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The bifurcation analysis of an eco-toxicant model with anti- predator behavior

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Abstract

In this study, the mathematical model of four differential equations for organisms that describe the effect of anti-predation behavior, age stage and toxicity have been analyzed. Local bifurcation and Hopf bifurcation have been studied by changing a parameter of a model to study the dynamic behavior determined by bifurcation curves and the occurrence states of bifurcation saddle node, transcritical and pitch fork bifurcation. The potential equilibrium point at which Hopf bifurcation occurs has been determined and the results of the bifurcation behavior analysis have been fully presented using numerical simulation.

Keywords: Prey-Predator, Local bifurcation, Global bifurcation, Hopf bifurcation.

1. Introduction

Several tuners have recently discussed the issue of stability of nonlinear control systems and its relationship to controllability. Nowadays, the prey - predator model is an important topic, it includes the study of some aspects in which there are different disciplines: ecology, biology, genetics, and other disciplines, among which is physics, for example [4, 5, 7].
A new practical way to distinguish between chaotic, periodic and quasi-cyclic cycles is presented to solve many problems in the environment that have been extensively studied in the past decades [8, 11]. It is either useful or has many potentials in fields such as engineering and power grids. In late differential equations, periodic solutions can be generated through Hopf bifurcation. To determine the nonlinear differential equations, the oscillatory solutions of the system and the steady state are considered, look [9, 10].

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Bifurcation theory in mathematics considers is a qualitative change that occurs in the behavior of a dynamic as a result of a change in one of its parameters [14, 1]. Physical example of this behavior is, when you press a piece of wood in the force with in the middle of it, and the force with which you press is considered the coeffient, so you see that the wood curves and changes its shape unit the force reaches a certain value called the bifurcation value then the behavior of the wood changes and it breaks. The point at which this behavior appears, the point of breaking the timber is called the bifurcation point this behavior is usually plotted in a diagram called a bifurcation diagram. Henri Poncare was the first to introduce the term bifurcation in (1885) [3]. As for method of calculating the location of this change in behavior, it is explained as follows.

There are many types of bifurcation, the most important (saddle-node, transcritical, pitchfork and hopf bifurcation) if the following differential equation $\dot{x} = f(x, \mu)$ had considerd the point at which qualitative change occurse in the behavior of this dynamic system or what is expressed mathematically in the term of bifurcation, see for example [15, 2, 12], it is first an equilibrium point and secondly the point at which the system layout becomes the (Jacobian matrix).

Local bifurcation analysis have been used with the help of the Sotomayor theorem [13] near the equilibrium points of the mathematical system (2.1), it consists of (first prey, second prey with age stages and only one predator) with toxicity and anti-predator. The hopf bifurcation effect of the positive equilibrium point has been studied.

2. Model formulation [6]

In this section, an ecological model consists of four species have been proposed : the first prey and second prey which have a stage- structure with only one predator , which are denoted to their populations sizes at time E_1(t), E_2(t), E_3(t) and E_4(t) respectively .

$$\begin{aligned}
\frac{\mathrm{d}E_1}{\mathrm{dt}} &= S_1E_1\left(1-\frac{E_1}{L_1}\right)-\frac{C_1E_1E_4}{m+E_1^2}\\
\frac{dE_2}{\mathrm{dt}} &= S_2E_3\left(1-\frac{E_3}{L_2}\right)-DE_2-C_2E_2E_4-\alpha_1{E_2}^2-K_1E_2,\\
\frac{dE_3}{\mathrm{dt}} &= DE_2-C_2E_3E_4-\alpha_2E_3^2-K_2E_3,\\
\frac{dE_4}{\mathrm{dt}} &= \frac{A_1E_1E_4}{m+E_1^2}+A_2E_2E_4+A_3E_3E_4-nE_1E_4-\alpha_3E_4-K_3E_4.
\end{aligned} \tag{2.1}$$

The positive parameters of system (2.1) can be described by the Table 1.

Table 1: The parameters of the system (2.1)

$S_i > 0, i = 1, 2.$	*The logistic growth rate of first prey and mature prey, respectively.*
$L_i > 0, i = 1, 2.$	*The carrying capacity of the first prey and the mature prey, respectively.*
$D > 0$	*The rate of transition of immature prey to mature prey.*
$C_1 > 0$	*The rate of predator attack on the first prey.*
$m > 0$	*The measuring extent to which the environment is provided to protect prey and predators.*
$C_i > 0, i = 2, 3.$	*The rate of predator attack on the second prey, mature and immature, respectively.*
$0 < A_i < 1,\ i = 1, 2, 3.$	*The conversion rates of food to a predator, respectively.*
$n > 0$	*The rate of anti-predator behavior of the first prey.*
$K_i > 0, i = 1, 2, 3.$	*The mortality rates of the second prey, mature and immature, and a predator, respectively.*
$\alpha_i > 0, i = 1, 2, 3.$	*The toxicity rates of the second prey, mature and immature and predatory, respectively.*

3. Local bifurcation analysis

In this section, the analysis of the local bifurcation of model (2.1) have been studied, focusing on the changes around each equilibrium point when the parameter values change in the dynamic

behavior. Our goal is to provide higher order conditions that ensure that the most common local bifurcations appear, with the help of Sotomayor's theorem.
Now, according to Jacobean matrix $J(E_1, E_2, E_3, E_4)$ of the system (2.1) which is given in [6] as follows:

$$J = [a_{ij}]_{4\times 4}$$
$$= \begin{bmatrix} \frac{S_1(L_1-2E_1)}{L_1} - \frac{C_1E_4\left(m-E_1^2\right)}{\left(m+E_1^2\right)^2} & 0 & 0 & -\frac{C_1E_1}{\left(m+E_1^2\right)} \\ 0, & -D-C_2E_4-2\alpha_1E_2-K_1 & \frac{S_2(L_2-2E_3)}{L_2} & -C_2E_2 \\ 0 & D & -C_3E_4-2\alpha_2E_3-K_2 & -C_3E_3 \\ \frac{A_1E_4\left(m-E_1^2\right)}{\left(m+E_1^2\right)^2} - nE_4 & A_2E_4 & A_3E_4 & \frac{A_1E_1}{\left(m+E_1^2\right)} + A_2E_2 + A_3E_3 - nE_1 - \alpha_3 - K_3 \end{bmatrix} \quad (3.1)$$

For any non- zero vector R $=(r_1, r_2, r_3, r_4)^T$:

$$\begin{aligned} &D^2F_\mu(X,\mu)(R,R) = [\alpha_{i1}]_{4\times 1}, \\ &\alpha_{11} = -2r_1\left(\frac{S_1}{L_1} + \frac{C_1E_1E_4(3E_1-m)}{(m+E_1^2)^3}r_1 + \frac{C_1(m-E_1^2)}{(m+E_1^2)^2}r_4\right), \\ &\alpha_{21} = -2\left(\alpha_1 r_2^2 + C_2r_4r_2 + \frac{S_2}{L_2}r_3^2\right), \\ &\alpha_{31} = -2r_3(\alpha_2r_3 + C_3r_4), \\ &\alpha_{41} = 2\left(\frac{A_1E_1E_4(3E_1-m)}{(m+E_1^2)^3}r_1^2 + \frac{A_1(m-E_1^2)}{(m+E_1^2)^2}r_4r_1 - nr_4r_1 + A_2r_4r_2 + A_3r_4r_3\right). \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} &D^3F_\mu(X,\mu)(R,R,R) = [\beta_{i1}]_{4\times 1} \\ &\beta_{11} = 2r_1\left(\frac{C_1E_1(3E_1^2-m)}{(m+E_1^2)^3}r_4r_1 - \frac{C_1E_4(14E_1^2m - m^2 - 9E_1^4)}{(m+E_1^2)^4}r_1^2\right), \\ &\beta_{21} = 0, \\ &\beta_{31} = 0, \\ &\beta_{41} = 2r_1\left(\frac{A_1E_4(14E_1^2m - m^2 - 9E_1^4)}{(m+E_1^2)^4}r_1^2 - \frac{A_1E_1(3E_1^2-m)}{(m+E_1^2)^3}r_4r_1\right). \end{aligned} \quad (3.3)$$

where $X = (E_1, E_2, E_3, E_4)$ and μ be any parameter.

Theorem 3.1. *System* (2.1) *with the parameter value* $\ddot{K}_2 = K_2 = \frac{S_2D}{(D+K_1)}$, *has a transcritical bifurcation at* $Q_1 = (L_1, 0, 0, 0)$.

Proof . *By the Jacobian matrix given in Eq. (4.3) in [6]* $\ddot{J}_1 = J_1\left(Q_1, \ddot{K}_2\right) = [\ddot{c}_{ij}]_{4\times 4}$, *where* $\ddot{c}_{ij} = c_{ij}$, *except* $\ddot{c}_{33} = -K_2$.

Then the characterise equation of $\ddot{J}_1$ *has a zero eigenvalue (say* λ_{1E_3}*)at* $\ddot{K}_2 = K_2$, *at the equilibrium point* Q_1.

Now, let $\ddot{R}^1 = (\ddot{r}_1^{[1]}, \ddot{r}_2^{[1]}, \ddot{r}_3^{[1]}, \ddot{r}_4^{[1]})^T$ *be the eigenvector corresponding to the eigenvalue* $\lambda_{1E_3} = 0$. *Thus,*$\left(\ddot{J}_1(Q_1) - \lambda_{1E_3}I\right)\ddot{R}^1 = 0$, *this gives:*

$$\ddot{r}_1^{[1]} = 0,\ \ddot{r}_2^{[1]} = \frac{S_2}{K_1+D}\ddot{r}_3^{[1]} \quad , \ddot{r}_4^{[1]} = 0$$

and $\ddot{r}_3^{[1]}$ *any non-zero real number .*

Let $\ddot{B}^{[1]} = (\ddot{b}_1^{[1]}, \ddot{b}_2^{[1]}, \ddot{b}_3^{[1]}, \ddot{b}_4^{[1]})^T$ *be the eigenvector associated with an eigenvalue* $\lambda_{1E_3} = 0$ *of the matrix* $\ddot{J}_1^{\,T}$*.Then,* $\left(\ddot{J}_1^{\,T}(Q_1) - \lambda_{1E_3} I\right) \ddot{B}^{[1]} = 0$

By solving this equation for , $\ddot{B}^{[1]} = \left(0, \frac{D}{D+K_1}\ddot{b}_3^{[1]}, \ddot{b}_3^{[1]}, 0\right)^T$ *, where* $\ddot{B}_3^{[1]}$ *any non-zero real number.*

Now, consider that:

$$\frac{\partial f}{\partial K_2} = f_{K_2}(X, K_2) = \left(\frac{\partial f_1}{\partial K_2}, \frac{\partial f_2}{\partial K_2}, \frac{\partial f_3}{\partial K_2}, \frac{\partial f_4}{\partial K_2}\right)^T = (0, , 0, -E_3, 0)^T .$$

So, $f_{K_2}\left(Q_1, \ddot{K}_2\right) = (0,0,0,0)^T$ *and hence* $\left(\ddot{B}^{[1]}\right)^T f_{K_2}\left(Q_1, \ddot{K}_2\right) = 0$

By using Sotomayor's theorem, the saddle- node bifurcation condition cannot be satisfied. Therefore, the first condition for transcritical bifurcation is satisfied. Now

$$Df_{K_2}(X, K_2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where, $Df_{K_2}(X, K_2)$ *represents the derivative of* $f_{K_2}(X, K_2)$ *with respect to* $X = (E_1, E_2, E_3, E_4)^T$ *. Furthermore, it is observed that:*

$$Df_{K_2}(Q_1, K_2)\,\ddot{R}^{[1]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{S_2}{K_1+D}\ddot{r}_3^{[1]} \\ \ddot{r}_3^{[1]} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\ddot{r}_3^{[1]} \\ 0 \end{bmatrix}$$

$$\left(\ddot{B}^{[1]}\right)^T \left[Df_{K_2}\left(Q_1, \ddot{K}_2\right)\ddot{R}^{[1]}\right] = \left(0, 0, -\ddot{r}_3^{[1]}, 0\right)\left(0, 0, \ddot{b}_3^{[1]}, 0\right)^T = \ddot{r}_3^{[1]}\ddot{b}_3^{[1]} \neq 0.$$

By substituting $\ddot{B}^{[1]}$ *in (1.3) we get:*

$$D^2F_\mu\left(Q_1, \ddot{K}_2\right)\left(\ddot{R}^{[1]}, \ddot{R}^{[1]}\right) = \begin{bmatrix} 0 \\ -2\left(\ddot{r}_3^{[1]}\right)^2 S_2\left(\frac{\alpha_1 S_2}{(D+K_1)^2} + \frac{1}{L_2}\right) \\ -2\alpha_2\left(\ddot{r}_3^{[1]}\right)^2 \\ 0 \end{bmatrix}$$

Hence, it was obtained

$$\left(\ddot{B}^{[1]}\right)^T \left[D^2F_\mu\left(Q_1, \ddot{K}_2\right)\left(\ddot{R}^{[1]}, \ddot{R}^{[1]}\right)\right] = -2\left(\ddot{r}_3^{[1]}\right)^2 \ddot{b}_3^{[1]}\left(\frac{\alpha_1 \ddot{K}_2^2 D}{(D+K_1)^3} + \frac{DS_2}{L_1(D+K_1)} + \alpha_2\right) \neq 0.$$

This means that system (2.1) has a transcritical bifurcation at Q_1 *with a parameter* $\ddot{K}_2 = K_2$*, and no pitch fork bifurcation can occurs at* $\ddot{K}_2 = K_2$ *.* □

Theorem 3.2. *Suppose that conditions (4.8b-4.8e) in [6] with the following conditions are satisfied:*

$$DS_2 > \left(C_3\overline{E}_4 + K_2\right)\left(D + C_4\overline{E}_4\right), \tag{3.4a}$$

$$14E_1^2 < m < \min\left\{3E_1, 3E_1^2\right\}, \tag{3.4b}$$

$$\overline{w}_1 \neq \overline{w}_2, \tag{3.4c}$$

$$\overline{w}_3 \neq \overline{w}_4, \tag{3.4d}$$

Then, system (2.1) with parameter value: $\overline{K}_1 = K_1 = \frac{DS_2 - \left(C_3\overline{E}_4 + K_2\right)(D + C_4\overline{E}_4)}{C_3\overline{E}_4 + K_2}$, *has a transcritical bifurcation at* Q_2.

Proof . *By the Jacobian matrix given in eq. (4.7) in [6]* $\overline{J}_2 = J_2\left(Q_2, \overline{K}_1\right) = [\overline{v}_{ij}]_{4\times 4}$, *where* $\overline{v}_{ij} = v_{ij}$ *, except* $\overline{v}_{22} = -D - C_2\overline{E}_4 - K_1$.

Then the characterise equation of $\overline{J}_1$ *has a zero eigenvalue (say*λ_{2E_2} *)at* $\overline{K}_1 = K_1$ *, at the equilibrium point* $Q_2 = (\overline{E}_1, 0, 0, \overline{E}_4)$.

Now, let $\overline{R}^{[2]} = \left(\overline{r}_1^{[2]}, \overline{r}_2^{[2]}, \overline{r}_3^{[2]}, \overline{r}_4^{[2]}\right)^T$ *be the eigenvector corresponding to the eigenvalue* $\lambda_{2E_2} = 0$. *Thus,* $\left(\overline{J}_2\left(Q_1\right) - \lambda_{2E_2} I\right)\overline{R}^{[2]} = 0$, *this gives:*

$$\overline{r}_1^{[2]} = I_1\overline{r}_2^{[2]},\ \overline{r}_3^{[2]} = I_2\overline{r}_2^{[2]}\ \ ,\overline{r}_4^{[2]} = I_3\overline{r}_2^{[2]}$$

where: $I_1 = -\frac{v_{14}}{v_{11}}I_3,\ I_2 = -\frac{v_{22}}{v_{23}},\ I_3 = -\frac{v_{11}(v_{42}+v_{43}I_2)}{v_{44}v_{11}-v_{41}v_{14}}$ *and* $\overline{r}_2^{[2]}$ *any non-zero real number.*

Let $\overline{B}^{[2]} = (\overline{b}_1^{[2]}, \overline{b}_2^{[2]}, \overline{b}_3^{[2]}, \overline{b}_4^{[2]})^T$ *be the eigenvector associated with an eigenvalue* $\lambda_{2E_2} = 0$ *of the matrix* $\overline{J}_2^{\,T}$*.Then,* $\left(\overline{J}_2^{\,T}\left(Q_2\right) - \lambda_{2E_2} I\right)\ \overline{B}^{[2]} = 0$.

By solving this equation for , $\overline{B}^{[2]} = \left(I_4\overline{b}_2^{[2]}, \overline{b}_2^{[2]}, I_5\overline{b}_2^{[2]}, I_6\overline{b}_2^{[2]}\right)^T$ *, where:*

$I_4 = -\frac{v_{14}}{v_{11}}I_5,\ \ I_5 = -\frac{v_{42}v_{33}-v_{32}v_{43}}{v_{43}v_{22}-v_{23}v_{42}},\ \ I_6 = -\frac{(v_{23}+v_{33}I_5)}{v_{43}}$ *and* $\overline{b}_2^{[2]}$ *any non-zero real number.*

Now, consider that:

$$\frac{\partial f}{\partial K_1} = f_{K_1}\left(X, K_1\right) = \left(\frac{\partial f_1}{\partial K_1}, \frac{\partial f_2}{\partial K_1}, \frac{\partial f_3}{\partial K_1}, \frac{\partial f_4}{\partial K_1}\right)^T = (0, -E_2, 0, 0)^T.$$

So, $f_{K_1}\left(Q_2, \overline{K}_1\right) = (0,0,0,0)^T$ *and hence* $\left(\overline{B}^{[2]}\right)^T f_{K_1}\left(Q_2, \overline{K}_1\right) = 0$.

By using Sotomayor's theorem, the saddle- node bifurcation condition cannot be satisfied. Therefore, the first condition for transcritical bifurcation is satisfied. Now

$$Df_{K_1}\left(X, K_1\right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where, $Df_{K_1}\left(X, K_1\right)$ *represents the derivative of* $f_{K_1}\left(X, K_1\right)$ *with respect to* $X = \left(E_1, E_2, E_3, E_4\right)^T$.

Furthermore, it is observed that:

$$DF_{K_1}(Q_2,K_1)\overline{R}^{[2]}=\begin{bmatrix}0&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}I_1\overline{r}_2^{[2]}\\\overline{r}_2^{[2]}\\I_2\overline{r}_2^{[2]}\\I_3\overline{r}_2^{[2]}\end{bmatrix}=\begin{bmatrix}0\\-\overline{r}_2^{[2]}\\0\\0\end{bmatrix}$$

$$\left(\overline{B}^{[2]}\right)^T\left[Df_{K_1}(Q_2,\overline{K}_1)\overline{R}^{[2]}\right]=\left(0,-\overline{r}_2^{[2]},0,0\right)\left(0,\overline{b}_2^{[2]},0,0\right)^T=-\overline{r}_2^{[2]}\overline{b}_2^{[2]}\neq 0$$

$$D^2F_\mu(Q_2,\overline{K}_1)\left(\overline{R}^{[2]},\overline{R}^{[2]}\right)=\begin{bmatrix}-2I_1\left(\overline{r}_2^{[2]}\right)^2\left(\frac{S_1}{L_1}+\frac{C_1E_1E_4(3E_1-m)}{(m+E_1^2)^3}+\frac{C_1(m-E_1^2)}{(m+E_1^2)^2}I_3\right)\\-2\left(\overline{r}_2^{[2]}\right)^2\left(\alpha_1+C_2I_3+\frac{S_2}{L_2}I_2\right)\\-2I_2\left(\overline{r}_2^{[2]}\right)^2(\alpha_2I_2+C_3I_3)\\2\left(\overline{r}_2^{[2]}\right)^2\left(I_1I_3\left(\frac{A_1(m-E_1^2)}{(m+E_1^2)^2}-n\right)+I_1^2\frac{A_1E_1E_4(3E_1-m)}{(m+E_1^2)^3}+I_3(A_2+A_3I_2)\right)\end{bmatrix},$$

Hence, it was obtained by conditions (4.8b-4.8d) in [6] and ((3.4b)-(3.4c)).

$$\left(\overline{B}^{[2]}\right)^T\left[D^2F_\mu(Q_2,\overline{K}_1)\left(\overline{R}^{[2]},\overline{R}^{[2]}\right)\right]=2\overline{b}_2^{[2]}\left(\overline{r}_2^{[2]}\right)^2(\overline{w}_1-\overline{w}_2)\neq 0.$$

$$\overline{w}_1=-I_1I_4\left(\frac{S_1}{L_1}+\frac{C_1E_1E_4(3E_1-m)}{(m+E_1^2)^3}+\frac{C_1(m-E_1^2)}{(m+E_1^2)^2}I_3\right)-I_2I_5(\alpha_2I_2+C_3I_3)$$
$$+I_1I_3I_6\left(\frac{A_1(m-E_1^2)}{(m+E_1^2)^2}-n\right),$$
$$\overline{w}_2=\left(\alpha_1+C_2I_3+\frac{S_2}{L_2}I_2\right)-I_6\left(I_1^2\frac{A_1E_1E_4(3E_1-m)}{(m+E_1^2)^3}+I_3(A_2+A_3I_2)\right).$$

This means that system (2.1) *has a transcritical bifurcation at Q_2 with a parameter $\overline{K}_1=K_1$. If condition* (3.4c) *not satisfied then. By substituting $\widetilde{R}^{[2]}$ in* (3.4a) *we get:*

$$D^3F_\mu(Q_2,\overline{K}_1)\left(\overline{R}^{[2]},\overline{R}^{[2]},\overline{R}^{[2]}\right)=\begin{bmatrix}2I_1^2\left(\overline{r}_2^{[2]}\right)^3\left(\frac{C_1E_1(3E_1^2-m)}{(m+E_1^2)^3}I_3-\frac{C_1E_4(m(14E_1^2-m)-9E_1^4)}{(m+E_1^2)^4}I_1\right)\\0\\0\\2I_1^2\left(\overline{r}_2^{[2]}\right)^3\left(\frac{A_1E_4(m(14E_1^2-m)-9E_1^4)}{(m+E_1^2)^4}I_1-\frac{A_1E_1(3E_1^2-m)}{(m+E_1^2)^3}I_3\right)\end{bmatrix},$$

Hence, it was obtained by conditions (4.8b-4.8d) in [6], (3.4b) *and* (3.4d).

$$\left(\overline{B}^{[2]}\right)^T\left[D^3F_\mu(Q_2,\overline{K}_1)\left(\overline{R}^{[2]},\overline{R}^{[2]},\overline{R}^{[2]}\right)\right]=2I_1^2\left(\overline{r}_2^{[2]}\right)^3\overline{b}_2^{[2]}(\overline{w}_3-\overline{w}_4)\neq 0$$
$$\overline{w}_3=\frac{I_3E_1(3E_1^2-m)}{(m+E_1^2)^3}(C_1I_4-A_1I_6),$$
$$\overline{w}_4=\frac{I_1E_4(m(14E_1^2-m)-9E_1^2)}{(m+E_1^2)^4}(C_1I_6-A_1I_4).$$

So, there is pitch fork bifurcation at Q_2 where $\overline{K}_1=K_1$. □

Theorem 3.3. *Suppose that the following conditions are satisfied:*

$$L_2 > 2\dot{E}_3, \tag{3.5a}$$

$$1 \neq \frac{\dot{E}_3}{L_2}. \tag{3.5b}$$

Then, system (2.1) *with parameter value:* $\dot{S}_2 = S_2 = \frac{(D+2\alpha_1\dot{E}_2+K_1)(2\alpha_2\dot{E}_3+K_2)L_2}{D(L_2-2\dot{E}_3)}$, *has a saddle-node bifurcation at* Q_5.

Proof . *By the Jacobian matrix given in eq. (4.10) in [6]* $\dot{J}_5 = J_5\left(Q_5, \dot{S}_2\right) = [\dot{u}_{ij}]_{4\times 4}$, *where* $\dot{u}_{ij} = u_{ij}$, *except* $\dot{u}_{23} = \frac{S_2(L_2-2\dot{E}_3)}{L_2}$. *Then the characterise equation of* $\dot{J}_5$ *a zero eigenvalue (say*λ_{5E_2} *)at* $\dot{S}_2 = S_2$, *at the equilibrium point* Q_5.

Now, let $\dot{R}^{[5]} = (\dot{r}_1^{[5]}, \dot{r}_2^{[5]}, \dot{r}_3^{[5]}, \dot{r}_4^{[5]})^T$ *be the eigenvector corresponding to the eigenvalue* $\lambda_{5E_2} = 0$. *Thus,* $\left(\dot{J}_5(Q_5) - \lambda_{5E_2} I\right)\dot{R}^{[5]} = 0$, *this gives:*

$$\dot{r}_1^{[5]} = 0,\ \dot{r}_3^{[5]} = Y_1\dot{r}_2^{[5]}\ \ ,\dot{r}_4^{[5]} = 0$$

where: $Y_1 = -\frac{u_{32}}{u_{33}}$, *and* $\dot{r}_2^{[5]}$ *any non-zero real number.*

Let $\dot{B}^{[5]} = (\dot{b}_1^{[5]}, \dot{b}_2^{[5]}, \dot{b}_3^{[5]}, \dot{b}_4^{[5]})^T$ *be the eigenvector associated with an eigenvalue* $\lambda_{5E_2} = 0$ *of the matrix* $\dot{J}_5{}^T$*.Then,* $\left(\dot{J}_5{}^T(Q_5) - \lambda_{5E_2} I\right)\ \dot{B}^{[5]} = 0$.

By solving this equation for , $\dot{B}^{[5]} = \left(0, \dot{b}_2^{[5]}, Y_2\dot{b}_2^{[5]}, Y_3\dot{b}_3^{[2]}\right)^T$, *where:* $Y_2 = -\frac{u_{22}}{u_{32}}, Y_3 = -\frac{(u_{24}+u_{34}Y_2)}{u_{44}}$. *and* $\dot{b}_2^{[5]}$ *any non-zero real number.*

Now, consider that:

$$\frac{\partial f}{\partial S_2} = f_{S_2}(X, S_2) = \left(\frac{\partial f_1}{\partial S_2}, \frac{\partial f_2}{\partial S_2}, \frac{\partial f_3}{\partial S_2}, \frac{\partial f_4}{\partial S_2}\right)^T = \left(0, E_3(1-\frac{E_3}{L_2}), 0, 0\right)^T.$$

So, $f_{S_2}\left(Q_5, \dot{S}_2\right) = \left(0, \dot{E}_3\left(1-\frac{\dot{E}_3}{L_2}\right), 0, 0\right)^T$, *and hence by condition* (3.5b).

$$\left(\dot{B}^{[5]}\right)^T f_{S_2}\left(Q_5, \dot{S}_2\right) = \dot{E}_3(1-\frac{\dot{E}_3}{L_2})\dot{b}_2^{[5]} \neq 0.$$

By using Sotomayor's theorem, the transcritical bifurcation condition cannot be satisfied. Therefore, the first condition for saddle- node bifurcation is satisfied.

Now By substituting $\dot{R}^{[5]}$ *in* (3.2) *we get:*

$$D^2F_\mu\left(Q_5, \dot{S}_2\right)\left(\dot{R}^{[5]}, \dot{R}^{[5]}\right) = \begin{bmatrix} 0 \\ -2\left(\dot{r}_2^{[5]}\right)^2\left(\alpha_1 + \frac{S_2}{L_2}Y_1^2\right) \\ -2\alpha_2 Y_1^2\left(\dot{r}_2^{[5]}\right)^2 \\ 0 \end{bmatrix},$$

Hence, it was obtained

$$\left(\dot{B}^{[5]}\right)^T\left[D^2F_\mu\left(Q_5, \dot{S}_2\right)\left(\dot{R}^{[5]}, \dot{R}^{[5]}\right)\right] = -2\left(\dot{r}_2^{[5]}\right)^2\dot{b}_2^{[5]}\left(\alpha_1 + \frac{\dot{S}_2}{L_2}Y_1^2 + \alpha_2 Y_1^2 Y_2\right) \neq 0.$$

So, system (2.1) *has asaddle- node bifurcation at* $\dot{S}_2 = S_2$.
But, opposite of condition (3.5b) *imply*

$$\left(\dot{B}^{[5]}\right)^T f_{S_2}\left(Q_5, \dot{S}_2\right) = \dot{E}_3(1 - \frac{\dot{E}_3}{L_2})\dot{b}_2^{[5]} \neq 0.$$

So,

$$Df_{S_2}(X, S_2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Where, $Df_{S_2}(X, S_2)$ *represents the derivative of* $f_{S_2}(X, S_2)$ *with respect to* $X = (E_1, E_2, E_3, E_4)^T$. *Furthermore, it is observed that:*

$$Df_{S_2}(Q_5, S_2)\dot{R}^{[5]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{r}_2^{[5]} \\ Y_1\dot{r}_2^{[5]} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\dot{B}^{[5]}\right)^T \left[Df_{S_2}\left(Q_5, \dot{S}_2\right)\dot{R}^{[5]}\right] = (0,0,0,0)\left(0, \dot{b}_2^{[5]}, 0, 0\right)^T = 0.$$

This means that system (2.1) *has no transcritical and pitch fork bifurcation at* Q_5 *with a parameter* $\dot{S}_2 = S_2$. □

Theorem 3.4. *Suppose that conditions (4.13b), (4.13e) and (4.13f) in [6] with the following conditions are satisfied:*

$$A_2\bar{\bar{E}}_2 + A_3\bar{\bar{E}}_3 > \alpha_3, \tag{3.6a}$$

$$A_2\delta_1 - \delta_4\left(\alpha_1\delta_1^2 + C_1\delta_1 + \frac{S_2}{L_2}\delta_2^2\right) - \alpha_2\delta_2^2\delta_5 \neq C_3\delta_2\delta_5 - A_3\delta_2 \tag{3.6b}$$

Then, system (2.1) *with parameter value* $\bar{\bar{K}}_3 = K_3 = \frac{\bar{\bar{w}}_1}{(e_{22}e_{33} - e_{23}e_{32})}$ *where:*

$$\bar{\bar{w}}_1 = -\left[(e_{23})(e_{34})(e_{42}) + (e_{32})(e_{24})(e_{43})\right] + (e_{22})(e_{34})(e_{43}) + \left(A_2\bar{\bar{E}}_2 + A_3\bar{\bar{E}}_3 - \alpha_3\right) + (e_{24})(e_{42})\left[(e_{33}) - (e_{32})\right],$$

has a saddle-node bifurcation at $Q_6 = \left(0, \bar{\bar{E}}_2, \bar{\bar{E}}_3, \bar{\bar{E}}_4\right)$.

Proof . *By the Jacobian matrix given in eq. (4.12) in [6]* $\bar{\bar{J}}_6 = J_6\left(Q_6, \bar{\bar{K}}_3\right) = [\bar{\bar{e}}_{ij}]_{4\times4}$, *where* $\bar{\bar{e}}_{ij} = e_{ij}$, *except* $\bar{\bar{e}}_{44} = A_2\bar{\bar{E}}_2 + A_3\bar{\bar{E}}_3 - \bar{\bar{K}}_3 - \alpha_3$.
Then the characteristic equation $\bar{\bar{J}}_6$ *having zero eigenvalue (say*$\lambda_{6E_4} = 0$*)if and if* $\bar{\bar{B}}_3 = 0$ *and, thus* Q_6 *is a non-hyperbolic equilibrium point.*
Note that, $\bar{\bar{K}}_3 > 0$ *provided that condition (4.13e-4.13f) in [6] and* (3.6a).
Now, let $\bar{\bar{R}}^{[6]} = (\bar{\bar{r}}_1^{[6]}, \bar{\bar{r}}_2^{[6]}, \bar{\bar{r}}_3^{[6]}, \bar{\bar{r}}_4^{[6]})^T$ *be the eigenvector corresponding to the eigenvalue* $\lambda_{6E_4} = 0$. *Thus,*$\left(\bar{\bar{J}}_6(Q_6) - \lambda_{6E_4}I\right)\bar{\bar{R}}^{[6]} = 0$, *this gives:*

$$\bar{\bar{r}}_1^{[6]} = 0,\ \bar{\bar{r}}_2^{[6]} = \delta_1\bar{\bar{r}}_4^{[6]}\quad, \bar{\bar{r}}_3^{[6]} = \delta_2\bar{\bar{r}}_4^{[6]}$$

where: $\delta_1 = -\frac{e_{43}}{e_{42}}\delta_2,\ \ \delta_2 = \frac{e_{24}e_{42}}{e_{22}e_{34}-e_{23}e_{42}}$, *and* $\bar{\bar{r}}_4^{[6]}$ *any non-zero real number.*
Let $\bar{\bar{B}}^{[6]} = (\bar{\bar{b}}_1^{[6]}, \bar{\bar{b}}_2^{[6]}, \bar{\bar{b}}_3^{[6]}, \bar{\bar{b}}_4^{[6]})^T$ *be the eigenvector associated with an eigenvalue* $\lambda_{6E_4} = 0$ *of the matrix* $\bar{\bar{J}}_6^{\,T}$*. Then,* $\left(\bar{\bar{J}}_6^{\,T}(Q_6) - \ \lambda_{6E_4} I\right)\ \bar{\bar{B}}^{[6]} = 0.$
By solving this equation for $\bar{\bar{B}}^{[6]} = \left(\delta_3\bar{\bar{b}}_4^{[6]}, \delta_4\bar{\bar{b}}_4^{[6]}, \delta_5\bar{\bar{b}}_4^{[6]}, \bar{\bar{b}}_4^{[6]}\right)^T$, *where:* $\delta_3 = -\frac{e_{41}}{e_{11}},\ \delta_4 = -\frac{e_{41}}{e_{11}}\ \delta_5,\ \delta_5 = \frac{e_{24}e_{42}}{e_{22}e_{34}-e_{24}e_{32}}$, *and* $\bar{\bar{b}}_4^{[6]}$ *any non-zero real number.*
Now, consider that:

$$\frac{\partial f}{\partial K_3} = f_{K_3}(X, k_3) = \left(\frac{\partial f_1}{\partial K_3}, \frac{\partial f_2}{\partial K_3}, \frac{\partial f_3}{\partial K_3}, \frac{\partial f_4}{\partial K_3}\right)^T = (0,0,0,-E_4)^T.$$

So, $f_{K_3}\left(Q_6, \bar{\bar{K}}_3\right) = \left(0,0,0,-\bar{\bar{E}}_4\right)^T$ *and hence* $\left(\bar{\bar{B}}^{[6]}\right)^T f_{K_3}\left(Q_6, \bar{\bar{K}}_3\right) = -\bar{\bar{E}}_4\bar{\bar{b}}_4^{[6]} \neq 0.$
By using Sotomayor's theorem, the transcritical bifurcation condition cannot be satisfied. Therefore, the first condition for saddle- node bifurcation is satisfied. Now By substituting $\bar{\bar{R}}^{[6]}$ *in* (3.2) *we get:*

$$D^2F_\mu\left(Q_6, \bar{\bar{K}}_3\right)\left(\bar{\bar{R}}^{[6]}, \bar{\bar{R}}^{[6]}\right) = \begin{bmatrix} 0 \\ -2\left(\bar{\bar{r}}_4^{[6]}\right)^2\left(\alpha_1\delta_1^2 + C_1\delta_1 + \frac{S_2}{L_2}\delta_2^2\right) \\ -2\delta_2\left(\bar{\bar{r}}_4^{[6]}\right)^2(\alpha_2\delta_2 + C_3) \\ 2\left(\bar{\bar{r}}_4^{[6]}\right)^2(A_2\delta_1 + A_3\delta_2) \end{bmatrix},$$

hence, it was obtained by conditions (4.13b),(4.13e) in [6] and (3.6b)

$$\left(\bar{\bar{B}}^{[6]}\right)^T\left[D^2F_\mu\left(Q_6, \bar{\bar{K}}_3\right)\left(\bar{\bar{R}}^{[6]}, \bar{\bar{R}}^{[6]}\right)\right] = 2\left(\bar{\bar{r}}_4^{[6]}\right)^2\bar{\bar{b}}_4^{[6]}\left(A_2\delta_1 - \delta_4\left(\alpha_1\delta_1^2 + C_1\delta_1 + \frac{S_2}{L_2}\delta_2^2\right) - \alpha_2\delta_2^2\delta_5 \right.$$
$$\left. - C_3\delta_2\delta_5 + A_3\delta_2\right) \neq 0.$$

This means that system (2.1) *has a saddle-node bifurcation at* Q_6 *with a parameter* $\bar{\bar{K}}_3 = K_3$*, and no pitch fork bifurcation at* Q_6 *where* $\bar{\bar{K}}_3 = K_3$*.* □

Theorem 3.5. *Suppose that conditions (4.15b), (4.15e) in [6] with the following conditions are satisfied:*

$$\left(\frac{A_1}{\left(m + \widetilde{E}_1^2\right)} - n\right)\widetilde{E}_1 + A_2\widetilde{E}_2 + A_3\widetilde{E}_3 > K_3, \tag{3.7a}$$

$$\frac{S_1\left(L_1 - 2\widetilde{E}_1\right)}{L_1} > \frac{C_1\widetilde{E}_4\left(m - \widetilde{E}_1^2\right)}{\left(m + \widetilde{E}_1^2\right)}, \tag{3.7b}$$

$$L_2 > \max\{2\widetilde{E}_1, 2\widetilde{E}_3\}, \tag{3.7c}$$

$$m < 3E_1 \tag{3.7d}$$

$$\widetilde{w}_2 \neq \widetilde{w}_3, \tag{3.7e}$$

where:

$$\widetilde{w}_2 = -t_1 t_4 \left(\frac{S_1 t_1}{L_1} + \frac{t_1 C_1 E_1 E_4 (3E_1 - m)}{(m + E_1^2)^3} + \frac{C_1 (m - E_1^2)}{(m + E_1^2)^2} \right) + t_1 \left(\frac{A_1 (m - E_1^2)}{(m + E_1^2)^2} - n \right)$$
$$+ t_1^2 \frac{A_1 E_1 E_4 (3E_1 - m)}{(m + E_1^2)^3} - t_5 \left(\alpha_1 t_2^2 + \frac{S_2}{L_2} t_3^2 \right) - \alpha_1 t_3^2 t_6,$$
$$\widetilde{w}_3 = t_2 (C_2 t_5 - A_2) - t_3 (A_3 - C_3 t_6),$$
$$t_1 = -\frac{h_{14}}{h_{11}},$$
$$t_2 = \frac{h_{34} h_{23} - h_{24} h_{33}}{h_{22} h_{33} - h_{23} h_{32}}, \; t_3 = -\left(\frac{h_{32} t_2 + h_{34}}{h_{33}} \right),$$
$$t_4 = -\frac{h_{41}}{h_{11}}, t_5 = \frac{h_{34} h_{32} - h_{24} h_{42}}{h_{22} h_{24} - h_{23} h_{32}}, t_6 = -\left(\frac{h_{23} t_5 - h_{43}}{h_{24}} \right).$$

Then, system (2.1) with parameter value: $\widetilde{\alpha}_3 = \alpha_3 = \frac{\widetilde{w}_1}{e_{11}(e_{23}e_{32})}$ *, where:*

$$\widetilde{w}_1 = \mu_{11}\mu_5 - h_{11}(\mu_{13} + \mu_{12}) - \mu_{10}(\mu_7 - \mu_3) - \mu_4\mu_6 + \left(\frac{A_1}{\left(m + \widetilde{E}_1^2\right)} - n \right) \widetilde{E}_1 + A_2 \widetilde{E}_2 + A_3 \widetilde{E}_3 - K_3$$

has a saddle-node bifurcation at $Q_8 = \left(\widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3, \widetilde{E}_4 \right)$.

Proof . *By the Jacobian matrix given by eq. (4.14) in [6]* $\widetilde{J}_8 = J_8(Q_8, \widetilde{\alpha}_3) = \left[\widetilde{h}_{ij}\right]_{4\times 4}$ *, where* $\widetilde{h}_{ij} = h_{ij}$ *,except* $\widetilde{h}_{44} = A_2 \widetilde{E}_2 + A_3 \widetilde{E}_3 - \alpha_3 - K_3$.
Then the characteristic equation $\widetilde{J}_8$ *having zero eigenvalue (say* $\lambda_{8E_4} = 0$*) if and if* $\widetilde{\rho}_4 = 0$ *and, thus* Q_8 *is a non-hyperbolic equilibrium point.*
Note that, $\widetilde{\alpha}_3 > 0$ *provided that conditions (1.7a-1.7c), (4.15b) and (4.15e) in [6].*
Now, let $\widetilde{R}^{[8]} = (\widetilde{r}_1^{[8]}, \widetilde{r}_2^{[8]}, \widetilde{r}_3^{[8]}, \widetilde{r}_4^{[8]})^T$ *be the eigenvector corresponding to the eigenvalue* $\lambda_{8E_4} = 0$. *Thus,* $\left(\widetilde{J}_8(Q_8) - \lambda_{8E_4} I \right) R^{[8]} = 0$*, this gives:*

$$\widetilde{r}_1^{[8]} = t_1 \widetilde{r}_4^{[8]} =, \; \widetilde{r}_2^{[8]} = t_2 \widetilde{r}_4^{[8]} \; , \widetilde{r}_3^{[8]} = t_3 \widetilde{r}_4^{[8]}$$

and $\widetilde{r}_4^{[8]}$ *any non-zero real number.*
Let $\widetilde{B}^{[8]} = \left(\widetilde{b}_1^{[8]}, \widetilde{b}_2^{[8]}, \widetilde{b}_3^{[8]}, \widetilde{b}_4^{[8]} \right)^T$ *be the eigenvector associated with to the eigenvalue* $\lambda_{8E_4} = 0$ *of the matrix* $\widetilde{J}_8^T$*.Then,* $\left(\widetilde{J}_8^T(Q_8) - \lambda_{8E_4} I \right) \widetilde{B}^{[8]} = 0$.
By solving this equation for , $\widetilde{B}^{[8]} = \left(t_4 \widetilde{b}_4^{[8]}, t_5 \widetilde{b}_4^{[8]}, t_6 \widetilde{b}_4^{[8]}, \widetilde{b}_4^{[8]} \right)^T$ *, and* $\widetilde{b}_4^{[8]}$ *any non-zero real number.*
Now, consider that:

$$\frac{\partial f}{\partial \alpha_3} = f_{\alpha_3}(X, \alpha_3) = \left(\frac{\partial f_1}{\partial \alpha_3}, \frac{\partial f_2}{\partial \alpha_3}, \frac{\partial f_3}{\partial \alpha_3}, \frac{\partial f_4}{\partial \alpha_3} \right)^T = (0, 0, 0, -E_4)^T.$$

So, $f_{\alpha_3}(Q_8, \widetilde{\alpha}_3) = \left(0, 0, 0, -\widetilde{E}_4 \right)^T$ *and hence* $\left(\widetilde{B}^{[8]} \right)^T f_{\alpha_3}(Q_8, \widetilde{\alpha}_3) = -\widetilde{E}_4 \widetilde{b}_4^{[8]} \neq 0.$

By substituting $\widetilde{R}^{[8]}$ *in* (3.2) *we get:*

$$D^2F_\mu\left(Q_8,\widetilde{\alpha}_3\right)\left(\widetilde{R}^{[8]},\widetilde{R}^{[8]}\right)=\begin{bmatrix}-2t_1\left(\widetilde{r}_4^{[8]}\right)^2\left(\frac{S_1t_1}{L_1}+\frac{t_1C_1E_1E_4(3E_1-m)}{\left(m+E_1^2\right)^3}+\frac{C_1\left(m-E_1^2\right)}{\left(m+E_1^2\right)^2}\right)\\ -2\left(\widetilde{r}_4^{[8]}\right)^2\left(\alpha_1t_2^2+C_2t_2+\frac{S_2}{L_2}t_3^2\right)\\ -2\left(\widetilde{r}_4^{[8]}\right)^2t_3\left(\alpha_1t_3+C_3\right)\\ 2\left(\widetilde{r}_4^{[8]}\right)^2\left(t_1\left(\frac{A_1\left(m-E_1^2\right)}{\left(m+E_1^2\right)^2}-n\right)+t_1^2\frac{A_1E_1E_4(3E_1-m)}{\left(m+E_1^2\right)^3}+t_2A_2+A_3t_3\right)\end{bmatrix}$$

Hence, it was obtained by conditions (4.15b), (4.15e) in [6] and (3.7b)-(3.7e).

$$\left(\widetilde{B}^{[8]}\right)^T\left[D^2F_\mu\left(Q_8,\widetilde{\alpha}_3\right)\left(\widetilde{R}^{[8]},\widetilde{R}^{[8]}\right)\right]=2\left(\widetilde{r}_4^{[8]}\right)^2\widetilde{b}_4^{[8]}(\widetilde{w}_2-\widetilde{w}_3)\neq 0.$$

This means that system (2.1) *has a saddle-node bifurcation at* Q_8 *with a parameter* $\widetilde{\alpha}_3=\alpha_3$*, and no pitch fork bifurcation at* Q_8 *where* $\widetilde{\alpha}_3=\alpha_3$*.* □

4. Hopf bifurcation analysis

In this section, it is explored and found the possibility of the occurrence of hopf bifurcation around positive equilibrium points of the system (2.1) as shown below.

Theorem 4.1. *Suppose that conditions (4.15a-4.15f) in [6] with the following conditions is satisfied:*

$$\widetilde{G}_1>\widetilde{G}_2, \tag{4.1a}$$
$$\widetilde{G}_4>\widetilde{G}_5, \tag{4.1b}$$
$$\widetilde{G}_7<\widetilde{G}_6, \tag{4.1c}$$
$$\widetilde{G}_8<\widetilde{G}_9, \tag{4.1d}$$
$$\mu_6\mu_4\left(m+\widetilde{E}_1^2\right)-C_1\widetilde{E}_4\left(m-\widetilde{E}_1^2\right)\widetilde{G}_{11}<\mu_6\mu_7\left(m+\widetilde{E}_1^2\right), \tag{4.1e}$$
$$\varepsilon_1\neq\varepsilon_2, \tag{4.1f}$$
$$\frac{\widetilde{\rho}_1^3}{4}>\left(\widetilde{\rho}_1\widetilde{\rho}_2-\widetilde{\rho}_3\right). \tag{4.1g}$$

Where:

$$\widetilde{G}_1=h_{44}\left(\mu_3+\mu_5+\mu_6\right)-\mu_0^2h_{44}-\mu_0\left(\mu_7-\mu_3-\mu_4-\mu_5\right)-\mu_0h_{44}^2$$
$$\widetilde{G}_2=\mu_3h_{22}-\left(\mu_{12}+\mu_{13}\right)-\mu_5h_{33},$$
$$\widetilde{G}_3=h_{44}\left(\mu_4-\mu_7\right)+\mu_0\mu_6+\mu_3h_{22}+\mu_5h_{33}-\left(\mu_{12}+\mu_{13}\right),$$
$$\widetilde{G}_4=\left(\mu_0+h_{44}\right)\left(\mu_4-\mu_7+\mu_3+\mu_5-\mu_0h_{44}\right),$$
$$\widetilde{G}_5=\mu_4h_{44}-h_{22}\left(\mu_7-\mu_3\right)-\left(\mu_{13}+\mu_{12}\right)+\mu_5h_{33},$$
$$\widetilde{G}_6=\left(-\mu_0^2-2\mu_0h_{44}-h_{44}^2+\mu_6\right)\left(\mu_4-\mu_7+\mu_3+\mu_5-\mu_0h_{44}\right)+\left(\mu_0-h_{44}\right)\widetilde{G}_3,$$
$$\widetilde{G}_7=-2\left(-\mu_0-h_{44}\right)\left(h_{22}\left(\mu_7-\mu_3\right)-\mu_4h_{44}+\left(\mu_{13}+\mu_{12}\right)-\mu_5h_{33}\right)+\mu_6\left(\mu_4-\mu_7\right),$$

$$\begin{aligned}
\widetilde{G}_8 &= \left(\mu_0^2 + 2\mu_0 h_{44} + h_{44}^2\right)\ \left(\mu_7 - \mu_3 + \mu_4 h_{44} + (\mu_{13} + \mu_{12}) - \mu_5 h_{33}\right) - \widetilde{G}_3\left(\mu_6 - \mu_0^2 - 2\mu_0 h_{44} - h_{44}^2\right)\\
&\quad - \left(\widetilde{G}_1 + \widetilde{G}_2\right)\left(\mu_4 - \mu_7 + \mu_3 + \mu_5 - \mu_0 h_{44}\right),\\
\widetilde{G}_9 &= -2\mu_6\left(\mu_4 - \mu_7\right)\left(\mu_0 + h_{44}\right),\\
\widetilde{G}_{10} &= \left(\mu_6 - 2\mu_0 h_{44} + h_{44}\right)\mu_6\left(-\mu_4 + \mu_7\right),\\
\widetilde{G}_{11} &= h_{22}\left(\mu_7 - \mu_3\right) - \mu_4 h_{44} - \left(\mu_{13} + \mu_{12}\right) - \mu_5 h_{33},\\
\widetilde{G}_{12} &= C_1\widetilde{E}_4\left(m - \widetilde{E}_1^2\right)\widetilde{G}_{11} - \mu_6\mu_4\left(m + \widetilde{E}_1^2\right) + \mu_6\mu_7\left(m + \widetilde{E}_1^2\right).
\end{aligned}$$

Then at the parameter value $\widetilde{S}_1 = S_1 = \frac{L_1\left(\widetilde{\rho}_3\Delta_1\left(m+\widetilde{E}_1^2\right)+\widetilde{\rho}_1^2\widetilde{G}_{12}\ \right)}{(L_1-2\widetilde{E}_1)(m+\widetilde{E}_1^2)\widetilde{\rho}_1^2\widetilde{G}_{11}}$, *the system* (2.1) *has a hopf bifurcation close to point*$\widetilde{e}_1$.

Proof . *The characteristic equation of system* (2.1) *at* $\widetilde{e}_1$ *which is given in [6]:*

$$\left[\lambda^4 + \widetilde{\rho}_1\lambda^3 + \widetilde{\rho}_2\lambda^2 + \widetilde{\rho}_3\ \lambda + \widetilde{\rho}_4\right] = 0, \tag{1.8h}$$

where $\widetilde{\rho}_i; i = 1,3,4$, *denotes the characteristic coefficients of eq. (1.8h) which are given in [6]. Then, applying the hopf bifurcation theorem, for (n=4) we need to find a parameter say* ($\widetilde{S}_1$), *it is clear that* $\widetilde{S}_1 > 0$ *provided that the condition (1.8e) to confirm the necessary conditions for the hopf bifurcation to achieve:*

$\widetilde{\rho}_i\left(\widetilde{S}_1\right) > 0\ \ ;\ i = 1\ ,3,4\ \ ,\ \ \Delta_1\left(\widetilde{S}_1\right) = \left(\widetilde{\rho}_1\widetilde{\rho}_2 - \widetilde{\rho}_3\right) > 0$, *Provided that conditions (4.15a - 4.15f) and* (4.1a). *While* $\widetilde{\rho}_1^3(\widetilde{S}_1)\ - 4\ \Delta_1(\widetilde{S}_1) > 0$, *provided the conditions* (4.1a) *and* (4.1g).
$\Delta_2\left(\widetilde{S}_1\right) = \left(\widetilde{\rho}_1\widetilde{\rho}_2 - \widetilde{\rho}_3\right)\widetilde{\rho}_3 - \widetilde{\rho}_1^2\widetilde{\rho}_4 = 0$, *Straightforward computation we get:*

$$\left[\tau_1(\widetilde{S}_1^3) + \tau_2(\widetilde{S}_1^2) + \tau_3(\widetilde{S}_1) + \tau_4\ \right] \tag{1.8i}$$

Where:

$$\begin{aligned}
\tau_1 &= \left(\frac{C_1 L_1\widetilde{E}_4\left(m - \widetilde{E}_1^2\right)}{\left(L_1 - 2\widetilde{E}_1\right)(m + \widetilde{E}_1^2)}\right)^3\left(\widetilde{G}_5 - \widetilde{G}_4\right),\\
\tau_2 &= \left(\frac{C_1 L_1\widetilde{E}_4\left(m - \widetilde{E}_1^2\right)}{\left(L_1 - 2\widetilde{E}_1\right)(m + \widetilde{E}_1^2)}\right)^2\left(\widetilde{G}_6 - \widetilde{G}_7\right),\\
\tau_3 &= \left(\frac{C_1 L_1\widetilde{E}_4\left(m - \widetilde{E}_1^2\right)}{\left(L_1 - 2\widetilde{E}_1\right)(m + \widetilde{E}_1^2)}\right)\left(\widetilde{G}_9 - \widetilde{G}_8\right),\\
\tau_4 &= \widetilde{G}_3(\widetilde{G}_1 - \widetilde{G}_2) + \widetilde{G}_{10},
\end{aligned}$$

Cleary, $\tau_i > 0, i = 1,2,$ *and* $\tau_i < 0, i = 3,4$ *under local stability conditions, see [6], as well as conditions* (4.1a)-(4.1d).
Now, at $\widetilde{S}_1 = S_1$ *the characteristic equation given by Eq. (1.8h) can be written as:*

$\left(\lambda^2+\frac{\widetilde{\rho}_3}{\widetilde{\rho}_1}\right)\left(\lambda^2+\ \widetilde{\rho}_1\lambda+\frac{\Delta_1}{\widetilde{\rho}_1}\right)=0$, *which has four roots:*

$$\lambda_{1,2}=\pm i\sqrt{\frac{\widetilde{\rho}_3}{\widetilde{\rho}_1}}\qquad and \qquad \lambda_{3,4}=\frac{1}{2}\left(-\widetilde{\rho}_1\pm\sqrt{\widetilde{\rho}_1^2-4\frac{\Delta_1}{\widetilde{\rho}_1}}\right).$$

Clearly, at $S_1=\widetilde{S}_1$ *there are two pure imaginary eigenvalues* (λ_1*and* λ_2) *and two real and negative eigenvalues. Now for all values of* S_1*in the neighbourhood of* $\widetilde{S}_1$ *.In general the roots of the following form.*

$$\lambda_1=\widetilde{\omega}_1+i\widetilde{\omega}_2\ \ ,\lambda_1=\widetilde{\omega}_1-i\widetilde{\omega}_2,\lambda_{3,4}=\frac{1}{2}\left(-\widetilde{\rho}_1\pm\sqrt{\widetilde{\rho}_1^2-4\frac{\Delta_1}{\widetilde{\rho}_1}}\right).$$

Clearly, $Re(\lambda_{1,2}(S_1))\mid_{S_1=\widetilde{S}_1}=\widetilde{\omega}_1(\widetilde{S}_1)=0$, *this means that the first condition of the necessary and sufficient hopf bifurcation is satisfied at* $S_1=\widetilde{S}_1$*.Now, that the transversality condition is verified, we must prove that:*

$$\widetilde{\Theta}\left(\widetilde{S}_1\right)\widetilde{\Psi}\left(\widetilde{S}_1\right)+\widetilde{\Gamma}\left(\widetilde{S}_1\right)\widetilde{\phi}\left(\widetilde{S}_1\right)\neq 0$$

Note that for $S_1=\widetilde{S}_1$ *we have* $\widetilde{\omega}_1(\widetilde{S}_1)=0$ *and* $\widetilde{\omega}_2(\widetilde{S}_1)=\sqrt{\frac{\widetilde{\rho}_3}{\widetilde{\rho}_1}}$ *, so give the following simplification:*

$$\widetilde{\Psi}\left(\widetilde{S}_1\right)=-2\widetilde{\rho}_3\left(\widetilde{S}_1\right),\qquad \widetilde{\phi}\left(\widetilde{S}_1\right)=2\frac{\widetilde{\omega}_2\left(\widetilde{S}_1\right)}{\widetilde{\rho}_1}\left(\widetilde{\rho}_1\widetilde{\rho}_2-2\widetilde{\rho}_3\right),$$
$$\widetilde{\Theta}\left(\widetilde{S}_1\right)=\widetilde{\rho}_4^{\prime}\left(\widetilde{S}_1\right)-\frac{\widetilde{\rho}_3}{\widetilde{\rho}_1}\widetilde{\rho}_2^{\prime}\left(\widetilde{S}_1\right),\qquad \widetilde{\Gamma}\left(\widetilde{S}_1\right)=\widetilde{\omega}_2\left(\widetilde{S}_1\right)\left(\widetilde{\rho}_3^{\prime}\left(\widetilde{S}_1\right)-\frac{\widetilde{\rho}_3}{\widetilde{\rho}_1}\widetilde{\rho}_1^{\prime}\left(\widetilde{S}_1\right)\right),$$

where

$$\widetilde{\rho}_1^{\prime}=\frac{d\widetilde{\rho}_1}{d\widetilde{S}_1}=-\frac{\left(L_1-2\widetilde{E}_1\right)}{L_1}\ ,\ \widetilde{\rho}_2^{\prime}=\frac{d\widetilde{\rho}_2}{d\widetilde{S}_1}=\frac{\left(L_1-2\widetilde{E}_1\right)}{L_1}\left(\mu_0+h_{44}\right),$$
$$\widetilde{\rho}_3^{\prime}=\frac{d\widetilde{\rho}_3}{d\widetilde{S}_1}=\frac{\left(L_1-2\widetilde{E}_1\right)}{L_1}\left(\mu_4-\mu_7+\mu_3+\mu_5-\mu_0h_{44}\right),$$
$$\widetilde{\rho}_4^{\prime}=\frac{d\widetilde{\rho}_4}{d\widetilde{S}_1}=\frac{\left(L_1-2\widetilde{E}_1\right)}{L_1}\left(h_{22}\left(\mu_7-\mu_3\right)-\left(\mu_{13}+\mu_{12}\right)-\mu_5h_{33}\right),$$

well, then we get the: $\widetilde{\Theta}\left(\widetilde{S}_1\right)\widetilde{\Psi}\left(\widetilde{S}_1\right)+\widetilde{\Gamma}\left(\widetilde{S}_1\right)\widetilde{\phi}\left(\widetilde{S}_1\right)=\varepsilon_1-\varepsilon_2\neq 0.$
Where:

$$\varepsilon_1=2\widetilde{\rho}_3\left(\frac{\left(L_1-2\widetilde{E}_1\right)}{L_1}\right)\left(\left(\mu_4-\mu_7+\mu_3+\mu_5-\mu_0h_{44}\right)\widetilde{\rho}_2+\frac{\widetilde{\rho}_3}{\widetilde{\rho}_1}\left(\mu_0+h_{44}\right)-\mu_5h_{33}\right),$$
$$\varepsilon_1=2\widetilde{\rho}_3\left(\frac{\left(L_1-2\widetilde{E}_1\right)}{L_1}\right)\left(h_{22}\left(\mu_7-\mu_3\right)+\left(\mu_{13}+\mu_{12}\right)+2\frac{\widetilde{\rho}_3}{\widetilde{\rho}_1}\left(\mu_4-\mu_7+\mu_3+\mu_5-\mu_0h_{44}\right)\right).$$

By conditions (4.15a-4.15f) in [6] and (4.1f) *hold, so we get the hopf bifurcation that occurs at the equilibrium point* $\widetilde{Q}_8$ *at parameter* $S_1=\widetilde{S}_1$. □

5. Numerical simulation

In this section the dynamic behavior of the system (2.1) have been studied. Calculations can be performed for a different set of parameters with different initial points to confirm the analytical results and the effect of the parameters on the dynamic model. Fig. (1) ($a-d$) it appears that the system (2.1) at the hypothetical set of parameters (5.1) has global positive equilibrium point.

$$\begin{aligned} &S_i = 0.3, i = 1,2,\ L_i = 0.6, i = 1,2,\ C_i = 0.4, i = 1,2,3,\ m = 0.6,\ D = 0.6, \\ &\alpha_i = 0.01,\ i = 1,2,3,\ K_i = 0.01,\ i =, 1,2,3\ A_i = 0.09,\ i = 1,2,3,\ n = 0.1. \end{aligned} \tag{5.1}$$

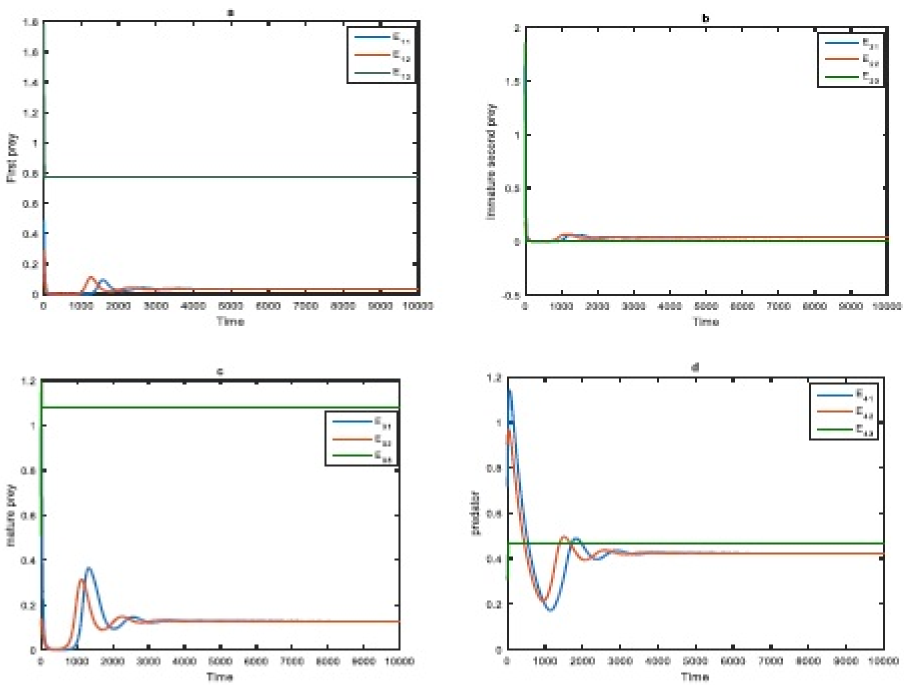

Figure 1: (a-d): Time series of the solution of system (2.1) start with different initial points (0.5 , 1.8 , 0.6 , 0.7) , (0.3, 0.2 , 0.1, 0.9) , and (1.9 , 2 , 0.4, 0.3) . (a) Path of E_1 as a function of time, (b) Path of e_2 as a function of time, (c) Path of E_3 as a function of time, (d) Path of E_4 as a function of time.

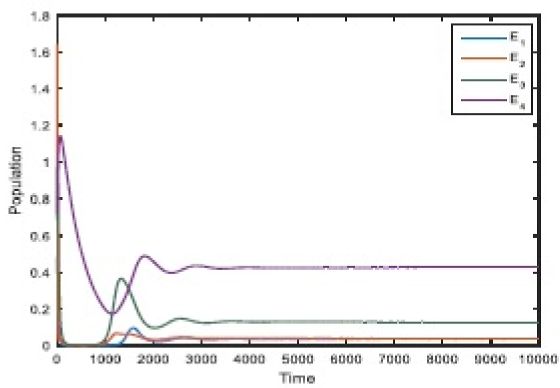

Figure 2: Graphical representation of the solution which approaches Q_8.

Now in order to study the effect of parameters on the dynamical behavior of the system (2.1), the system (2.1) has been numerically resolved to the data given in (5.1) with changing one parameter each time the results are obtained.
The effect of others parameters on the dynamics summarized in Table 2.

Table 2: The bifurcation point of system (2.1).

Range of parameter	*The stable point*	*The bifurcation point*
$0.1 \leq L_i, \alpha_i \leq 1, = 1, 2.$	Q_8	
$0.1 \leq n, m \leq 1.5$	Q_8	
$0.01 \leq K_2 < 0.096$	Q_8	$K_2 = 0.096$
$0.096 \leq K_2 \leq 1$	Q_5	
$0.1 \leq S_2 < 0.27$	Q_8	$S_2 = 0.27$
$0.27 \leq S_2 < 0.44$	Q_5	$S_2 = 0.44$
$0.44 \leq S_2 < 2$	Q_1	
$0.1 \leq C_i < 0.42$	Q_8	
$0.42 \leq C_i \leq 1 \ , i = 1, 2.$	Q_5	
$0.01 \leq A_3 < 0.017$	Q_8	
$0.017 \leq A_3 \leq 0.3$	Q_5	
$0.01 \leq \alpha_3, K_3 < 0.077$	Q_5	$\alpha_3, K_3 = 0.077$
$0.01 \leq \alpha_3, K_3 < 1$	Q_5	
$0.01 \leq K_1 < 0.42$	Q_1	$K_1 = 0.42$
$0.42 \leq K_1 < 0.61$	Q_5	$K_1 = 0.61$
$0.61 \leq K_1 \leq 1$	Q_8	
$.1 \leq S_1 < 0.23$	Q_5	$S_1 = 0.23$
$0.23 \leq S_1 < 1$	Q_8	
$0.1 \leq D < 0.169$	Q_5	
$0.169 \leq D \leq 1$	Q_8	
$0.01 \leq A_i \leq 0.3, i = 1, 2.$	Q_8	
$0.1 \leq C_3 < 1$	Q_8	

The effect of varying the parameter S_2 in the range $0.1 \leq S_2 < 0.27$ the solution approaches to Q_8, as shown in Fig.(3) (a) , increasing further in the range $0.27 \leq S_2 < 0.44$ the solution approaches to Q_5, as shown in Fig.(3) (b), but in the $0.44 \leq S_2 < 2$ the solution approaches to Q_1, as shown in Figure 3 (c).

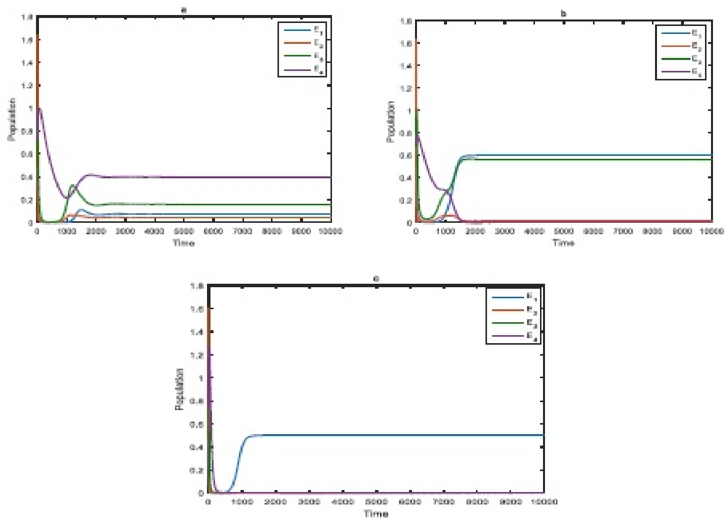

Figure 3: (a-c): (a) Time series of the solution of system (2.1) with $S_2 = 0.1$, which approaches to $Q_8 = (0.360, 0.039, 0.167, 0.020)$, and (b) time series of the solution of system (2.1) with $S_2 = 0.28$, which approaches to $Q_5 = (0.140,\ 0.010, 0.032, 0)$, and (c) time series of the solution of system (2.1) with $S_2 = 0.43$, which approaches to $Q_1 = (0.5, 0, 0, 0)$.

For the parameter K_1 in the range $0.01 \leq K_1 < 0.42$ the solution approaches to Q_1, as shown in Fig.(4) (a), increasing further in the range $0.42 \leq K_1 < 0.61$ the solution approaches to Q_5, as shown in Fig.(4) (b), but in the $0.61 \leq k_1 \leq 1$ the solution approaches to Q_8 , as shown in Fig.(4) (c) .

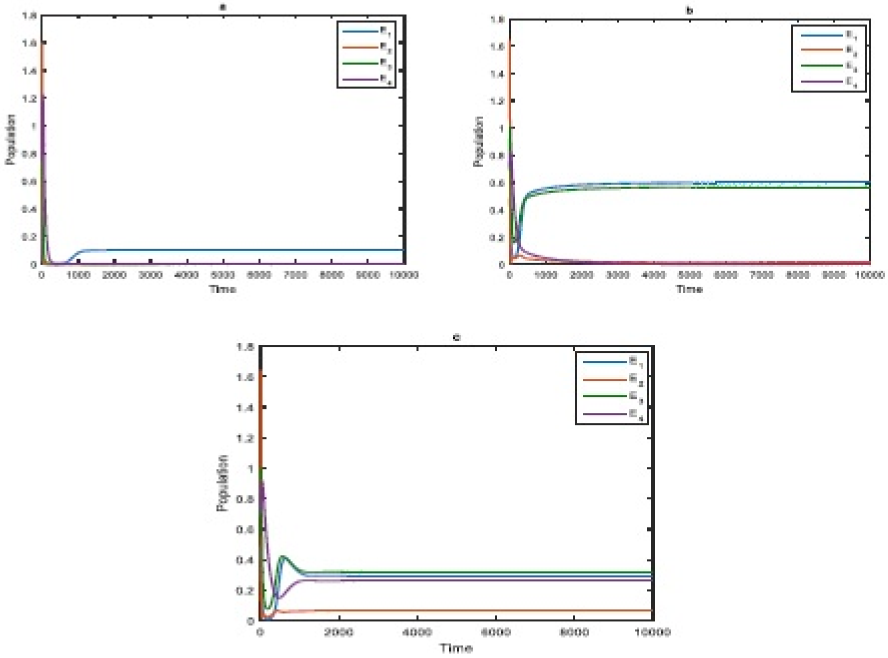

Figure 4: (a-c): (a) Time series of the solution of system (2.1) with $\mathrm{K}_1 = 0.01$,which approaches to $Q_1 = (0.5, 0, 0, 0)$, and (b) time series of the solution of system (2.1) with $K_1 = 0.43$,which approaches to $Q_5 = (0.410,\ 0.032, 0.140, 0)$, and (c) time series of the solution of system (2.1) with $K_1 = 0.61$,which approaches to $Q_8 = (0.360, 0.065, 0.157.080)$.

6. Conclusions and Discussions

In this study, the mathematical model of four differential equations for living organisms that describe the characteristic of the effect of anti-predation behavior in the mathematical model containing toxicity and its life stages have been analyzed, which are proposed to focus on age logically. Local bifurcation and Hopf bifurcation have been studied by changing a parameter of a model to study the dynamic behavior determined by bifurcation curves and the occurrence states of saddle node bifurcation occurring at points Q_5, Q_6, Q_8 and transcritcal bifurcation occurring at points Q_1, Q_2 The pitch fork bifurcation that occurs at point is determined Q_2 . The positive equilibrium point at which Hopf bifurcation occurs has been determined and the results of the bifurcation behavior analysis have been fully presented using numerical simulation. With data given in Eq. (5.1). Which are summarized as follow:

1. There is no periodic dynamics for system (2.1).
2. The parameters $A_3, D, \alpha_3, S_i, C_i, i = 1, 2\ and\ K_i, i = 1, 2, 3$, play an important role on the dynamics of system (2.1), while at others parameters $L_i, \alpha_i,\ \mathrm{A}_i, i = 1, 2, n, m$, the solution is still approaching the positive equilibrium point.

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