Int. J. Nonlinear Anal. Appl. 13 (2022) 1, 1763-1768
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.5795

Some separation axioms On Γ-algebra

Mohaimen M. Abbood[a,*], Ali Al-Fayadh[b], Saba N. Al-Khafaji[c]

[a] *Ministry of Education, General Directorate of Education in Diyala, Iraq*
[b] *Department of Mathematicssand Computer Applications, College of Science, Al-Nahraini University, Baghdad, Iraq*
[c] *Department of Mathematics,College of Education, Al-Zahraa University for Women, Karbala, Iraq*

(Communicated by Madjid Eshaghi Gordji)

Abstract

In this paper, we define and study some separation axioms on Γ-algebra space (gamma algebra space). The relationships between various separation axioms in Γ-algebra space are proved. In addition, the measurable function between two measurable spaces is introduced and some results are discussed.

Keywords: algebra, σ–field, σ–algebra, Γ-algebra, measurable function.

1. Introduction

Measure theory has shown to be very useful for its applications in analysis and probability theory. The theory of measurement is used in modeling the physical world and has been extensively studied [6, 7, 3, 4, 5, 2, 8].

Let Ω be a nonempty set. The collection of all subsets of a set Ω, denoted by $\mathcal{P}(\Omega)$, and it is called a power set of Ω. We assume that the complement of a set Ω is the empty set $\varnothing$.

The concept of ring was studied by [5], where a collection $\wp \subseteq \mathcal{P}(\Omega)$ is called ring if whenever $E,\ F \in \wp$, then $E \cup F \in \wp$ and $E - F \in \wp$, where $E \cup F$ denotes the union of E and F, and $E - F$ denotes the difference of E and F.

[6] studied the concept of σ–field, where significant results have been demonstrated in the measure theory. A collection $\wp \subseteq \mathcal{P}(\Omega)$ is called σ–field if and only if $\Omega \in \wp$ and $\wp$ is closed under countable union and complementation. A measurable space is defined as a pair $(\Omega, \wp)$ where Ω be a nonempty set and $\wp$ is σ–field of Ω. [1] introduced the concept of Γ-algebra (Γ-field) and studied its properties.

In this paper we define the notion of separation axioms on Γ-algebra space and then study the relationships between various separation axioms for Γ-algebra spaces. In addition, the measurable function between two measurable spaces is introduced and some results are discussed.

*Corresponding author

Email addresses: mohaimen.m.abbood35502@st.tu.edu.iq (Mohaimen M. Abbood), aalfayadh@yahoo.com (Ali Al-Fayadh), saba.nazar@alzaraa.edu.iq (Saba N. Al-Khafaji)

Received: September 2021 *Accepted:* October 2021

2. Basic Definitions and Results Preliminaries

In this section, we review basic definitions relative to the work. Various separation axioms such as M_0 , M_1 , M_2, M_3, and M_4 are defined on Γ-algebra space.

Definition 2.1. *[1] Let Ω be a nonempty set and $\wp \subseteq \mathcal{P}(\Omega)$. A nonempty collection $\wp$ of subsets of a set Ω is called Γ-algebra or (Γ-field) if the following conditions are satisfied:*

1. $\varnothing, \Omega \in \wp$.
2. *If $D \in \wp$ and there exist $\varnothing \neq E_i \subset D \subset \Omega$ then at least one of $E_i's \in \wp$.*
3. *If $D_1, D_2, \ldots \in \wp$, then $\bigcup_{i=1}^{\infty} D_i \in \wp$.*

Example 2.2. *Let $\Omega = \{a, b, c\}$. Let $\wp = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \Omega\}$. Then $(\Omega, \wp)$ is a Γ-algebra of a set Ω.*

Definition 2.3. *[1] If $\wp$ is a Γ-algebra on a set Ω. A pair $(\Omega, \wp)$ is called measurable space relative to the Γ-algebra $\wp$ and the elements of $\wp$ are called measurable sets.*

We will denote to the measurable set by μ-measurable set.

Definition 2.4. *A Γ-algebra on a set Ω is said to be discrete-Γ-algebra provided, if $A \subseteq \Omega$, then A is a μ-measurable set.*

Definition 2.5. *Let $\wp$ be a Γ-algebra on a set Ω. The complement of a μ-measurable set A is $\Omega - A$ and denoted by μ^*-measurable set.*

In the following we define various separation axioms on Γ-algebra spaces and study the relationships between them.

Definition 2.6. *A Γ-algebra space $(\Omega, \wp)$ is said to be M_0 if for any two distinct points of a Γ-algebra space $(\Omega, \wp)$, at least one of them has μ-measurable set which does not contain the other point.*

Example 2.7. *Let $\Omega = \{a, b, c\}$ is a Γ-algebra on a set Ω. Define the collection $\wp$ as follows;*
$\wp = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \Omega\}$. Then $(\Omega, \wp)$ is an M_0 Γ-algebra space.

Example 2.8. *Let $\Omega = \{a, b, c\}$ is a Γ-algebra on a set Ω. Define the collection $\wp$ as follows;*
$\wp = \{\varnothing, \{a\}, \Omega\}$. Then $(\Omega, \wp)$ is Not an M_0. Note that $b \neq c$ but there does not exist μ-measurable set which contains b and does not contain c or which contains c and does not contain b.

Definition 2.9. *A Γ-algebra space $(\Omega, \wp)$ is said to be M_1 if for any two distinct points of a Γ-algebra space $(\Omega, \wp)$, each has μ-measurable set not containing the other point.*

Example 2.10. *Let $\Omega = \{a, b, c\}$ is a Γ-algebra on a set Ω. Define the collection $\wp$ and as follows;*
$\wp = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \Omega\}$. Then $(\Omega, \wp)$ is an M_1.

Example 2.11. *Let $\Omega = \{a, b, c\}$ is a Γ-algebra on a set Ω. Define the collection $\wp$ and as follows;*
$\wp = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \Omega\}$. Then $(\Omega, \wp)$ is a Not an M_1 space. Note that $b \neq c$ but there does not exist μ-measurable set which contains c and does not contain b.

Theorem 2.12. *If a* Γ*-algebra space* $(\Omega, \wp\ )$ *is an* M_1 *then it is an* M_0 *space.*
Proof . *Let* $(\Omega, \wp\ )$ *be an* M_1 *space. Suppose* $p \neq q \in \Omega$*, then by Definition 2.9, each of* p *and* q *has a* μ*-measurable set not containing the other point. Thus* $(\Omega, \wp\ )$ *be an* M_0 *space.* □

Remark 2.13. *The converse of Theorem 2.12 need Not be true as Example 2.11 shows.*

Theorem 2.14. *If a* Γ*-algebra space* $(\Omega, \wp\ )$ *is an* M_1 *and* $x \in \Omega$*, then* $\{x\}$ *is a* μ^**-measurable set.*
Proof . *Suppose* $(\Omega, \wp\ )$ *is an* M_1 *space, and* $x \in \Omega$*. Let* $p \in \Omega - \{x\}$*, then* $p \neq x$*. Since* $(\Omega, \wp\ )$ *is an* M_1 *space, there is a* μ*-measurable set* U_p *such that* $p \in U_p$ *and* $x \notin U_p$ *. Then* $p \in U_p \subset \Omega - \{x\}$*. Therefore, for each* $p \in \Omega$*, there exists a* μ*-measurable set* U_p *of* Ω *which contains* p *and is contained in the complement of* $\{x\}$*. It remains to show that* $\Omega - \{x\} = \cup\{U_p :\ p \in \Omega - \{x\}\}$*. Let* $y \in \Omega - \{x\}$*. there is a* μ*-measurable subsets* U_y *of* Ω *such that* $y \in U_y$ *which is contained in* $\Omega - \{x\}$*. Thus* $y \in \cup\{U_p :\ p \in \Omega - \{x\}\}$*, and hence* $\Omega - \{x\} \subset \cup\{U_p :\ p \in \Omega - \{x\}\}$*.*
Now, let $y \in \cup\{U_p :\ p \in \Omega - \{x\}\}$*. Then, there is a* U_p *such that* $y \in U_p \subset \Omega - \{x\}$*, and hence* $y \in \Omega - \{x\}$*. Therefore,* $\cup\{U_p :\ p \in \Omega - \{x\}\} \subset \Omega - \{x\}$*. Hence* $\Omega - \{x\} = \cup\{U_p :\ p \in \Omega - \{x\}\}$*. Since* U_p *is a* μ*-measurable set of* Ω*, then* $\cup\{U_p :\ p \in \Omega - \{x\}\}$ *is a* μ*-measurable subsets of* Ω*. This implies that* $\Omega - \{x\}$ *is a* μ*-measurable set of* Ω*. Then* $\{x\}$ *is a* μ^**-measurable set.* □

Definition 2.15. *A* Γ*-algebra space* $(\Omega, \wp\ )$ *is said to be* M_2*, if for any two distinct points of a* Γ*-algebra space* $(\Omega, \wp\ )$*, each has a* μ*-measurable set which does not intersect the other.*

Example 2.16. *Let* $\Omega = \{a, b, c, d\}$ *is a* Γ*-algebra on a set* Ω*. Define the collection* $\wp$ *as follows;*
$\wp = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{b, c\}, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}, \{a, b, c\}, \Omega\}$*.*
Then $(\Omega, \wp\ )$ *is an* M_2 *space.*

Example 2.17. *Let* $\Omega = \{a, b, c, d\}$ *is a* Γ*-algebra on a set* Ω*. Define the collection* $\wp$ *as follows;*
$\wp = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \Omega\}$*. Then* $(\Omega, \wp\ )$ *is Not an* M_2 *space. Note that* $a \neq d$ *but there are no two disjoint* μ*-measurable sets* U *and* V *such that* $a \in U$ *and* $d \in V$*.*

Theorem 2.18. *If a* Γ*-algebra space* $(\Omega, \wp\ )$ *is a* M_2*, then it is an* M_1 *space.*
Proof . *Suppose* $(\Omega, \wp\ )$ *is an* M_2 *space and* $p \neq q \in \Omega$ *. Since* $(\Omega, \wp)$ *is an* M_2 *space, there are two disjoint* μ*-measurable sets* U *and* V *such that* $p \in U$ *and* $q \in V$*. But* $U \cap V = \varnothing$ *implies that* $q \notin U$*. Thus there is a* μ*-measurable set* U *such that* $p \in U$ *and* $q \notin U$*. Thus* $(\Omega, \wp\ )$ *is an* M_1 *space.* □

Remark 2.19. *The converse of Theorem 2.18 need Not be true as the following example shows.*

Example 2.20. *Let* $\Omega = \{a, b, c, d\}$ *is a* Γ*-algebra on a set* Ω*. Define the collection* $\wp$ *as follows;*
$\wp = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}, \{a, b, c\}, \Omega\}$*.*
Then $(\Omega, \wp\ )$ *is an* M_1 *but it is Not an* M_2 *space. Note that* $c \neq d$ *but there are no two disjoint* μ*-measurable sets* U *and* V *such that* $c \in U$ *and* $d \in V$*.*

Definition 2.21. *A* Γ*-algebra space* $(\Omega, \wp\ )$ *is said to be* M_3*, if* $(\Omega, \wp\ )$ *is an* M_1 *space and such that if* $p \in \Omega$ *and* F *is a* μ^**-measurable set with* $p \notin F$*, then there exists two disjoint* μ*-measurable sets* U *and* V *with* $p \in U$ *and* $F \subseteq V$*.*

Note that, a Γ-algebra space $(\Omega, \wp\ )$ in Example 2.16 is an M_3 space, while a Γ-algebra space $(\Omega, \wp\ )$ in Example 2.17 is Not an M_3 space since $\{b, c, d\}$ is a μ^*-measurable set with $a \notin \{b, c, d\}$ but there are no two disjoint μ-measurable sets U and V such that $a \in U$ and $\{b, c, d\} \subseteq V$.

Theorem 2.22. *If a Γ-algebra space $(\Omega, \wp)$ is an M_3, then it is an M_2 space.*
Proof . *Suppose $(\Omega, \wp)$ is an M_3, and $p \neq q \in \Omega$. Since $(\Omega, \wp)$ is an M_3 space, then $(\Omega, \wp)$ is an M_1 space. Thus by Theorem 2.14, $\{p\}$ is a μ^*-measurable set. Therefore, there are two disjoint μ-measurable sets U and V such that $q \in U$ and $\{p\} \subseteq V$. Since $\{p\} \subseteq V$, $p \in V$. So $q \in U$, $p \in V$ and $U \cap V = \varnothing$. Therefore, $(\Omega, \wp)$ is an M_2 space.* $\square$

Remark 2.23. *The converse of Theorem 2.22 need Not be true as the following example shows.*

Example 2.24. *Let $\Omega = \{1, 2, 3, 4, 5\}$ is a Γ-algebra on a set Ω. Define the collection $\wp$ as follows;*

$$\wp = \left\{ \begin{array}{c} \varnothing, \{1\}, \{3\}, \{5\}, \{1,3\}, \{1,5\}, \{3,5\}, \{1,2\}, \{2,3\}, \{4,5\}, \{1,2,5\}, \{1,2,3\}, \{2,3,5\}, \\ \{3,4,5\}, \{1,3,5\}, \{1,4,5\}, \{2,3,4\}, \{1,2,3,4\}, \{2,3,4,5\}, \{1,2,4,5\}, \{1,2,3,5\}, \\ \{1,3,4,5\}, \Omega \end{array} \right\}$$

Then $(\Omega, \wp)$ is an M_2. The set of all μ^*-measurable sets is,

$$\left\{ \begin{array}{c} \varnothing, \{2,3,4,5\}, \{1,2,4,5\}, \{1,2,3,4\}, \{2,4,5\}, \{2,3,4\}, \{1,2,4\}, \{3,4,5\}, \{1,4,5\}, \{1,2,3\}, \\ \{3,4\}, \{4,5\}, \{1,4\}, \{1,2\}, \{2,4\}, \{2,3\}, \{1,5\}, \{5\}, \{1\}, \{3\}, \{4\}, \{2\}, \{1,5\}, \Omega \end{array} \right\}$$

Then $(\Omega, \wp)$ is an M_2 but it is Not an M_3 since $\{1, 5\}$ is a μ^*-measurable set with $4 \notin \{1, 5\}$ but there are no two disjoint μ-measurable sets U and V such that $4 \in U$ and $\{1, 5\} \subseteq V$.

Definition 2.25. *A Γ-algebra space $(\Omega, \wp)$ is said to be M_4, if $(\Omega, \wp)$ is an M_1 space and such that for any two disjoint μ^*-measurable sets F and G, then there exists two disjoint μ-measurable sets U and V with $F \subseteq U$ and $G \subseteq V$.*

The following example is an M_4 space.

Example 2.26. *Let $\Omega = \{1, 2, 3, 4\}$ is a Γ-algebra on a set Ω. Define the collection $\wp$ as follows;*
$\wp = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,4\}, \{3,4\}, \{1,2,4\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}, \Omega\}$.
The set of all μ^*-measurable sets is,
$\{\Omega, \{2,3,4\}, \{1,3,4\}, \{1,2,4\}, \{3,4\}, \{2,4\}, \{1,4\}, \{2,3\}, \{1,2\}, \{3\}, \{4\}, \{2\}, \{1\}, \varnothing\}$.
Then $(\Omega, \wp)$ is an M_4 space.

Example 2.27. *Let $\Omega = \{1, 2, 3, 4\}$ is a Γ-algebra on a set Ω. Define the collection $\wp$ as follows;*
$\wp = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}, \Omega\}$. *The set of all μ^*-measurable sets is,*
$\{\Omega, \{2,3,4\}, \{1,3,4\}, \{1,2,4\}, \{3,4\}, \{2,4\}, \{1,4\}, \{1,2\}, \{4\}, \{2\}, \{1\}, \varnothing\}$.
Then $(\Omega, \wp)$ is Not an M_4 space because it is not an M_1.

Example 2.28. *Let $\Omega = \{1, 2, 3, 4\}$ is a Γ-algebra on a set Ω. Define the collection $\wp$ as follows;*
$\wp = \{\varnothing, \{1\}, \{2\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \{2,4\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \Omega\}$.
Then $(\Omega, \wp)$ is an M_1. The set of all μ^*-measurable sets is,
$\{\varnothing, \{2,3,4\}, \{1,3,4\}, \{3,4\}, \{2,4\}, \{1,4\}, \{4\}, \{1,3\}, \{3\}, \{2\}, \{1\}, \Omega\}$.
Then $(\Omega, \wp)$ is Not an M_4 space since $\{1, 4\}$ and $\{3\}$ are two disjoint μ^*-measurable sets but there are no two disjoint μ-measurable sets U and V such that $\{3\} \subseteq U$ and $\{1, 4\} \subseteq V$.

Theorem 2.29. *If a Γ-algebra space $(\Omega, \wp)$ is an M_4, then it is an M_3 space.*
Proof . *Suppose $(\Omega, \wp)$ is an M_4, and $a \in \Omega$. Let G be a μ^*-measurable set such that $a \notin G$. Since $(\Omega, \wp)$ is an M_4 space, then $(\Omega, \wp)$ is an M_1 space, and $\{a\}$ is a μ^*-measurable set by Theorem 2.14. Then there exist two disjoint μ-measurable sets U and V such that $\{a\} \subseteq U$ and $G \subseteq V$. But $\{a\} \subseteq U$ implies that $a \in U$. Thus $a \in U$ and $G \subseteq V$ and $U \cap V = \varnothing$. Hence by Definition 2.21, $(\Omega, \wp)$ is a M_3 space.* $\square$

3. Measurable Functions on Γ-algebra space

Definition 3.1. *Let (X, P) and (Y,W) are two Γ-algebra spaces and $f : X \to Y$ be a function. Then f is said to be a measurable function provided, if U is a μ-measurable subset of Y, then $f^{-1}(U)$ is a μ-measurable subset of X.*

Example 3.2. *Let $X = \{a, b, c\}$. Define the collection P on X as follows;*
$P = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Let $Y = \{r, s, t\}$. Define the collection W on Y as follows; $W = \{\varnothing, Y, \{r\}, \{r, s\}\}$. Define $f : X \to Y$ as the following; $f(a) = r; f(b) = s; f(c) = t$. Then f is a measurable function.
Proof . *It is clear that (X, P) and (Y,W) are two Γ-algebra spaces. The set of all μ-measurable subsets of X is $\{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$, and the set of all μ-measurable subsets of Y is $\{\varnothing, Y, \{r\}, \{r, s\}\}$. Then; $f^{-1}(\varnothing) = \varnothing; f^{-1}(Y) = X;\ f^{-1}(\{r\}) = \{a\};\ f^{-1}(\{r, s\}) = \{a, b\}$.*
Note that for every μ-measurable subset U of Y, then $f^{-1}(U)$ is a μ-measurable subset of X. Therefore f is a measurable function. □

Example 3.3. *Let $X = \{a, b, c\}$. Define the two collection P on X as follows;*
$P = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Let $Y = \{r, s, t\}$. Define the collection W on Y as follows;
$W = \{\varnothing, Y, \{r\}, \{r, s\}\}$. Define $g : Y \to X$ as the following;
$g(r) = b; g(s) = a; g(t) = c$. Then g is Not a measurable function.
Proof . *By definition of inverse; $g^{-1}(\varnothing) = \varnothing; g^{-1}(X) = Y;\ g^{-1}(\{a\}) = \{s\};\ g^{-1}(\{b\}) = r; g^{-1}(\{a, b\}) = \{r, s\}; g^{-1}(\{b, c\}) = \{r, t\}$.*
Note that $\{b, c\}$ is a μ-measurable subset of X but $g^{-1}(\{b, c\}) = \{r, t\}$ is not a μ-measurable subset of Y. Therefore g is Not a measurable function. □

Definition 3.4. *Let (X, P) and (Y,W) are two Γ-algebra spaces. Let $f : X \to Y$ be a function and $x \in X$. Then f is said to be a measurable function at x provided, if given any μ-measurable subset V of Y, $f(x) \in V$, then there exists a μ-measurable subset U of X such that $x \in U$ and $f(U) \subset V$.*

Example 3.5. *Let (X, P) and (Y, W) be defined as in Example 3.3. Define $g : Y \to X$, by $g(r) = b;\ g(s) = a;\ g(t) = c$. Then g is a measurable function at a point and is Not a measurable function.*
Proof . *Since $\{r\}$ is μ-measurable subset of Y, and each μ-measurable subset of X which contains $g(r) = b$ also contains $g(\{r\}) = \{b\}$, then g is a measurable function at $r \in Y$. It was shown, however, that g is Not a measurable function.* □

Theorem 3.6. *If (X, P) and (Y, W) are two Γ–algebra spaces, then $f : X \to Y$ is a measurable if and only if f is a measurable function at each point of X.*
Proof . *Let (X, P) and (Y, W) are two Γ–algebra spaces and $f : X \to Y$.*
Suppose that f is a measurable function. Let $x \in X$. Let V be a μ-measurable subset of Y such that $f(x) \in V$. Since $f(x) \in V$, then $x \in f^{-1}(V)$. But f is a measurable function, therefore $f^{-1}(V)$ is a μ-measurable subset of X. Thus $x \in f^{-1}(V)$ and $f(f^{-1}(V)) \subset V$.
Therefore f is a measurable at x. Suppose that f is a measurable function at each point $x \in X$. Let U be a μ-measurable subset of Y. Let $x \in f^{-1}(U)$. Then $f(x) \in U$. By Definition 3.1, there is a μ-measurable subset $G_x \subset X$ such that $x \in G_x$, and $G_x \subset f^{-1}(U)$. Such a μ-measurable subset can be found for each x in X. It remains to show that $f^{-1}(U) = \cup\{G_x : x \in f^{-1}(U)\}$. Let $y \in f^{-1}(U)$. Then $f(y) \in U$. So $y \in G_y \subset \cup\{G_x : x \in f^{-1}(U)\}$. Thus $f^{-1}(U) \subset \cup\{G_x : x \in f^{-1}(U)\}$.

Let $y \in \cup\{G_x : x \in f^{-1}(U)\}$ *. Then there is an* $x \in X$ *such that* $y \in G_x \subset f^{-1}(U)$*. Therefore* $y \in f^{-1}(U)$ *. Hence* $\cup\{G_x : x \in f^{-1}(U)\} \subset f^{-1}(U)$*. So* $f^{-1}(U) = \cup\{G_x : x \in f^{-1}(U)\}$*. Since each* G_x *is a* μ*-measurable subset,* $f^{-1}(U)$*, which is a union of* μ*-measurable subsets, is a* μ*-measurable subset. Thus* f *is measurable.* □

Theorem 3.7. *If* (X, P) *and* (Y, W) *are two* $\Gamma-$*algebra spaces and* f*, mapping* X *into* Y*, is measurable, and* F *is a* μ^* *measurable subset of* Y*, then* $f^{-1}(F)$ *is a* μ^* *measurable subset of* X*.*

Proof . *Let* (X, P) *and* (Y, W) *are two* $\Gamma-$*algebra spaces. Let* f*, mapping* X *into* Y*, be measurable. Let* F *be a* μ^* *measurable subset of* Y*. Since* F *is a* μ^* *measurable subset, then* $Y - F$ *is a* μ*-measurable subset of* Y*. Thus* $f^{-1}(Y - F)$ *is a* μ*-measurable subset of* X*, and* $f^{-1}(Y - F) = f^{-1}(Y) - f^{-1}(F) = X - f^{-1}(F)$*.*

Since $X - f^{-1}(F) = f^{-1}(Y - F)$ *, then* $X - f^{-1}(F)$ *is a* μ*-measurable subset of* X*. But then* $f^{-1}(F) = X - (X - f^{-1}(F))$ *is a* μ^* *measurable subset of* X*.* □

Theorem 3.8. *If* (X, P) *and* (Y, W) *are two* $\Gamma-$*algebra spaces, and* $f : X \to Y$ *is such that, if* g *is a* μ^**-measurable subset of* Y*, then* $f^{-1}(g)$ *is a* μ^**-measurable subset of* X*, then* f *is a measurable function.*

Proof . *Let* (X, P) *and* (Y, W) *are two* $\Gamma-$*algebra spaces. Let* $f : X \to Y$*, be such that, if* g *is a* μ^**-measurable subset of* Y*, then* $f^{-1}(g)$ *is a* μ^**-measurable subset of* X*.*

Let U *be a measurable subset of Y. Then* $U = Y - (Y - U)$ *and* $(Y - U)$ *is a* μ^**-measurable subset of* Y*. So* $f^{-1}(U) = f^{-1}(Y - (Y - U)) = f^{-1}(Y) - f^{-1}(Y - U) = X - f^{-1}(Y - U)$*. Since* $(Y - U)$ *is a* μ^**-measurable subset of* Y*,* $f^{-1}(Y - U)$ *is a* μ^**-measurable subset of* X*. Then* $f^{-1}(U) = X - f^{-1}(Y - U)$*, which is a* μ*-measurable subset of* X*. Therefore* f *is a measurable.* □

4. Conclusions

In this study, we introduce the notion of separation axioms on Γ-algebra spaces (gamma algebra spaces), and then we investigate the relationships between them. We proved that each M_i axiom is M_{i-1} $(i = 1, 2, 3, 4)$ and the converse need not true. In addition, the concept of measurable function between two measurable spaces is introduced and some results are discussed.

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