Int. J. Nonlinear Anal. Appl. 4 (2013) No. 2, 68-77
ISSN: 2008-6822 (electronic)
http://www.ijnaa.semnan.ac.ir

# Existence of three positive solutions for nonsmooth functional involving the p-biharmonic operator

M.B. Ghaemi[a,*], S. Mir[b]

[a]*Department of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran*
[b]*Department of Mathematics, Faculty of Basic Sciences, Payame Noor University, Tehran, Iran*

**Abstract**

This paper is concerned with the study of the existence of positive solutions for a Navier boundary value problem involving the p-biharmonic operator; the right hand side of problem is a nonsmooth functional with variable parameters. The existence of at least three positive solutions is established by using nonsmooth version of a three critical points theorem for discontinuous functions. Our results also yield an estimate on the norms of the solutions indepent of the parameters.

*Keywords:* p-biharmonic, nonsmooth nonlinearity, critical points.
*2010 MSC:* Primary 35J40; Secondary 35J55.

## 1. Introduction

It is well known that the mathematical modelling of important equations in different fields of research, such as mechanical engineering, control systems, economics, computer science and many others, leads naturally to the consideration of nonlinear differential equations. In particular, the deformations of an elastic beam in an equilibrium state or study travelling waves in suspension bridges, can be described by fourth-order boundary value problems, as Timoshenko and Gare [14] have pointed out that a classical fourth-order equation arising in the beam column theory and Lazer and Mckenna [7], pointed out that this type of nonlinearity furnishes a model to study travelling waves in suspension bridges. So, in recent years, these type of equations have attracted much attention owing to its interest to a large class of physical phenomena.

It is the purpose of this paper to investigate the following nonlinear, nonsmooth, Navier boundary value problem involving the p-biharmonic operator

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) \in \lambda\partial F(x,u) + \mu a(x)|u|^{p-2}u & in\Omega \\ u \geq 0 & in\Omega \\ u = \Delta u = 0 & on\ \partial\Omega \end{cases} \tag{1.1}$$

*Corresponding author
*Email addresses:* mghaemi@iust.ac.ir (M.B. Ghaemi ), mir@phd.pnu.ac.ir (S. Mir)

*Received:* January 2012 *Revised:* June 2013

where $a(x) \in L^{\infty}(\Omega)$, $\lambda, \mu \in [0, +\infty)$, $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a non-empty bounded open set with smooth boundary $\partial\Omega$, $1 < p < \frac{N}{2}$ and $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that $F(., u)$ is measurable for every $u \in \mathbb{R}$ and $F(x, .)$ is locally Lipschitz for a.e. $x \in \Omega$. Also $\partial F(x, u)$ denotes the generalized Clarke gradient of $F(x, u)$ at $u \in \mathbb{R}$.

In many papers, the right-hand side nonlinearity is a continuously defferentiable with respect to the real variable and the technical approach adopted is based on the three-critical-points theorem obtained by Ricceri [12]. We refer to [6] and [8]-[10]. But, in many applications, we encounter problems with inequality constraints and deal with functionals defined on a closed and convex subset of a Banach space. Such inequalities arise in problems of mechanics and engineering, when one wants to describe more realistic laws of nonmonotone and multivalued nature. This leads to nonsmooth (locally Lipschitz) and nonconvex energy functionals; for example see [3]-[6].

In the beam column theory and in one-dimentional case, Gyulove and Morosanu in [3], investigate the problem

$$(|u''|^{p-2}u'')'' - (a(t)|u'|^{p-2}u')' + b(t)|u|^{p-2}u \in \partial F(t, u) \qquad t \in [0, 1] \tag{1.2}$$

with a general boundary condition and proved the problem has a solution by nonsmooth critical point theory.

On the other hand, Iannizotto in [4], by extension from three-critical-point theorem of Ricceri [12], considered the p-Laplacian problem

$$\begin{cases} -\Delta_p u \in \lambda \partial F(x, u) + \mu \partial G(x, u) & in\Omega \\ u \geq 0 & in\Omega \\ u = 0 & on\ \partial\Omega \end{cases} \tag{1.3}$$

with a nonsmooth nonlinearity and proved the existence of at least three positive solutions.

In present study, we prove the existence of at least three positive solutions for problem (1.1) and obtain an estimate on the norms of the solutions. Our approach is chiefly based on the main critical point theorem given in [4], and related to the results obtained in [1]. We achieve our goal under different assumptions on $F$ ( see Theorem 3.1) with respect to those adopted on [4]. Also, problem (1.1) is generated to p-biharmonic with a certain function $G$. In particular, our assumptions can state in a more general form.

## 2. Preliminaries

Let $X$ be a Banach space whose dual is denoted by $X^*$. We recall that the generalized directional derivative $\Phi^\circ(u; v)$ of a locally Lipschitz function $\Phi : X \to \mathbb{R}$ at a point $u \in X$ and in the direction $v \in X$ is defined by

$$\Phi^\circ(u; v) = \limsup_{w \to u, \tau \to 0^+} \frac{\Phi(w + \tau v) - \Phi(w)}{\tau}.$$

The set $\partial\Phi(u) := \{u^* \in X^* :< u^*, v >\leq \Phi^\circ(u; v)\ for\ all\ v \in X\}$ denotes the generalized $\partial\Phi(u)$ of the function $\Phi$ (in the sense of Clarke [2]).
The following Lemma summarizes some basic properties of the generalized gradients that can find in [[2], chapter 2].

**Lemma 2.1.** *Let $\Phi, H : X \to \mathbb{R}$ be locally Lipschitz functionals. Then, for every $u, v \in X$ the following conditions hold:*

*(1) $\partial\Phi(u)$ is convex and weakly$^*$- compact;*

*(2) the set-valued mapping* $\partial\Phi(u) : X \to 2^{X^*}$ *is weakly* upper semicontinuous;*

*(3)* $\Phi^\circ(u; v) = \max_{u^* \in \partial\Phi(u)} < u^*, v > \leq L||v||$;

*(4)* $\partial(\lambda\Phi)(u) = \lambda\partial\Phi(u)$ *for every* $\lambda \in \mathbb{R}$;

*(5)* $\partial(\Phi + H)(u) \subseteq \partial\Phi(u) + \partial H(u)$;

*(6)* $\partial(\varphi \circ \Phi)(u) \subseteq \{\xi u^* : \xi \in \partial\varphi(\Phi(u)), u^* \in \partial\Phi(u)\}$ *for*

*every locally Lipschitz* $\varphi : \mathbb{R} \to \mathbb{R}$.

The classical Lagrange Mean value theorem is extended to the nonsmooth framework by the following result (the Lebourge Mean value theorem).

**Theorem 2.2 ([4]).** *Let* $\Phi : X \to \mathbb{R}$ *be a locally Lipschitz functional. Then, for every* $u, v \in X$ *there exists* $w \in [u, v]$, $w^* \in \partial\Phi(u)$ *such that* $\Phi(u) - \Phi(v) =< w^*, u - v >$.

To prove of our main result, we need the following lemmas and definitions.

**Lemma 2.3 ([4], Lemma 6).** *Let* $\Phi : X \to \mathbb{R}$ *be a locally Lipschitz functional with compact gradient. Then* $\Phi$ *is sequentially weakly continuous.*

**Definition 2.4 ([11]).** *Let* $X$ *be a Banach space,* $I : X \to (-\infty, +\infty]$ *is a Motreanu-Panagiotopoulos-type functional, where* $I = h + \Psi$ *such that* $h : X \to \mathbb{R}$ *is locally Lipschitz and* $\Psi : X \to (-\infty, +\infty]$ *is convex, proper and lower semicontinuous.*

We have the following Definitions from [[4], section 2].

**Definition 2.5.** *Let* $\Phi : X \to \mathbb{R}$ *be a locally Lipschitz functional and* $\chi : X \to \mathbb{R} \cup \{+\infty\}$ *be a proper, convex, lower semicontinuous functional whose restriction to the set* $dom(\chi) = \{x \in X : \chi(u) < +\infty\}$ *is continuous. Then* $\Phi + \chi$ *is a Motreanu-Panagiotopoulos functional.*

**Definition 2.6.** *Let* $\Phi + \chi$ *be a Motreanu-Panagiotopoulos functional. A vector* $u \in X$ *is said to be a critical point of the functional* $\Phi + \chi$, *if the following inequality holds*

$$\Phi^\circ(u; v - u) + \chi(v) - \chi(u) \geq 0, \qquad \forall v \in X.$$

**Definition 2.7.** *Let* $\Phi : X \to \mathbb{R}$ *be locally Lipschitz and* $C$ *be a nonempty, closed, convex subset of* $X$. *The indicator of* $C$ *is the function* $\chi_C : X \to \mathbb{R} \cup \{+\infty\}$ *defined by putting for every* $u \in X$, $\chi_C = \begin{cases} o & if\ u \in C \\ +\infty & if u \notin C \end{cases}$ *(it is easily seen that* $\chi_C$ *is proper, convex and lower semicontinuous), while its restriction to* $dom(\chi_C) = C$ *is the constant 0; clearly* $u \in X$ *is a critical point for the Motreanu-Panagiotopoulos functional* $\Phi + \chi_C$ *iff* $u \in C$ *and the following condition holds*

$$\Phi^\circ(u; v - u) \geq 0 \qquad for\ every\ v \in C.$$

**Definition 2.8.** *A mapping* $A : X \to X^*$ *is of type* $(S)_+$ *if, for every sequence* $\{u_n\}$ *such that* $u_n \rightharpoonup u \in X$ *and* $\limsup_{n\to+\infty} < A(u_n), u_n - u > \leq 0$, *one has* $u_n \to u$.

Now, Here and in the sequel $\Omega$ is an open bounded subset of $\mathbb{R}^N$, $1 < p < \frac{N}{2}$, while $X$ denotes the space $W^{2,p}(\Omega) \cap W_0^{2,p}(\Omega)$ endowed with the norm $||u|| = (\int_\Omega |\Delta u(x)|^p)^{\frac{1}{p}}$ for all$u \in X$ which is equivalent to usual intersection norm $||u|| = \max\{||u||_{W^{2,p}}, ||u||_{W_0^{2,p}}\}$ and $(X, ||.||)$ is a reflexive uniformly convex Banach space and its dual $(X^*, ||.||_*)$ is strictly convex.

The Rellich Kondrachov theorem assures that $X$ is compactly embedded in $L^s(\Omega)$ s.t. $s < \frac{Np}{N-2p}$ and

$$k := \sup_{u \in X \setminus \{0\}} \frac{||u||_{L^s(\Omega)}}{||u||} < +\infty. \tag{2.1}$$

The hypotheses on the nonsmooth potential $F : \Omega \times \mathbb{R} \to \mathbb{R}$ are the following.

$(F_1)$ for all $u \in \mathbb{R}$, $x \to F(x,u)$ is measurable;

$(F_2)$ for almost all $x \in \Omega$, $u \to F(x,u)$ is locally Lipschitz;

$(F_3)$ for almost all $x \in \Omega$, all $u \in \mathbb{R}$ and all $\xi \in \partial F(x,u)$, we have $|\xi| \le \alpha(x) + c|u|^{s-1}$ with $\alpha \in L^\infty(\Omega)$, $c > 0$, $1 \le s < p^* = \frac{NP}{N-2p}$;

$(F_4)$ for almost every $x \in \Omega$ and every $u \in \mathbb{R}$, $F(x,u) \le \beta(x)(1 + |u|^q)$ $(\beta \in L^1(\Omega), 1 < q < p)$.

For our aim, the following theorem is the main tool that contained in [[4], Theorem 14] and is the key to show the existence of solutions.

**Theorem 2.9.** *Let $(X, ||.||)$ be a reflexive Banach space, $\Lambda \subseteq \mathbb{R}$ an interval, $C$ a nonempty, closed, convex subset of $X$, $\mathcal{N} \in C^1(X,\mathbb{R})$ a sequentially weakly lower semicontinuous functional, bounded on any bounded subset of $X$, such that $\mathcal{N}'$ is of type $(S)_+$, $\mathcal{F} : X \to \mathbb{R}$ a locally Lipschitz functional with compact gradient, and $\rho_1 \in \mathbb{R}$. Assume also that the following conditions hold:*

*(1)* $\sup_{\lambda\in\Lambda}\inf_{u\in C}[\mathcal{N}(u) + \lambda(\rho_1 - \mathcal{F}(u))] < \inf_{u\in C}\sup_{\lambda\in\Lambda}[\mathcal{N}(u) + \lambda(\rho_1 - \mathcal{F}(u))]$;

*(2)* $\lim_{||u||\to+\infty}[\mathcal{N}(u) - \lambda\mathcal{F}(u)] = +\infty$ *for every* $\lambda \in \Lambda$.

*Then, there exists $\lambda', \lambda'' \in \Lambda(\lambda' < \lambda'')$ and $\sigma_1 > 0$ such that for every $\lambda \in [\lambda', \lambda'']$ and every locally Lipschitz functional $\mathcal{G} : X \to \mathbb{R}$ with compact gradient, there exists $\mu_1 > 0$ such that for every $\mu \in ]0, \mu_1[$ the functional $\mathcal{N} - \lambda\mathcal{F} - \mu\mathcal{G} + \chi_C$ has at least three critical points whose norms are less than $\sigma_1$.*

**Proposition 2.10 ([13], Proposition 3.1).** *Let $X$ be a nonempty set, and $\Phi$, $\Psi$ two real function on $X$. Assume that there are $r > 0$ and $u_0, u_1 \in X$ such that*

$$\Phi(u_0) = \Psi(u_0) = 0, \quad \Phi(u_1) > r, \quad \sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) < r\frac{\Psi(u_1)}{\Phi(u_1)}.$$

*Then, for each $h$ satisfying*

$$\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) < h < r\frac{\Psi(u_1)}{\Phi(u_1)},$$

*one has*

$$\sup_{\lambda\ge0}\inf_{u\in X}(\Phi(u) + \lambda(h - \Psi(u))) < \inf_{u\in X}\sup_{\lambda\ge0}(\Phi(u) + \lambda(h - \Psi(u))).$$

## 3. Existence and multiplicity

For our approach we will use the functionals $\mathcal{N}, \mathcal{F}, \mathcal{G} : X \to \mathbb{R}$ defined by putting $\mathcal{N}(u) := \frac{1}{p}||u||^p, \mathcal{F}(u) := \int_\Omega F(x, u(x))dx, \mathcal{G}(u) := \int_\Omega pa(x)|u|^p dx$ and set $G(x, u) := pa(x)|u|^p$ that $\partial G(x, u) = a(x)|u|^{p-2}u$.
Let $C = \{u \in X : u(x) \geq 0 \; a.e \; on \; \Omega\}$ (the positive cone in the Sobolev space) and for every $\lambda, \mu > 0$, put $I = \mathcal{N} - \lambda\mathcal{F} - \mu\mathcal{G} + \chi_C$ and $\Lambda = [0, +\infty)$.

Moreover, by a weak solution of (1.1) , we mean a function $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that for a.e $x \in \Omega$,

$$\Delta_p^2 u \in \lambda\partial F(x, u) + \mu\partial G(x, u), \quad u(x) \geq 0.$$

It is obvious that our goal is to find critical points of the functional $\mathcal{N} - \lambda\mathcal{F} - \mu\mathcal{G}$ on the closed and convex cone $C$.

Let $m := \sup dist_{x\in\Omega}(x, \partial\Omega)$. Fix $x_0 \in \Omega$ such that $B(x_0, m) \subseteq \Omega$, where $B(x_0, m)$ denotes the open ball of center $x_0$ and radius $m$. Also, denote

$$L := \frac{\Gamma(1 + \frac{N}{2})}{\pi^{\frac{N}{2}}(m^N - (\frac{m}{2})^N)} (\frac{3m^2}{8kN})^p, \tag{3.1}$$

where $\Gamma$ being the Gamma function and $k$ is defined in (2.1).

The main our result reads as follows.

**Theorem 3.1.** *Let $\Omega, p, F, G$ be as above and $(F_1), (F_2), (F_3)$ and $(F_4)$ be satisfied. Also, assume that there exist two positive constants $\gamma$ and $d$, with $d > L^{\frac{1}{p}}\gamma$ such that*

*(i) $F(x, \tilde{t}) \geq 0$ for each $(x, \tilde{t}) \in (\bar{\Omega}\backslash B(x_0, \frac{m}{2})) \times [0, d]$; and $F(x, 0) = 0$ for almost every $x \in \Omega$.*

*(ii) $\displaystyle\frac{1}{\gamma^p} \sup_{||u||_s \leq \gamma} \int_\Omega F(x, u(x))dx < \frac{L}{d^p} \int_{B(x_0, \frac{m}{2})} F(x, d)dx.$*

*Then, there exists $\lambda', \lambda'' > 0 \quad (\lambda' < \lambda'')$ and $\sigma_1 > 0$ such that for every $\lambda \in [\lambda', \lambda'']$, there exists $\mu_1 > 0$ such that for every $\mu \in ]0, \mu_1[$, problem (2.1) admits at least three solutions whose norms are less than $\sigma_1$.*

Before giving the proof of theorem 3.1, we give some lammas.

**Lemma 3.2.** *$\mathcal{N} \in C^1(X, \mathbb{R})$ and its gradient defined for every $u, v \in X$ by*

$$< N'(u), v >= \int_\Omega |\Delta u(x)|^{p-2}\Delta u(x)\Delta v(x)dx$$

*is of type $(S)_+$.*

**Proof .** Let $(u_n)_n$ be a sequence in $X$ such that $u_n \rightharpoonup u$ and $\limsup_{n\to+\infty} < \mathcal{N}'(u_n), u_n - u >\leq 0$. Since $\mathcal{N}'(u)$ is an element of $X^*$, then by weakly convergence $\lim_{n\to+\infty} < \mathcal{N}'(u), u_n - u >= 0$ and so

$$\begin{aligned} 0 &\geq \limsup_{n\to+\infty} < \mathcal{N}'(u_n) - \mathcal{N}'(u), u_n - u > \\ &= \limsup_{n\to+\infty} \int_\Omega (|\Delta u_n|^{p-2}\Delta u_n(x) - |\Delta u|^{p-2}\Delta u(x))(\Delta u_n - \Delta u)dx \end{aligned}$$

$$= ||\Delta u_n||_p^p + ||\Delta u||_p^p - \int_\Omega |\Delta u_n|^{p-2}\Delta u_n \Delta u dx - \int_\Omega |\Delta u|^{p-2}\Delta u \Delta u_n dx$$
$$\geq \limsup_{n\to+\infty}(||\Delta u_n||_p^{p-1} - ||\Delta u||_p^{p-1})(||\Delta u_n||_p + ||\Delta u||_p) \geq 0.$$

Then $||\Delta u_n||_p \to ||\Delta u||_p$ and due to the uniform convexity of $X$, we have $u_n \to u$ in $X$. $\Box$

**Lemma 3.3.** *Assume that $F : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies $(F_1), (F_2)$ and $(F_3)$, Then the functional $\mathcal{F}(u)$ is locally Lipschitz with compact gradient for every $u \in X$. Moreover, if $u^* \in \partial\mathcal{F}(u)$ then $u^* \in L^{s'}(\Omega)$ and satisfies $u^*(x) \in \partial F(x, u(x))$ for a.e $x \in \Omega$.*

**Proof .** By Theorem 2.2 and from $(F_3)$, we deduce that for a.e $x \in \Omega$ and every $u \in \mathbb{R}$,

$$|F(x, u)| \leq \alpha_1(x) + c_1|u|^s; \quad \alpha_1 \in L^\infty(\Omega), \quad c_1 > 0.$$

So $F(., u(.)) \in L^1(\Omega)$ and it shows that $F$ is well defined.

Now, by the compactly embedding of $X$ in $L^s(\Omega)$ and assumption $(F_2)$, it follows that $F$ is indeed locally Lipschitz. So, if $u \in X$, $u^* \in \partial\mathcal{F}(u)$, we have for every $v \in X$, $< u^*, v > \leq \mathcal{F}^\circ(u; v)$ and $\mathcal{F}^\circ(u; .) : L^s(\Omega) \to \mathbb{R}$. Moreover, $u^* \in X^*$ is continuous also with respect to the topology induced on $X$ by the norm $||.||_s$. So, we can represent $u^*$ as an element of $L^{s'}(\Omega)$ with $\frac{1}{s} + \frac{1}{s'} = 1$ and write for every $v \in L^s(\Omega)$, $< u^*, v >= \int_\Omega u^*(x)v(x)dx$.

For the proof of the inclusion

$$u^*(x) \in \partial F(x, u(x)) \quad \text{for a.e } x \in \Omega. \tag{3.2}$$

we refer the reader to Clarck [[2],section 2.7].

Now, we prove that $\partial\mathcal{F}$ is compact: let $\{u_n\}_{n\geq1}$ be a sequence in $X$ such that $||u_n|| \leq M$ for every $n \in \mathbb{N}$ $(M > 0)$ and choose $u_n^* \in \partial\mathcal{F}(u_n)$ for every $n \in \mathbb{N}$. Then by (3.2) and $(F_3)$, we get for every $n \in \mathbb{N}, v \in X$,

$$< u_n^*, v > \leq \int_\Omega |u_n^*(x)||v(x)|dx \leq \int_\Omega (\alpha(x) + c|u_n^*|^{s-1})|v(x)|dx$$
$$\leq (||\alpha||_\infty + c_2||u||^{s-1})||v|| \ (with\ c_2 > 0) \leq (M_1 + c_2M_2)||v||.$$

So $||u_n^*||_* \leq M_1 + c_2M_2$ (with $M_1, M_2 > 0$) and it follows that $\{u_n\}_{n\geq1}$ is bounded and by passing to a subsequence, if necessary, we may assume that $u_n^* \rightharpoonup u^*$ in $X^*$. So for proving that $\partial\mathcal{F}(u)$ is compact, we shall prove the convergence of $u_n^*$ is strong.

We argue by contraction: assume that there is some $\varepsilon > 0$ such that for every $n \in \mathbb{N}$, $||u_n^* - u^*||_* > \varepsilon$ and hence for every $n \in \mathbb{N}$ there is a $v_n \in B(0, 1)$ such that

$$< u_n^* - u^*, v_n >> \varepsilon. \tag{3.3}$$

Then $\{v_n\}_{n\geq1}$ is a bounded sequence and up to subsequence, $v_n \rightharpoonup v$ in $X$, and $||v_n - v||_s \to 0$ in $L^s(\Omega)$ by compact embedding of $X$ into $L^s(\Omega)$.So for $n$ big enough,

$| < u_n^* - u^*, v > | < \dfrac{\varepsilon}{3}$ (by $u_n^* \rightharpoonup u^*$ in $X^*$)

$| < u^*, v_n - v > | < \dfrac{\varepsilon}{3}$ (by $v_n \rightharpoonup v$ in $X$) and $||v_n - v||_s < \dfrac{\varepsilon}{3(M_1 + c_2M_2)}$.

This implies

$$< u_n^* - u^*, v_n > \leq < u_n^* - u^*, v > + < u_n^*, v_n - v > - < u^*, v_n - v >$$
$$\leq \frac{\varepsilon}{3} + \int_\Omega |u_n^*||v_n - v|dx + \frac{\varepsilon}{3}$$

$$\leq \frac{2\varepsilon}{3} + ||u_n^*||_{s'}||v_n - v||_s \leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

contradicting (3.3). $\square$

**Lemma 3.4.** *Assume $\mathcal{G} : X \to \mathbb{R}$ is the functional defined in the first of section 3, then $\mathcal{G}$ is a locally Lipschitz functional and its gradient defined for every $u, v \in X$, by $< \mathcal{G}'(u), v >= \int_\Omega a(x)|u|^{p-2}uvdx$, is compact.*

**Proof .**We know $\mathcal{G}(u) = \int_\Omega pa(x)|u|^p dx$ is locally continuous on each bounded subset of $X$ : indeed, let $u, v \in B(0, M)(M > 0)$ so

$$|\mathcal{G}(u) - \mathcal{G}(v)| = |\int_\Omega (pa(x)|u|^p - pa(x)|v|^p)dx| \leq \int_\Omega |pa(x)(|u|^p - |v|^p)|dx$$

$$\leq p||a||_\infty \int_\Omega (|u|^{p-1} - |v|^{p-1})|u(x) - v(x)|dx \leq 2M^{p-1}p||a||_\infty||u - v||_p$$

$$\leq c_3||u - v||.$$

Hence $\mathcal{G}$ is locally Lipschitz. Also, $\mathcal{G}(u) \in C^1(X, \mathbb{R})$ at $u \in X$, so $\partial\mathcal{G}(u) = \{\mathcal{G}'(u)\}$ and $\mathcal{G}' : X \to X^*$ is compact. $\square$

**Lemma 3.5.** *Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies $(F_1), (F_2)$ and $(F_3)$. Then, for every $\lambda, \mu > 0, I : X \to \mathbb{R} \cup \{+\infty\}$ , is a Motreanu-Panagiotopoulos functional and if $u \in X$ is a critical point of $I$, then $u$ is a solution of problem* (1.1).

**Proof .**By Lemmas 3.2, 3.3 and 3.4, the functional $J = \mathcal{N} - \lambda\mathcal{F} - \mu\mathcal{G}$ is locally Lipschitz; besides, $C$ is a closed convex cone and $0 \in C$ : thus, $I = \mathcal{N} - \lambda\mathcal{F} - \mu\mathcal{G} + \chi_C$ is a Motreanu-Panagiotopoulos functional.

Let $u \in X$ be a critical point of $I$ : then $u \in C$ and by Definition 2.7,

$$J^0(u; v - u) \geq 0. \tag{3.4}$$

If we take $v = u + sw$ $(s > 0)$, in the inequality (3.4), we easily get

$$< \mathcal{N}'(u), w > -\lambda\mathcal{F}^0(u; w) - \mu < \mathcal{G}'(u), w >\geq 0, \tag{3.5}$$

for all $w \in X$. So, inequality (3.5) reads

$$-\lambda\mathcal{F}^0(u; w) \geq -\int_\Omega |\Delta u|^{p-2}\Delta u\Delta wdx + \mu\int_\Omega a(x)|u|^{p-2}uwdx \ \forall w \in X.$$

Now, by putting $l(w) = -\int_\Omega |\Delta u|^{p-2}\Delta u\Delta wdx + \mu\int_\Omega a(x)|u|^{p-2}uwdx$ we have

$$-\lambda\mathcal{F}^0(u; w) \geq l(w). \tag{3.6}$$

Moreover, the estimate

$$-\lambda\mathcal{F}^0(u; w) \leq \lambda\mathcal{F}^0(u; w) \leq \lambda c_4||w|| \quad \forall w \in X.$$

holds with $c_4 > 0$ being a Lipschitz constant of $\mathcal{F}$ in a vicinity of $u$ as in Lemma 2.1, (3). Hence,

$$|l(w)| \leq c_5||w|| \ (with\ c_5 > 0) \quad \forall w \in X,$$

showing that $l$ is continuous. The inequality (3.6) yields that $l \in -\lambda\partial\mathcal{F}(u)$ and hence there is some $u_l \in \partial\mathcal{F}(u)$ such that by Lemma 3.3, $u_l \in L^{s'}(\Omega)$, $u_l(x) \in \partial F(x, u(x))$ for a.e $x \in \Omega$ that $l = -\lambda u_l$ and $< l, w >= \int_\Omega l(x)w(x)dx$ that shows

$$\int_\Omega |\Delta u|^{p-2}\Delta u \Delta w dx - \mu \int_\Omega a(x)|u|^{p-2}uwdx - \lambda \int_\Omega u_l w dx = 0,$$

for all $w \in X$ that is, $u$ is a weak solution of the Navier problem

$$\begin{cases} \Delta_p^2 u = \lambda u_l + \mu a(x)|u|^{p-2}u & in\ x \in \Omega \\ u = \Delta u = 0 & on\ x \in \partial\Omega. \end{cases}$$

Recalling Lemmas 3.3 and 3.4, get for a.e $x \in \Omega$

$$\Delta_p^2 u \in \lambda\partial F(x, u(x)) + \mu\partial G(x, u(x)),$$

and $u$ is a solution of (1.1). $\square$

Now we can give the proof of our main result.

*Proof of the Theorem 3.1*

**Proof .** We are going to apply Theorem 2.9. Under the conditions $(F_1) - (F_4)$ and Lemmas 3.2-3.5, put $\Lambda = [0, +\infty)$ and observe that $X$ is a reflexive Banach space; $C \neq \emptyset$ is closed and convex, $\mathcal{N} \in C^1(X, \mathbb{R})$ is continuous and convex and hence sequentially weakly lower semicontinuous and obviously, bounded on each bounded subset of $X$. In particular, $\mathcal{N}'$ is of type $(S)_+$. Also, $\mathcal{F}$ is a locally Lipschitz functional with compact gradient.

We wish to prove condition (1) in Theorem 2.9. We explicitly observe that, in view of (2.1), it follows that, for every $t > 0$,

$$\mathcal{N}^{-1}(-\infty, t] := \{u \in X \ : \ \mathcal{N}(u) \leq t\} \subseteq \{u \in L^s(\Omega) \ : \ ||u||_{L^s} \leq k(pt)^{\frac{1}{p}}\} \tag{3.7}$$

Next, put

$$\tilde{u} := \begin{cases} 0 & x \in \bar{\Omega}\backslash B(x_0, m) \\ \dfrac{1}{m^2 - (\frac{m}{2})^2}[\sum_{i=1}^{N}(x_i - x_{0i})^2]d & x \in B(x_0, m)\backslash B(x_0, \frac{m}{2}) \\ d & x \in B(x_0, \frac{m}{2}). \end{cases}$$

We have,

$$\frac{\partial \tilde{u}(x)}{\partial x_i} = \begin{cases} 0 & x \in (\bar{\Omega}\backslash B(x_0, m)) \cup (B(x_0, \frac{m}{2})) \\ \dfrac{4}{3m^2}[-2(x_i - x_{0i})]d & x \in B(x_0, m)\backslash B(x_0, \frac{m}{2}) \end{cases}$$

$$\frac{\partial^2 \tilde{u}(x)}{\partial x_i^2} = \begin{cases} 0 & x \in (\bar{\Omega}\backslash B(x_0, m)) \cup (B(x_0, \frac{m}{2})) \\ \dfrac{-8}{3m^2}d & x \in B(x_0, m)\backslash B(x_0, \frac{m}{2}) \end{cases}$$

$$\sum_{i=1}^{N} \frac{\partial^2 \tilde{u}(x)}{\partial x_i^2} = \begin{cases} 0 & x \in (\bar{\Omega}\backslash B(x_0, m)) \cup (B(x_0, \frac{m}{2})) \\ \dfrac{-8}{3}\dfrac{d}{m^2}N & x \in B(x_0, m)\backslash B(x_0, \frac{m}{2}). \end{cases}$$

It is easy to verify that $\tilde{u} \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and in particular, one has $\mathcal{N}(\tilde{u}) = \dfrac{1}{p}||\tilde{u}||^p = \dfrac{1}{p}\dfrac{\pi^{\frac{N}{2}}}{\Gamma(1 + \frac{N}{2})}\int_{\frac{m}{2}}^{m} |\frac{8}{3}\frac{d}{m^2}N|^p r^{N-1} dr$, then

$$\mathcal{N}(\tilde{u}) = \frac{1}{p}(\frac{8N}{3m^2})^p \frac{\pi^{\frac{N}{2}}}{\Gamma(1 + \frac{N}{2})} d^p (m^N - (\frac{m}{2})^N) = \frac{1}{p}\frac{d^p}{k^p L}. \tag{3.8}$$

Put $t := \frac{1}{p}(\frac{\gamma}{k})^p$. Now, from $d > L^{\frac{1}{p}}\gamma$, it follows that $\mathcal{N}(\tilde{u}) > \frac{1}{p}\frac{L\gamma^p}{k^p L} = \frac{1}{p}\frac{\gamma^p}{k^p} = t$. Moreover, taking (3.7) into account,

$$\eta(t) = \sup\{\mathcal{F}(u) \ : \ u \in C, \mathcal{N}(u) \leq t\} \subseteq \sup\{\mathcal{F}(u) \ : \ u \in C, ||u||_{L^s} \leq \gamma\} = \eta'(t),$$

so

$$\frac{\eta(t)}{t} \leq pk^p \frac{\eta'(t)}{\gamma^p}. \tag{3.9}$$

At this point, by definition of $\tilde{u}$, clearly we can write

$$\int_\Omega F(x, \tilde{u}(x))dx = \int_{B(x_0,m)\setminus B(x_0,\frac{m}{2})} F(x, \tilde{u}(x))dx + \int_{B(x_0,\frac{m}{2})} F(x, \tilde{u}(x))dx.$$

Moreover, owing to $0 \leq \tilde{u}(x) \leq d$, for each $x \in \Omega$, $\tilde{u} \in C$ and by using (i) in Theorem 3.1, we have

$$\int_\Omega F(x, \tilde{u}(x))dx \geq \int_{B(x_0,\frac{m}{2})} F(x, \tilde{u}(x))dx, \tag{3.10}$$

and by (3.8)-(3.12) and (ii) in Theorem 3.1, it follows that

$$\frac{\mathcal{F}(\tilde{u})}{\mathcal{N}(\tilde{u})} \geq pk^p L \frac{\int_{B(x_0,\frac{m}{2}} F(x, d)dx}{d^p} > \frac{\eta'(t)}{\gamma^p} \geq \frac{\eta(t)}{t},$$

that is $\sup_{u \in N^{-1}(-\infty, r]} \mathcal{F}(u) < t\frac{\mathcal{F}(\tilde{u})}{\mathcal{N}(\tilde{u})}$. Then, there is some $h \in \mathbb{R}$ such that

$$\sup_{u \in \mathcal{N}^{-1}(-\infty, r]} \mathcal{F}(u) < h < t\frac{\mathcal{F}(\tilde{u})}{\mathcal{N}(\tilde{u})}.$$

So, conditions of Proposition 2.10, are verified taking $u_1 := \tilde{u}$ and $u_0 = 0$ since $\mathcal{N}(0) = \mathcal{F}(0) = 0$. Hence condition (1) from Theorem 2.9 is satisfied taking $\rho_1 := h$, $C = X$ in Proposition 2.10.

Next, we prove condition (2) in Theorem 2.9 by using $(F_4)$, one has

$$\mathcal{F}(u) = \int_\Omega F(x, u(x))dx \leq ||\beta||_{L^1(\Omega)}(meas(\Omega) + c_5||u||^q),$$

and hence for every $\lambda \in \Lambda$

$$\mathcal{N}(u) - \lambda\mathcal{F}(u) \geq \frac{||u||^p}{p} - \lambda||\beta||_{L^1(\Omega)}(meas(\Omega) + c_5||u||^q).$$

Therefore, owing to $q < p$, the following relation holds

$$\lim_{||u|| \to +\infty} \mathcal{N}(u) - \lambda\mathcal{F}(u) = +\infty,$$

for every $\lambda > 0$.

Finally, all the assumption of Theorem 2.9 are satisfied. Hence there exists $\lambda', \lambda'' \in \Lambda$ and $\sigma_1 > 0$ as in Theorem 2.9. Then for $G$ that is locally Lipschitz functional with compact gradient (Lemma 3.4), there exists $\mu_1 > 0$ that for every $\mu \in (0, \mu_1)$ there exists three solutions $u_0, u_1, u_2 \in C \cap B(0, \sigma_1)$ for problem (1.1) and the proof is completed. $\square$

**Remark 3.6.** *We point out that hypothesis (ii) in Theorem 3.1, can be given in a more general form. Precisely, fix $x_0 \in \Omega$ and pick $r_1, r_2 \in \mathbb{R}$ with $r_1, r_2 > 0$, such that $B(x_0, r_1) \subset B(x_0, r_2) \subseteq \Omega$. Moreover, denote*

$$L_{r_1,r_2} := \frac{\Gamma(1+\frac{N}{2})}{\pi^{\frac{N}{2}}}(\frac{r_2^2 - r_1^2}{2Nk})^p \frac{1}{r_2^N - r_1^N}. \tag{3.11}$$

*If $d$ and $\gamma$ in Theorem 3.1 are satisfying $d > L^{\frac{1}{p}}\gamma$, hypothesis (ii) can be replaced by the following assumption:*

$$\frac{1}{\gamma^p} \sup_{||u||_s \leqslant \gamma} \int_\Omega F(x, u(x))dx < \frac{L_{r_1,r_2}}{d^p} \int_{B(x_0,m/2)} F(x, d)dx. \tag{3.12}$$

**Remark 3.7.** *A comparison between our main result (Theorem 3.1) and some of those the previously cited ones, is now in order: in the present paper, we extended the main result of Candito ([2] Theorem 3.1) to a class of perturbed Motrreanu-Panagiotopoulos functionals, this feature gains a remarkable importance in the applications. Moreover, it is worth noticing that, since parameter $a(x)$ in problem (1.1), is variable , causes that the fourth-order problem is investigate in a complete form. Also, owing to estimate of norms of solutions, several applications can be improved.*

*On the other hand, the main difference between Theorem 3.1 above and the main result of Iannizzotto [4] in applications (Theorem 22) consists in different assumption about the $F(x, u)$, that it is possible to obtain a well determined interval of values of parameter $\lambda$ for which the problem depending on $\lambda$ admits at least three weak solutions.*

## References


[1] P. Candito and G. M. Bisci, *Multiple solutions for a Navier boundary value problem involving the p-biharmonic operator,* Discrete and Continuous Dynamical Systems Series, **5** (2012), no 4, 741-751.

[2] F.H. Clarke, *Optimization and Nonsmooth Analysis,* New York, 1983. ISBN: 0-89871-256-4

[3] T. Gyulov and G. Moroşanu, *On a class of boundary value problems involving the p-biharmonic operator,* J. Math. Anal. Appl. **367** (2010) 43-57.

[4] A. Iannizzotto, *Three critical points for perturbed nonsmooth functionals and applications,* J. Nonlinear Analysis **72** (2010) 1319-1338.

[5] A. Iannizzotto and N. S. Papageorgiou, *Existence of three nontrivial solutions for nonlinear Neumann hemivariational inequalities,* J. Nonlinear Analysis **70** (2009) 3285-3297.

[6] S.Th. Kyritsi and N. S. Papageorgiou, *Nonsmooth critical point theory on closed convex sets and nonlinear hemivariational inequalities,* J. Nonlinear Analysis **61** (2005) 373-403.

[7] A.C. Lazer and P. J. Mckenna, *Large amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis,* SIAM ReV. **32** (1990) 537-578.

[8] C. Li and C. Tang, *Three solutions for a Navier boundary value problem involving the p-biharmonic,* J. Nonlinear Analysis **72** (2010) 1339-1347.

[9] L. Lin, *Existence of three solutions for p-biharmonic equation,* J. Nonlinear Analysis and Application, **2011** (2011) 1-9.

[10] L. Lin and C. Tang, *Existence ot three solutions for (p,q)-biharmonic systems,* J. Nonlinear Analysis **73** (2010) 796-805.

[11] D. Motreanu and P.D. Panagiotopoulos, *Minimax Theorems and Qualitative properties of the Solutions of Hemivariational Inequalities,* Kluwer Academic Publishers, Dordrecht, The Netherlands, 1999. ISBN: 9780792354567.

[12] B. Ricceri, *A three critical points theorem revisited,* J. Nonlinear Analysis **70** (2009) 3084-3089.

[13] B. Ricceri, *Existence of three solutions for a class of elliptic eigenvalue problems,* Math. Comp. Model. **32** (2000) 1485-1494.

[14] S. Timoshenko and J. Gere, *Theory of Elastic Stability,* Second ed., McGraw-Hill, New York, 1961. ISBN: 0-07-Y85821-7