$, $|z| > |k| > \frac{2}{a_2 |c|}$ *and* $|z| > |k| > \frac{2}{a_3 |a|}$, $a_1 |k| |b| - 1 > 1$. *Hence* $|z_n| \to \infty$ *when* $n \to \infty$. $\square$

Corollary 3.8. *Let* $|z_m| > \max \left\{ \left| \frac{2}{a_2 |c|} \right|, \frac{2}{a_1 |b|}, \frac{2}{a_3 |a|} \right\}$, *for some* $m \geq 0$. *Since* $a_1 |bk| > 2$, $a_2 |ck| > 2$ *and* $a_3 |ak| > 2$, $|z_{m+n}| > (a_1 |bk| |z|)^{m+n}$. *Therefore* $|z_n| \to \infty$ *as* $n \to \infty$.

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