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A new bond portfolio optimization model as two-stage stochastic programming problems in U.S. market

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Abstract

We formulate a new bond portfolio optimization model as a two-stage stochastic programming problem in which a decision maker can optimize the cost of bond portfolio selection while deciding which bonds to sell, which bonds to hold, and which bonds to buy from the market, as well as determine the quantity of additional cash in period t under different scenarios and varying assumptions, The model proved its efficiency by finding the optimal values and giving an investment plan that, it will reduce the cost of the portfolio.

Keywords: Stochastic Portfolio Programming model, linear programming, nonlinear programming, constrained optimization.

1. Introduction

It is difficult to create and pick the best bond for a portfolio in modern financial mathematics. Markowitz was the first to resolve this issue in 1952. Simple models and extensions were used to create the optimal portfolio with the lowest expenses and highest cash to achieve the best investment strategy in the various circumstances. Since a portfolio is a collection of securities, the idea is to pick the best portfolio from the available portfolios. Markowitz used the simple regression model to solve the portfolio the most. That is why this study seeks to translate the attributes of each individual

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security into a random linear model with two phases, using exact mathematical hypotheses and procedures.

that assuming the term "two-stage stochastic linear programming," or "TSLP," refers to issues that have two decision stage by use linear programming (LP) with stochastic Recourse model. Surveys like [6, 12] and publications like [6, 13] and others show the relevance of this topic for managerial aims, properties, solution approaches, and applications. For stochastic optimization issues, a variety of approaches can be utilized. These include resilient optimization [4], chance constraint optimization [11, 14], sampling-based methods [8] and scenario-based optimization [6, 8]. The last strategy will be the subject of this paper. The fundamental TSLP issue, on the other hand, is a risk-neutral linear optimization problem with continuous variables. For example, TSLP problems with mixed-integer variables [1, 3] and multi-stage versions of the TSLP problem [9, 10] have been presented, as have TSLP problems that incorporate risk aversion [7, 10], and Monte Carlo sample based methods for two stage bond portfolio optimization problem [8] etc. Methods of improvement has been discussed to choice the best available decisions that are able to be executed regards to selling and purchasing a number of assets by the time, considering a number of primary conditions and the investor's goal that is determined previously. This is a very important problem which has been discussed intensively and accurately over the years. Some common methods will be recount here briefly, noting the reason of the unsuitability of these methods for our specific problem. It is also requiring a general discussion of different goals which may the investor has, and how the definition of the goal can change our available options, and looking forward to determine what we think that is the appropriate goal for investors facing the problem what we are facings.

2. Stochastic Two Stage Linear Programming with fixed recourse function

$$\begin{aligned} &\min z = c^T + E_\xi Q(X, W) \\ \text{S.t} \\ &Ax = b \\ &x > 0 \end{aligned} \tag{2.1}$$

Where represent stage 2

$$\begin{aligned} &Q(x, w) = \min \ q(w)^T y \\ \text{S.t} \\ &(w)x + W(w)y = h(w)x \\ &y \geq 0 \end{aligned} \tag{2.2}$$

Where $C \in R^{n1}$, $q \in R^{n_2}$, $A \in R^{n_1 * m_1}$, $b \in R^{m_1}$, $W(w) \in R^{m_2 * n_2}$,
$T(w) \in \{T_1(w), T_2(w), \ \ldots\ldots . T_{m_2}(w)\} \in m_2 * n_1$, $h(w) \in R^{m_2}$,
$\xi^\tau(w) = [q(w),\ h(w),\ T_1(w), T_2(w),\ \ldots, T_{m_2}(w)]$
The form above can be rewritten

$$\begin{aligned} &\min z = c^T + E_\xi \left[\min \ q(w)^T y(w)\right] \\ \text{S.t} \\ &Ax = b \\ &T(w)x + W(w)y = h(w)x \\ &x \geq 0 \\ &y(w) \geq 0 \end{aligned} \tag{2.3}$$

Where $C, \ R^{n_1} \ vecter, \ A\&W \ Matrix \ m_1 * n_1 \ \& \ m_2 * n_2$,
$W =$ recourse matrix, it here maybe is fixed or random.

Here X is a polyhedral set, characterized by a finite number of linear constraints, and X is the first-stage decision vector. First, we must make a "here-and-now" judgment prior to the arrival of the uncertain data viewed as a random vector, while the realization of this data is known. At the second stage, after the opportunity to utilize ξ has presented itself, we pursue a certain optimization task. In the beginning, we just care about getting the first-stage decision and its associated cost of it, and then care to find the expected cost of the (probability) second-stage decision. Alternatively, we can regard the solution to the second-stage problem as a recourse action in which the term Wy represents the systematic discrepancy of the system $Tx < h$ and qTy represents the cost of this recourse action. Due to the nature of the objectives and restrictions, the examined two-stage issue is linearly posed. This is conceptually unnecessary and one can think of two-stage stochastic programming in a more comprehensive way. A good example of this is an integer (combinatorial) problem where X is the feasible set (finite) [14].

3. Model building for securities market

After examining on the previous determined portfolios models and how to build it and examining random portfolios and quadratic portfolios, it is noted that there is special hypothesis for each portfolio built depends on the nature of the market. And hypothesis related to the portfolio built are:

(1) Short sale is not included because it will result negative values in the financial asset which is incompatible with non-negative variables of the model.

(2) Additional investments are not existing.

(3) No bond is recalled in any circumstances.

(4) The portfolio contains cash Z_0.

(5) all Bonds are purchased in primary stage and the purchased bond is not sold in the same stage.

(6) There is a simple restriction on the inventory balance for each asset in the portfolio which states that the quantity we keep is the same quantity we have in addition to the quantity we buy subtract the quantity we sell.

(7) The is no additional cost on transactions.

(8) Evaluate the portfolio each six months.

(9) Cash flows are obtained immediately.

(10) The time plan horizon of the problem 5 year.

There are a number of hypothesis which can be used depending on available data and information.

3.1. Deterministic linear programming models of portfolio

The portfolio model will be built considering all information is confirmed means the linear model and portfolio commitments are not random. The goal behind this is to reduce purchasing cost and primary investment cost and to clarify absolute linear programming method for the portfolio model with fixed income. The essential portfolio model is considered as cost model that offers a purchasing strategy with lower cost for bonds and which used cash flows from its income in order to commit monetary obligations. Then the essential model will be developed (PREKOPA ,2014). At the end of this chapter, practical applications will be discussed.

Indices

$j = (1, \dots, N)$ bonds

"N = bonds set of purchased steage1" "

"$t = (1, 2, ..., T)$ time period divided into two parts"

For stage one, $t = (t = 1, ..., t^*)$.

For stage two, $t = (t = t^* + 1, ..., T)$

T denotes a time sequence.

Parameters

cj = price for bond j

f_{jt}=cash flow getting bond j in stage 1 at period t (from capons and principal re –payment)

L_t cash requirement or (liability) to be paid at time t.

ρ_t= re- investment for period t in stage 1 $(t = 1, \dots, t^*)$

i_t = interest rate in stage 1 for period t

q_j = Minimum purchase allowed of bond

Q_j = Maximum purchase allowed of bonds.

Decision variables

X_j = number of bond j to purchased measured in thousands.

Z_t = cash surplus to be acuminated at period t.

$d_j = 1$, selecting bond in the portfolio

$d_j = 0$, not selecting bond in the portfolio

3.2. Deterministic Model (linear programming)

Minimizing Object function

$$\sum_{j=1}^{N} C_j x_j + Z_0 \tag{3.1a}$$

Subject to

$$\sum_{j=1}^{N} f_{jt}\ x_j + Z_t(1 + \ r_{t-1}) - Z_t = L_t \qquad for\ t = 1, \dots, t^* - 1 \ \ , j \in N \tag{3.1b}$$

$$\sum_{j=1}^{N} f_{jt}\ x_j + Z_t(1 + \ i_t) = L_t \qquad when\ j = T \tag{3.1c}$$

$$x_j \geq 0, \quad z_t \geq 0 \qquad q_j d_j \leq x_j \leq Q d_j \tag{3.1d}$$

It is noted that the objective function is total bonds purchases $\sum_{j=1}^{N} C_j x_j$ and $j = 1, \ldots, N$, in addition to the primary investment value Z_0 because the portfolio in cash back guaranteed $f_0 Z_0$ at the end of the first period within planning horizon of the portfolio where initial cash investments required in the portfolio in order to redeem early periods cash obligations before starting to return coupon principal repayment.

Constrain (3.1b) is the monetary balance function or what is called obligations constrain which starts with monetary coupons and redeem requirements in addition to reinvestment function $\rho_t = (1 + \ i_t)$ product to the surplus cash for early period subtract the present period cash from it to commit requirements L_{t} . The value ρ_{t} has a very small values in some cases.
As for constrain (3.1c), it is a constrain to not transfer cash surplus to another level to keep it in the portfolio in order to keep cash in the portfolio.
And the constrain (3.1d) is to ensure that the model works within threshold limits and to guarantee implementable and realistic results.

3.3. Two-stage Stochastic programming model

The previous studies focused on deterministic mathematical models to choice fixed income portfolio, however, these models describe the stable situation in the financial market or all available and certain information and this in the nature of the market does not exist because financial market characterized by randomness specially in the present period of time, therefore, it is preferable for investors to resort to build two-step linear random portfolio. The first step is the linear step where information is certain then moving to the second step for random model through adding return function to the previous linear model for portfolio and converting portfolio from deterministic linear programming to the linear stochastic programming and some hypothesis will be changed which have been imposed previously specially by deterministic linear programming assuming that parameters are certain (certainty). The portfolio supposed to redistribute at the end of each period of time because of existing randomness in real for bonds market, in other words, the obligation size will be varied from fixed to randomness, and new computers have been very useful for solving these complex issues.

3.3.1. Extension to the Basic Models of Portfolios

New Indices:

M : set of bonds that can be purchased in stage1 within any scenarios
K : scenarios of the uncertain future
E : number(quantity) of bonds purchased in stage1 ,where have $t \leq t^*$, $(K = 1, 2, \ldots, k)$
t^* : last time period when all parameters know with certainly

New parameter

L_{kt} : cash Obligation(liability) to be Fulfilling in time t within scenario k.
f_{jkt} : cash flow to be Fulfilling in time t within scenario k.
ρ_{kt} : re investment to be Fulfilling in time t within scenario k.
i_{kt} : interest rate in time t within scenario k.
p_k : probability occur scenario k .
v_{kt} : change factor for price in period t within scenario k.
Q_{jkt} : selling price of bond j within scenario k at time t, $j \in N \cup M$
p_{jk} : buying price of bond j purchased within scenario k at time t.

New decision variables:

A_{kt} : quantity cash of required in period t within scenario k

z_{kt} : cash surplus accumulated at the each of the period t within scenario k.

s_{jkt} : Number of bond j which will sold in stage 2 under scenario k (that has not yet reached the maturity).

y_{jk} : number of units purchased for j in stage 2 within scenario k.

To get on two stage stochastic programming with recourse, can be written as : (Object Function):

$$\sum_{j=1}^{N} c_j \ x_j + \sum_{k=1}^{K} \sum_{t=t^*+1}^{T} p_k a_{kt} A_{kt} + \sum_{k=1}^{K} \sum_{j\in m} p_k p_{jk} y_{jk} - \sum_{k=1}^{K} \sum_{j\in E} \sum_{t=t^*+1}^{T} p_k Q_{jkt} S_{jkt} + z_0 \tag{3.2a}$$

where

$E = N - N^*$, number bonds which have maturity $\geq t^*$.

$N =$ quantity of bonds purchase stage 1.

$N^* =$ quantity of bonds purchase stage 1 but it has maturity $\leq t^*$.

Subject to

$$\sum_{j=1}^{N} f_{\mathrm{jt}} x_j + (1+\ i_t) z_{t-1} - z_t = L_t \ \ where \qquad t = 1, \dots, t^* \tag{3.2b}$$

Constraint of transition when period $t = t^* + 1, k = 1, \dots, K$:

$$\sum_{j\in E} f_{j(t^*+1)} x_j \ - \ \sum_{j\in E} f_{j(t^*+1)} S_{jk(t^*+1)} + (1+\ i_{k(t^*+1)}) \ \sum_{j\in E} Q_{jk(t^*+1)} S_{jk(t^*+1)}$$
$$+ \sum_{j\in M} f_{jk(t^*+1)} y_{jk} + (1+\ i_{k(t^*+1)}) z_{t^*} - z_{k(t^*+1)} + A_{k(t^*+1)} = L_{k(t^*+1)} \tag{3.2c}$$

When $t = t^* + 2, \dots, T, \quad k = 1, \dots, K$

$$\sum_{j\in E} f_{jt} x_j \ - \ \sum_{t=t^*+2}^{T} \sum_{j\in E} f_{jt} S_{jkt} + \ (1+\ i_{kt}) \ \sum_{t=t^*+2}^{T} \sum_{j\in E} Q_{jkt} S_{jkt} + \ \sum_{t=t^*+2}^{T} \sum_{j\in M} f_{jkt} y_{jk}$$
$$+ (1+\ i_{kt}) z_{t^*} - z_{kt} + A_{kt} = L_{kt} \tag{3.2d}$$

$$\sum_{t=t^*+1}^{T} s_{jkt} \leq y_j, \quad j \in E \ \ , \ k = 1, \ \dots, \ K \tag{3.2e}$$

$$q_j d_j \leq x_j \leq Q d_j \qquad for \ \ j = 1, 2, \dots, N \tag{3.2f}$$

$$x_j \ \ , \ y_{jk} \ , \ S_{jkt} \ \ , \ Z_t \ \ , \ \ Z_{kt} \ , \ \ M_{kt} \quad \geq \ \ 0 \tag{3.2g}$$

3.3.2. Explanation Objective function

We notice that the objective function (3.2a) contains the cost reduction part for the bonds of the first stage $\sum_{j=1}^{N} c_j x_j$ In addition to the expected value of discounted costs $\sum_{k=1}^{K} \sum_{t=t^*+1}^{T} p_k v_{kt} A_{kt}$ of the additional cash A_{kt}that will be required in a later period to met the obligations.

The discount factor a_{kt} can be calculated from The equation:

$$a_{kt} = \frac{1}{\prod_{t=t^*+1}^{T} \rho_{kt}} \tag{3.3}$$

Where ρ_{kt} it is the investment rate that took place $\rho_t = (1 + i_t)$ in the first stage As for the second stage, it is calculated $\rho_{kt} = (1 + i_{kt})$ where i_t, i_{kt}, the rate of return for the first and second stages, respectively.

To reduce the expected buying price of those bonds purchased in the second stage $\sum_{k=1}^{K} \sum_{j\in m} p_k p_{jk} y_{\mathrm{jk}}$ which represents the expected value of purchasing bonds in the second stage where p_k it is the probability of the occurrence of the scenario$k$. Under the scenario$k = k_1, \ldots, K$ is the purchase price of the bond j in the second stage within the scenario k.

Then we find the expected profit from selling bonds in the second stage that has not reached the stage of maturity. Also $EQ_{jkt}S_{jkt} = \sum_{k=1}^{K} \sum_{j\in E} \sum_{t=t^*+1}^{T} p_k Q_{jkt} S_{jkt}$ which represents the expected value of selling bonds.

The general objective of the portfolio is to reduce the cost of purchasing bonds in the first stage, And reduce the cost of purchasing bonds in the second stage within the different scenario, in addition to reducing the expected value of the discounted value, which expresses the deficit in the portfolio, which can be reduced by maximizing the sales profit in the second stage within the different scenarios In addition to reducing the initial cash of the portfolio, the investment amount is Z_0. In order to achieve these goals, the values must be well-studied and express the objectives and fulfill the conditions sec (3.2) in addition to the different scenarios that the portfolio manager must take to buy, hold or sell bonds during the planning period timeline

3.3.3. Explanation Constraints of model

First Constraint (3.2b): Restrictions are also called liability constraints we also notice that these constraints are similar to the constraints of the obligations in the deterministic linear model (3.1b) because we assume that the information is certainty for the period $t = 1, 2, \ldots, t^*$, We notice during that period that all the information is perfect and known with certainty.

As for constraint (3.2c) it are a obligations constraints as too, but these constraints are stochastic constraints that depend on scenarios within the period $t = t^* + 1, \ldots, T$ to fulfill the stochastic (probabilistic) liabilities in the second stage because it is not possible to ascertain the fulfillment of obligations as in the deterministic model where obligations can be fulfilled by obtaining a cash flows to bonds, that can be obtained by multiplying the bond price c_j in the first stage by coupon rate I_j for those bonds that have not reached maturity date, i.e. $t < M_j$ where $t = 1, 2, \ldots, t^*$ as for the bonds that have reached maturity date $t = \mathrm{MD}_j$ where, $t = 1, 2, \ldots, t^*$, also the cash flow for it is equal to the price of the bond multiplied by $(1 + I_j)$, as for the cash flows of bonds in the second stage through the price of the bond p_{jk} in the coupon rate I_j for the period $t = t^* + 1, \ldots, T$ for bonds that have not reached the stage of maturity date $t < \mathrm{MD}_j$ where $t = t + 1, \ldots, T$, as for the cash flows of bonds that have reached the stage of maturity $t = \mathrm{MD}_j$, then cash flows for them is equal to the price of the bond multiplied by $(1 + I_j)$,the cash flows of bonds that cross the stage of maturity date $t > \mathrm{MD}_j$ equal to zero in both two-stages.

Since obligations are probabilities that cannot be certain fulfill during the period $t = t^* + 1, \ldots, T$ through cash flows and bond investment and the rate of reinvestment from cash surpluses, a recourse variable has been added A_{kt} that represents the amount of cash added for the period t to fulfill liabilities under scenario k.

The reinvestment rate of the surplus cash for each period can be calculated by depending on the rate of interest directly through the equation $p_{kt} = 1 + i_{kt}$ and the discount factor.

$$v_t = \frac{1}{\prod_{t=1}^{T} \rho_t} \quad stage\ 1 \tag{3.4}$$

$$v_{kt} = \frac{1}{\prod_{t=1}^{T} \rho_{kt}} \quad stage\ 2 \tag{3.5}$$

To the second stage, the constraint (3.2d) is the transition period of constraint when $t = t^* + 1$ from the first deterministic stage to stochastic stage , in this period the financial analyst must decide whether to sale the bonds or keep the bonds purchased in the first stage or buy new bonds in the period $t^* + 1$ (which represents the transition from the first period to the second stage) to meet the requirements for fulfilling obligations, in addition to determining the amount of cash to be added to that period $A_{k(t^*+1)}$, we note that the transitional constraint consists of a set of parts as the first part is the flow of bonds purchased in the first stage that have not reached the maturity stage, the second part is the financial flow of the bonds sold in the second stage, the third part is the financial flow from purcheses bonds in the second stage, and the fourth part is the reinvestment of risk free cash Under different scenarios k buying, selling and holding.

As for constraint (3.2e), it is a condition of the portfolio that the amount of sold bonds that did not reach maturity in the second stage is less than the amount of bonds purchased in the first stage the constraint (3.2g) is the constraint of the threshold limit that ensures that the solution falls within the region of the feasible solution

Introduction Bond Data

We get all the data of bonds are composed finra - arkets [4], a bonds the candidates were only US Corporate Debentures in Table 1 and classification of bonds in Table 2.

Table 1: Take the Active of most invested bonds

NO.	Bond name	Symbol bond	Issue size dollar amount in thousands	Par value	Maturity data	Value traded	Nominal vale traded	Transactions	Interest
1	B A T CAP CORP	BTI4773079	2,477,391	5551000	08/15/2024	5.89E+08	5.55E+08	9	3.222%
2	VERIZON COMMUNI-CATIONS INC	VZ5148420	4,250,000	11157000	12/21/2030	1.15E+09	206707.4	20	2.550%
3	CONAGRA BRANDS INC	CAG5059244	1,000,000	2782000	11/01/2027	2.73E+08	2.78E+08	20	1.375%
4	APPLE INC	AAPL5231623	2,300,000.00	24382000	08/05/2028	2.43E+09	179718.8	18	1.400%
5	ORACLE CORP	ORCL.GP	1,250,000	1840000	07/08/2039	2.55E+08	264460.2	19	6.125%
6	FISERV INC	FISV4845277	2,000,000	443000	07/01/2026	48178147	44300000	17	3.200%
7	SOUTHWESTERN ENERGY CO	SWN5241640	1,200,000.00	15290000	03/15/2030	1.59E+09	1.53E+09	21	5.375%
8	AMGEN INC	AMGN523527!	1,250,000.00	43768000	08/15/2028	4.37E+09	4.38E+09	28	1.650%
9	MOODYS CORP	MCO5238038	600,000	3084000	08/19/2031	3.07E+08	3.08E+08	12	2.000%
10	CF INDS INC	CF4105321	750,000.00	19024000	03/15/2034	2.38E+09	1.9E+09	31	5.150%

Table 2: classification of bonds

Type of bonds	***US Corporate Debentures***
Type of Debts	***Senior Unsecured Note***
Callable	***Yes***

4 To plan horizontally for the next 5 years (T=10 sim-annual), there are assumptions that must be taken into account:

1. Depend on the horizon of the plan then data maturity will be Table 3

Table 3: distribution of the horizon plan period by depending on the maturity date

Maturity data	*08/15/ 2024***	*12/21/ 2030*	*11/01/ 2027*	*08/05/ 2028*	*07/08/ 2039*	*07/01/ 2026****	*03/15/ 2030*	*08/15/ 2028*	*08/19/ 2031*	*03/15/ 2024**
Period	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Period of planning	*1/2022*	*7/2022*	*1/2023*	*7/2023*	*1/2024**	*7/2024**	*1/2025*	*7/2025*	*1/2026*	*7/2026****

2. Observing bond prices and recording changes in price during the year through the highest and lowest price, in addition to the bid price(offering) of the bond are found in Table 4.

Table 4: Investment Grade Top 10 Most Active Bonds with changes bond's prices for 2021 year

Symbol	***Highest price***	***Normal price(offering)***	***Lowest price***
BTI4773079	***154.83***	***99.33***	***132.25***
VZ5148420	***99.07***	***99.59***	***104.35***
FISV4845277	***107.3***	***99.7***	***112.39***
CAG5059244	***101.06***	***98.82***	***94***
AAPL5231623	***100.46***	***99.77***	***99.2***
ORCL.GP	***154.8***	***99.33***	***132.25***
SWN5241640	***105.36***	***100.02***	***100.19***
AMGN5235279	***99.1***	***99.91***	***100.3***
MCO5238038	***99.74***	***98.67***	***98.76***
CF4105321	***124.85***	***99.48***	***113.69***

3. The paper will divide the five years into ten periods depending on the coupon rate and it is paid every six months (semi-annual) and considering the first four market periods as fixed T = 4, this means the first two years in the portfolio are fixed, therefore we build the linear model which represents the first stage, while the other remained periods T = 6 are uncertainty in the market which represent the second stage, the uncertainty is the change in the interest rates, which will study of three scenarios k=3 high, normal and low-interest rates. also, we will divide the bonds into two parts, the first part consist of four bonds are fixed in the market trading, while the other six bonds are not fixed in interest.

4. Stage one bonds: B A T CAP CORP, VERIZON COMMUNICATIONS INC, CONAGRA BRANDS INC, APPLE INC.

5. Stage two bonds: ORACLE CORP, FISERV INC, SOUTHWESTERN ENERGY CO, AMGEN INC, MOODYS CORP, CF INDS INC

6. To calculate interest rates in stage 1 for period t is

i_t=coupon rate +Interest issued by the US Federal Reserve

Where:

Interest issued by the US Federal Reserve are values that range between [0.25-1.75] %, it's any following a uniform distribution [0.25-1.75] %, therefore randomly generated it (see Table 5).

The Federal Open Market Committee meets eight times each year to set the interest rate, and after the meeting, the Federal Reserve publishes a statement containing the interest rate decision. The decision to set interest rates depends mostly on inflation.

The rapid deployment of a vaccine (covid 19) in the United States and the passage of a $1.9 trillion fiscal stimulus package boosted the expected economic recovery. Looking ahead, US longer-dated interest rates have soared rapidly, with the yield on 10-year Treasuries increasing from less than 1% at the start of the year to more than 1.75% in mid-March.[1,2]

Table 5: interest rates in stage 1

Period	***1***	***2***	***3***	***4***
Interest rate	***0.025***	***0.019***	***0.013***	***0.115***

To calculate interest rates in stage 2 for period t under k of scenario is i_{kt}= coupon rate +Interest issued by the US Federal Reserve

Where:

Interest issued by the US Federal Reserve are values that range between [0.25-1.75] %, it's any following a uniform distribution [0.025-0.175] %, therefore randomly generated it.

Table 6: interest rates belong to any scenario in stage 2

K \ Period	***5***	***6***	***7***	***8***	***9***	***10***
1	***0.107***	***0.045***	***0.047***	***0.063***	***0.151***	***0.063***
2	***0.147***	***0.061***	***0.164***	***0.077***	***0.054***	***0.0626***
3	***0.117***	***0.095***	***0.077***	***0.149***	***0.112***	***0.107***

7. The cost of bonds in stage 1 is the same par value because of information certainty (see Table 7)

Table 7: cost of bonds in stage 1

j	***Bond name***	***Symbol bond***	***Cost of bonds***
1	***B A T CAP CORP***	***BTI4773079***	***5551000***
2	***VERIZON COMMUNICATIONS INC***	***VZ5148420***	***11157000***
3	***CONAGRA BRANDS INC***	***CAG5059244***	***2782000***
4	***APPLE INC***	***AAPL5231623***	***24382000***

8. Now to calculated prices of buying (cost) to stage 2 by depended on different scenario k (shown 9)

At the start, we will calculate the value of a change factor (impact factor) in price through historical data, where:

$$v_{jk} = \frac{|price\ normal - price\ (high\ or\ low)|}{\text{price normal}} * 100$$

Table 8: change factor for price

j	*Bond name Y_j*	*Symbol bond*	*Change factor v_{j1}*	*Change factor v_{j2}*	*Change factor v_{j3}*
5	ORACLE CORP	ORCL.GP	0.55844	1	0.33142
6	FISERV INC	FISV4845277	0.07623	1	0.12728
7	SOUTHWESTERN ENERGY CO	SWN5241640	0.05339	1	0.0017
8	AMGEN INC	AMGN5235279	0.008107	1	0.0039
9	MOODYS CORP	MCO5238038	0.01084	1	0.00091
10	CF INDS INC	CF4105321	0.25503	1	0.14284

After calculated v_{jk} will be:
p_{jk}=par value*v_{jk} look at the Tables (4-8).

Table 9: prices of buying (cost) of stage 2

j	*Bond nameY_j*	*Symbol bond*	*Buying under scenario 1 p_{j1}*	*Buying under scenario 2 p_{j2}*	*Buying under scenario 3 p_{j3}*
5	ORACLE CORP	ORCL.GP	1027530	1840000	609812.8
6	FISERV INC	FISV4845277	33769.89	443000	56385.04
7	SOUTHWESTERN ENERGY CO	SWN5241640	816333.1	15290000	25993
8	AMGEN INC	AMGN5235279	354827.2	43768000	170695.2
9	MOODYS CORP	MCO5238038	33430.56	3084000	2806.44
10	CF INDS INC	CF4105321	4851691	19024000	2717388

9. to calculate the value movements of the financial of a future interest of the bonds (cash flows) in each period of time in stage 1, see at table (10)

1. Cash Flow of stage 1 = cost bond * coupon bond when timeless than maturity data
2. Cash Flow of stage 1 = cost bond *(1+coupon bond) when time equal maturity data
3. Otherwise equal zero

Table 10: Cash flows of stage 1

j	*Bounds \ Times*	*1*	*2*	*3*	*4*	*5*	*6*
1	B A T CAP CORP	178742.2	178742.2	178742.2	178742.2	178742.2	2557162.99
2	VERIZON COMMUNICATIONS INC	284503.5	284503.5	284503.5	284503.5	284503.5	284503.5
3	CONAGRA BRANDS INC	38391.6	38391.6	38391.6	38391.6	38391.6	38391.6
4	APPLE INC	341348	341348	341348	341348	341348	341348

Now to calculate cash flows for each bond in stage 2

1. Cash Flow of stage 2= price of buying under any scenario * coupon bond when timeless than maturity data
2. Cash Flow of stage 2 = price of buying under any scenario *(1+coupon bond) when time equal maturity data
3. Otherwise equal zero

Table 11: Cash flows of stage 2 for ORACLE CORP

K \ Period	5	6	7	8	9	10
1	62987.56448	62987.56448	62987.56	62987.56	62987.56	62987.56
2	112792	112792	112792	112792	112792	112792
3	37381.52464	37381.52464	37381.52	37381.52	37381.52	37381.52

Table 12: Cash flows of stage 2 for FISERV INC

K \ Period	5	6	7	8	9	10
1	1080.63648	1080.63648	1080.636	1080.636	34850.53	2064000
2	14176	14176	14176	14176	457176	2064000
3	1804.32128	1804.32128	1804.321	1804.321	58189.36	2064000

Table 13: Cash flows of stage 2 for SOUTHWESTERN ENERGY CO

K \ Period	5	6	7	8	9	10
1	43918.72078	43918.72078	43918.72	43918.72	43918.72	43918.72
2	822602	822602	822602	822602	822602	822602
3	1398.4234	1398.4234	1398.423	1398.423	1398.423	1398.423

Table 14: Cash flows of stage 2 for AMGEN INC

K \ Period	5	6	7	8	9	10
1	58546.48404	58546.48404	58546.48	58546.48	58546.48	58546.48
2	7221720	7221720	7221720	7221720	7221720	7221720
3	28164.708	28164.708	28164.71	28164.71	28164.71	28164.71

Table 15: Cash flows of stage 2 for MOODYS CORP

K \ Period	5	6	7	8	9	10
1	668.6112	668.6112	668.6112	668.6112	668.6112	668.6112
2	61680	61680	61680	61680	61680	61680
3	56.1288	56.1288	56.1288	56.1288	56.1288	56.1288

Table 16: Cash flows of stage 2 for CF INDS INC

K \ Period	5	6	7	8	9	10
1	5101552.792	5101552.792	0	0	0	0
2	20003736	20003736	0	0	0	0
3	2857333.65	2857333.65	0	0	0	0

10. To find price of sell for bonds of stage 2 which have maturity data

To make table of selling price, we start calculate reinvestment rate in stage 1 and 2, it which depend addition one to interest rate. either price of sell bond is cost of bond remained which has maturity date a great than period t in stage 1 divided by the product of the reinvestment rate for the previous period.

$$Q_{jkt} = \frac{c_j}{\prod_{t=1}^{t^*} \rho_t \prod_{t^*+1}^{t-1} \rho_{kt}} \; for\ remained\ bonds\ from\ stage\ 1\ to\ stage\ 2$$

$$Q_{jkt} = \frac{p_{jk}}{\prod_{t^*+1}^{t-1} \rho_{kt}} for\ bonds\ from\ stage\ 2$$

Table 17: reinvestment rate of stage 1

Periods	1	2	3	4
ρ	1.025	1.019	1.013	1.115

Table 18: Reinvestment rate of stage 2

K \ Period	5	6	7	8	9	10
1	1.107	1.045	1.047	1.063	1.151	1.063
2	1.147	1.061	1.164	1.077	1.054	1.0626
3	1.117	1.095	1.077	1.149	1.112	1.107

Table 19: Selling price of stage 2

Name bounds	Period / K	5	6	7	8	9	10
B AT	1	4325515	3907421	0	0	0	0
CAP	2	4325515	3771156	0	0	0	0
CORP	3	4325515	3872440	0	0	0	0
VERIZON	1	8693889	7853558	7515366	7178000	6752587	5866714
COMMUNICATIONS	2	8693889	7579676	7143898	6137370	5698579	5406622
INC	3	8693889	7783248	7107989	6599805	5743955	5165427
CONAGRA	1	2167823	1958286	1873958	1789836	1683759	1462866
BRANDS	2	2167823	1889994	1781332	1530354	1420942	1348142
INC	3	2167823	1940754	1772378	1645662	1432256	1288000
APPLE	1	18999228	17162808	16423740	15686475	14756797	12820849
INC	2	18999228	16564279	15611950	13412328	12453415	11815385
	3	18999228	15533477	14422912	12552578	11288290	10197190
ORACLE	1	1027529.6	928211	888240.2	848366.964	798087.5	693386.1
CORP	2	1840000	1604185	1511956	1298930.88	1206064	1144273
	3	609812.8	545938	498573.6	462928.097	362317	362317
FISERV	1	33769.89	30505.77	50479	27881.6874	26229.24	0
INC	2	443000	386224.9	364019.7	312731.728	290373	0
	3	56385.04	50479	46099.54	42803.6592	37252.97	0
SOUTHWESTERN	1	816333.1	737428.3	23270.37	673995.215	634050.1	550868.9
ENERGY	2	15290000	13330427	12564022	10793833.2	10022129	9508662
CO	3	25993	23270.37	21251.48	19732.1047	17173.29	15443.6
AMGEN	1	354827.176	320530.4	152815.8	292958.62	275596.1	239440.5
INC	2	43768000	38158675	35964821	30897612.3	28688591	27218777
	3	170695.2	152815.8	139557.8	129580.101	112776.4	101417.6
MOODYS	1	33430.56	30199.24	2512.48	27601.5238	25965.69	22559.24
CORP	2	3084000	2688753	2534169	2177121.1	2021468	1917901
	3	2806.44	2512.48	2294.502	2130.45697	1854.184	1667.431
CF	1	4851690.72	0	0	0	0	0
ENDS	2	19024000	0	0	0	0	0
INC	3	2717388.16	0	0	0	0	0

11. Discount factor

Table 20: Discount factor of stage 1

Period	1	2	3	4
a_t	0.975609756	0.957419	0.945132	0.847652

Table 21: Discount factor of stage 2

Period / K	5	6	7	8	9	10
1	0.903342	0.864442	0.862791	0.776705	0.674809	0.634816
2	0.87184	0.821715	0.705941	0.65547	0.621888	0.585251
3	0.895255	0.817585	0.759131	0.660689	0.594145	0.536716

12. To find cash obligations (liability) to happen of the first time period), we will assume that the obligation is a percentage of the cash flows of stage 1, see table (22).

Table 22: Cash obligations at stage1

Periods	***1***	***2***	***3***	***4***
Obligation	***14999***	***128141***	***105161***	***146183***

13. To find cash obligations (liability) to happen in the time t under any scenario k, we will assume that the obligation is a percentage of the cash flows of stage 2 (all bonds), see table (23)

Table 23: Cash obligations at stage 2

K \ Period	***5***	***6***	***7***	***8***	***9***	***10***
Obligation 1	***16366 000***	***16354200***	***144522000***	***13122950***	***10374000***	***2998000***
obligation 2	***1946300***	***1282000***	***90758800***	***16509800***	***1670200***	***3046800***
obligation 3	***16010505***	***90782540***	***8644900***	***1650980***	***16070***	***1663100***

4. Results of analysis

After solving the proposed stochastic linear programming model 1 For the portfolio by use code C++ with CEPLEX By linking the two programs, the optimal results were as follows see to Table 24.

Table 24: The optimal solution of a model (2.1)

Objective Value = 41800997.091227	
X[2]= 160.000000	***Z[2][9]=139918958.497728***
X[3]= 30.000000	***Z[2][10]= 145631085.536537***
Z[1]= 4652231.804688	***Z[3][5]= 169070630.079847***
Z[2]= 9279714.010822	***Z[3][6]= 94349799.735885***
Z[3]= 13962420.100139	***Z[3][7]= 92969834.276182***
Z[4]= 20089146.245470	***Z[3][8]= 105171359.949068***
Z[1][5]= 167058184.909302	***Z[3][9]= 116934482.626949***
Z[1][6]= 160125278.236890	***Z[3][10]= 127783372.358640***
Z[1][7]= 23129166.056330	***S[2][1][5]= 160.000000;***
Z[1][8]= 11463353.541315	***S[2][2][5]= 160.000000;***
Z[1][9]= 2820319.846794	***S[2][3][5]= 160.000000***
Z[2][5]= 188105685.741160	***S[3][1][5]= 30.000000***
Z[2][6]= 198298132.700308	***S[3][2][5]= 30.000000;***
Z[2][7]= 140060227.361440;	***S[3][3][5]= 30.000000***
Z[2][8]= 134335064.809833	***A[1][6]= 1903674.707946***

4.1. To clarify the results (investment plan)

We note that the model (1) gave an optimal investment plan for the purchase amount of 160 units of the bond VERIZON COMMUNICATIONS INC and 30 units of the CONAGRA BRANDS INC bond in the first stage within the first period. while in the first period of the second stage (the fifth period representing the portfolio revaluation) the possible sale quantity under any scenario was the sale of all previous units that were purchased, i.e. 160 units sold from VERIZON COMMUNICATIONS INC under any scenario K=1,2,3 also 30 units of CONAGRA BRANDS INC sold under any scenario k=1,2,3 as for the second period of the second stage (sixth period), an amount of cash was added in the amount of 190,3674.7 to the portfolio. This addition resulted from changes in the investment plan and to confirm the sale of the previous units of bonds and to avoid the cash deficit that occurred in the tenth period under the first scenario (low interest). The cash is added to the obligations in the sixth period under the first scenario. It is possible that the deficit was caused by an error in the estimation of the financial obligations that were estimated based on historical data, the optimal cost for the model (1) was 41800997.091, to clarify the investment plan through Table (25), which gives the purchase and sale quantities, their prices and the amount of financial flows of the bonds nominated for investment. Table (26) and (27) shows the investment plan for the stage 1 of the new cash flows resulting from changes in the portfolio and the total interest Total Income and accumulated cash at the beginning and end of the stage assuming the information is certainty as for Table (28) and (29) it shows the investment plan for the second phase under any scenario of new cash flows resulting from changes in the portfolio, the total interest, and the quantity of units sold in the fifth period.

Table 25: Explain the plan horizon of investment a model (2.1)

Period		*Purchases*	*Sales*	*Price of buying($)*	*Price of selling($)*	*Flow of cash*	*Cost of unit*	*Purchases*	*Sales*	*Price of buying($)*	*Price of selling ($)*	*Flow of cash*	*Cost of unit*
Nam bond		*VERIZON COMMUNICATIONS INC*						*CONAGRA BRANDS INC*					
1		*160*				*284503.5*		*30*				*38391.6*	
2					*284503.5*						*38391.6*		
3					*284503.5*						*38391.6*		
4					*284503.5*						*38391.6*		
	S_1	*0*	*160*								*30*		
5	S_2	*0*	*160*								*30*		
	S_3	*0*	*160*								*30*		
	S_1	*0*											
6	S_2	*0*											
	S_3	*0*											
	S_1	*0*											
7	S_2	*0*											
	S_3	*0*											
	S_1	*0*											
8	S_2	*0*											
	S_3	*0*											
	S_1	*0*											
9	S_2	*0*											
	S_3	*0*											
	S_1	*0*											
10	S_2	*0*											
	S_3	*0*											

Table 26: The plan horizon of investment for bond VERIZON COMMUNICATIONS INC at a model 1 for stage 1

Period	*Interest Rate*	*Obligation ($)*	*Z_t at begin of period ($) (Cash accumulated)*	*Flow of cash ($)*	*Total interest($)*	*Total Income ($)*	*Z_t at end of period ($)*
Name bond			***VERIZON COMMUNICATIONS INC***				
1	*0.025*	*14999*	*0*	*45520460*	*0*	*45520460*	*4652231.80*
2	*0.019*	*128141*	*4652231.80*	*45520460*	*4740624.20*	*9261084.2*	*9279714.01*
3	*0.013*	*105161*	*9279714.01*	*45520460*	*9400350.29*	*13920810.29*	*13962420.10*
4	*0.115*	*146183*	*13962420.10*	*45520460*	*15568098.41*	*61088558.41*	*20089146.245*

Table 27: The plan horizon of investment for bond CONAGRA BRANDS INC at a model 1 for stage 1

Period	*Interest Rate*	*Obligation ($)*	*Z_t at begin of period ($) (Cash accumulated)*	*Flow of cash ($)*	*Total interest($)*	*Total Income ($)*	*Z_t at end of period ($)*
Name bond			***CONAGRA BRANDS INC***				
1	*0.025*	*14999*	*0*	*1151748*	*0*	*1151748*	*4652231.80*
2	*0.019*	*128141*	*4652231.80*	*1151748*	*4740624.20*	*5892372*	*9279714.01*
3	*0.013*	*105161*	*9279714.01*	*1151748*	*9400350.29*	*10552098.29*	*13962420.10*
4	*0.115*	*146183*	*13962420.10*	*1151748*	*15568098.41*	*61088558.41*	*20089146.245*

Table 28: The plan horizon of investment for bond VERIZON COMMUNICATIONS INC at a model 1 for stage 2

Period		*Interest Rate*	*Obligation ($)*	*Z_t at begin of period ($) (Cash accumulated)*	*Flow of cash ($)*	*Total interest($)*	*Total Income ($)*	*Z_t at end of period ($)*
Name bond				***VERIZON COMMUNICATIONS INC***				
	S1	*0.107*	*16366 000*	*167058184.909302*	*0*	*1391022240*	*184933410.7*	*184933410.7*
5	*S2*	*0.147*	*1946300*	*88105685.741160*	*0*		*101057221.5*	*101057221.5*
	S3	*0.117*	*16010505*	*169070630.079847*	*0*		*188851893.8*	*188851893.8*
	S1	*0.045*	*16354200*	*160125278.236890*	*0*	*0*	*167330915.8*	*167330915.8*
6	*S2*	*0.061*	*1282000*	*198298132.700308*	*0*	*0*	*210394318.8*	*210394318.8*
	S3	*0.095*	*90782540*	*94349799.735885*	*0*	*0*	*103313030.7*	*103313030.7*
	S1	*0.047*	*144522000*	*23129166.056330*	*0*	*0*	*24216236.86*	*24216236.86*
7	*S2*	*0.164*	*90758800*	*140060227.361440*	*0*	*0*	*163030104.6*	*163030104.6*
	S3	*0.077*	*8644900*	*92969834.276182*	*0*	*0*	*100128511.5*	*100128511.5*
	S1	*0.063*	*13122950*	*11463353.541315*	*0*	*0*	*12185544.81*	*12185544.81*
8	*S2*	*0.077*	*16509800*	*134335064.809833*	*0*	*0*	*144678864.8*	*144678864.8*
	S3	*0.149*	*1650980*	*105171359.949068*	*0*	*0*	*120841892.6*	*120841892.6*
	S1	*0.151*	*10374000*	*2820319.846794*	*0*	*0*	*3246188.144*	*3246188.144*
9	*S2*	*0.054*	*1670200*	*139918958.497728*	*0*	*0*	*147474582.3*	*147474582.3*
	S3	*0.112*	*16070*	*116934482.626949*	*0*	*0*	*130031144.7*	*130031144.7*
	S1	*0.063*	*2998000*	*0*	*0*	*0*	*0*	*0*
10	*S2*	*0.0626*	*3046800*	*145631085.536537*	*0*	*0*	*154747591.5*	*154747591.5*
	S3	*0.107*	*1663100*	*127783372.358640*	*0*	*0*	*141456193.2*	*141456193.2*

Table 29: The plan horizon of investment for bond CONAGRA BRANDS INC at a model 1 for stage 1

Period		*Interest Rate*	*Obligation ($)*	*Z_t at begin of period ($) (Cash accumulated)*	*Flow of cash ($)*	*Total interest($)*	*Total Income ($)*	*Z_t at end of period ($)*
Name bond				*CONAGRA BRANDS INC*				
	S1	*0.107*	*16366 000*	*167058184.909302*	*0*	*65034690*	*184933410.7*	*184933410.7*
5	*S2*	*0.147*	*1946300*	*88105685.741160*	*0*		*101057221.5*	*101057221.5*
	S3	*0.117*	*16010505*	*169070630.079847*	*0*		*188851893.8*	*188851893.8*
	S1	*0.045*	*16354200*	*160125278.236890*	*0*		*167330915.8*	*167330915.8*
6	*S2*	*0.061*	*1282000*	*198298132.700308*	*0*		*210394318.8*	*210394318.8*
	S3	*0.095*	*90782540*	*94349799.735885*	*0*		*103313030.7*	*103313030.7*
	S1	*0.047*	*144522000*	*23129166.056330*	*0*		*24216236.86*	*24216236.86*
7	*S2*	*0.164*	*90758800*	*140060227.361440*	*0*		*163030104.6*	*163030104.6*
	S3	*0.077*	*8644900*	*92969834.276182*	*0*		*100128511.5*	*100128511.5*
	S1	*0.063*	*13122950*	*11463353.541315*	*0*		*12185544.81*	*12185544.81*
8	*S2*	*0.077*	*16509800*	*134335064.809833*	*0*		*144678864.8*	*144678864.8*
	S3	*0.149*	*1650980*	*105171359.949068*	*0*		*120841892.6*	*120841892.6*
	S1	*0.151*	*10374000*	*2820319.846794*	*0*		*3246188.144*	*3246188.144*
9	*S2*	*0.054*	*1670200*	*139918958.497728*	*0*		*147474582.3*	*147474582.3*
	S3	*0.112*	*16070*	*116934482.626949*	*0*		*130031144.7*	*130031144.7*
	S1	*0.063*	*2998000*	*0*	*0*		*0*	*0*
10	*S2*	*0.0626*	*3046800*	*145631085.536537*	*0*		*154747591.5*	*154747591.5*
	S3	*0.107*	*1663100*	*127783372.358640*	*0*		*141456193.2*	*141456193.2*

5. Conclusion

This paper developed two-stage stochastic programming with recourse model for bond portfolio optimization problem, the model applied on US Corporate Debentures of portfolio to handle when random variables L_{kt} has discrete distribution, also the results was very good and it may give investment plan for 5 years.

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