A survey of $\mathcal{C}$−class and pair upper-class functions in fixed point theory

A. H. Ansari[a,b,c], Anita Tomar[d,*], Meena Joshi[e]

[a]*Department of Mathematics, Islamic Azad University, Karaj Branch, Karaj, Iran*
[b]*Department of Mathematics, Payame Noor University, Tehran, Iran*
[c]*Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, Pretoria, Medunsa-0204, South Africa*
[d]*Sri Dev Suman Uttarakhand Vishwavidyalay, P. L. M. S. Campus, Rishikesh- 249201, Uttarakhand, India*
[e]*S. G. R. R. (P. G.) College Dehradun, India*

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Abstract

We demonstrate that the $\mathcal{C}$−class functions, pair (h, f) upper-class functions, cone $\mathcal{C}$−class functions, $1-1-$ up-class functions, multiplicative $\mathcal{C}$−class functions, inverse$-\mathcal{C}$−class functions, $\mathcal{C}_F$−simulation functions, $\mathcal{C}^*$−class functions are powerful and fascinating weapons for the generalization, improvement, and extension of considerable conclusions obtained in the fixed point theory. Towards the end, we point out some open problems whose answers could be interesting.

Keywords: $\mathcal{C}$−class, pair upper-class, up-class, multiplicative $\mathcal{C}$−class, inverse$-\mathcal{C}$−class.
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1. Introduction and Preliminaries

The Banach contraction principle [13] publicized in 1922 and included in the Ph.D. dissertation of Stefan Banach, is the commencement of metric fixed point theory. This elemental principle is utilized in managing different practical and theoretical issues emerging in various branches of science and consequently fascinated numerous researchers (see references). Extensions and generalizations of Banach's fixed point theorem is centred essentially around three components:

1. by changing the distance structure

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*Anita Tomar
Email addresses: analsisamirmath2@gmail.com; mathanalsisamir4@gmail.com (A. H. Ansari), anitatmr@yahoo.com (Anita Tomar), joshimeena35@gmail.com (Meena Joshi)

2. by changing the conditions on the mapping under consideration, and
3. by changing the contractive condition.

Some of the interesting class of functions, $\mathcal{C}-$class, pair (h, f) upper-class, cone $\mathcal{C}-$class, $1-1-$up-class, multiplicative $\mathcal{C}-$class, inverse $\mathcal{C}-$class, $\mathcal{C}_F-$simulation, $\mathcal{C}^*-$class were familiarized by Ansari [6]- [7], Ansari and Shukla [10], Ansari et al. [11], Ampadu et al. [5], Saleem et al. [42], Khojasteh et al. [32] and Chandok et al. [17], respectively, that cover a major class of contractive conditions. The aim of this work is to demonstrate the development of these functions and that these are powerful and fascinating weapons for the generalization, improvement, and extension of significant results in the metric fixed point theory. These subsume most of the celebrated and contemporary contractive conditions and perform a magnificent role in the evolution of fixed point theory.

Definition 1.1. *A self-mapping $\mathcal{A}$ of a metric space $(\mathcal{U}, d)$ is a contraction on $\mathcal{U}$ if there exists $k \in [0, 1)$ satisfying*

$$d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}) \leq kd(\mathfrak{u}, \mathfrak{v}), \ \mathfrak{u}, \mathfrak{v} \in \mathcal{U}. \tag{1.1}$$

Theorem 1.2. *A contraction mapping $\mathcal{A}$ of a non-empty complete metric space $(\mathcal{U}, d)$ admits a unique fixed point $\mathfrak{u}^* \in \mathcal{U}$. Additionally, $\mathfrak{u}^*$ may be established on starting with $\mathfrak{u}_0 \in \mathcal{U}$ and defining a sequence $\{\mathfrak{u}_n\}$ as: $\mathfrak{u}_n = \mathcal{A}\mathfrak{u}_{n-1}$, then $\mathfrak{u}_n \to \mathfrak{u}^*$.*

Example 1.3. *Let a self-mapping $\mathcal{A}$ on $\mathbb{R}$ be defined as:*

$$\mathcal{A}\mathfrak{u} = (\mathfrak{u} - 3)(\mathfrak{u} - 4)(\mathfrak{u} - 5) - 3(\mathfrak{u} - 2)(\mathfrak{u} - 4) - 4(\mathfrak{u} - 3)(\mathfrak{u} - 5) + 5(\mathfrak{u} - 3)(\mathfrak{u} - 4).$$

Then,

$$\mathcal{A}3 = 3 \quad and \quad \mathcal{A}4 = 4.$$

Noticeably, $81 = |\mathcal{A}3 - \mathcal{A}0| > |3 - 0| = 3$ and $\mathcal{A}$ does not have a unique fixed point. Clearly, 3 and 4 are the two fixed points of $\mathcal{A}$.

Example 1.4. *Let a self-mapping $\mathcal{A}$ on $\mathbb{R}$ be defined as:*

$$\mathcal{A}\mathfrak{u} = \frac{3\mathfrak{u} + 1}{4}.$$

Then,

$$\mathcal{A}1 = 1.$$

Noticeably,

$$|\mathcal{A}\mathfrak{u} - \mathcal{A}\mathfrak{v}| \leq k|\mathfrak{u} - \mathfrak{v}|, \quad \frac{3}{4} < k < 1,$$

and $\mathcal{A}$ has a unique fixed point at $\mathfrak{u} = 1$.

Throughout the paper $(\mathcal{U}, d)$ is a metric space, $\mathcal{A}$ is a self -mapping on $\mathcal{U}$ and $\mathfrak{u}, \mathfrak{v} \in \mathcal{U}$.

The Geraghty contraction was introduced by Geraghty [21] in 1973 for the generalization of the Banach contraction utilizing a control function. Let S symbolize the class of functions $\beta : \mathbb{R}^+ \to [0, 1)$ so that:

1. $\mathbb{R}^+ = \{\mathfrak{t} \in \mathbb{R} : \mathfrak{t} > 0\}$;
2. $\beta(\mathfrak{t}_n) \to 1$ implies $\mathfrak{t}_n \to 0$.

Definition 1.5. *[21] A self-mapping $\mathcal{A}$ is a Geraghty contraction if there exists $\beta \in S$ satisfying*

$$d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}) \leq \beta(d(\mathfrak{u}, \mathfrak{v}))d(\mathfrak{u}, \mathfrak{v}). \tag{1.2}$$

Khan et al.[31] introduced a control function, namely altering distance function, which is useful in establishing a fixed point and symbolize the set of altering distance functions by $\mathbf{\Psi}$.

Definition 1.6. *[31] A function $\psi : [0, \infty) \to [0, \infty)$ is an altering distance function if the subsequent properties hold true:*

(i) $\psi(\mathfrak{t}) = 0 \iff \mathfrak{t} = 0$,
(ii) ψ *is continuous and strictly increasing.*

To establish a fixed point Khan et al. [31] utilized a contraction mapping

$$\psi(d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v})) \leq c\psi(d(\mathfrak{u}, \mathfrak{v})), \quad c \in (0, 1). \tag{1.3}$$

Rhoades [39] (see, also Alber and Guerre-Delabriere [3]) considered weakly contractive mapping as:

$$d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}) \leq d(\mathfrak{u}, \mathfrak{v}) - \psi(d(\mathfrak{u}, \mathfrak{v})). \tag{1.4}$$

Dutta and Choudhury [19] utilized the inequality

$$\psi(d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v})) \leq \psi(d(\mathfrak{u}, \mathfrak{v})) - \varphi(d(\mathfrak{u}, \mathfrak{v})), \tag{1.5}$$

where, $\psi, \varphi \in \mathbf{\Psi}$.

Remark 1.7. *On taking $\psi(\mathfrak{t}) = (1 - k)\mathfrak{t}, \ 0 \leq k < 1$, weak contraction (1.4) diminishes to Banach contraction (1.1).*

Definition 1.8. *[6] A continuous function $\varphi : [0, \infty) \to [0, \infty)$ is an ultra-altering distance function if $\varphi(0) \geq 0$, $\varphi(\mathfrak{t}) > 0$ and $\mathfrak{t} > 0$.*

2. $\mathcal{C}-$class functions

In 2014, the subsequent observations

1. $d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}) \leq cd(\mathfrak{u}, \mathfrak{v}) \leq d(\mathfrak{u}, \mathfrak{v})$;
2. $d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}) \leq \beta(d(\mathfrak{u}, \mathfrak{v}))d(\mathfrak{u}, \mathfrak{v}) \leq d(\mathfrak{u}, \mathfrak{v})$; and
3. $\psi(d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v})) \leq \psi(d(\mathfrak{u}, \mathfrak{v})) - \varphi(d(\mathfrak{u}, \mathfrak{v})) \leq \psi(d(\mathfrak{u}, \mathfrak{v}))$,

guided to the introduction of $\mathcal{C}$-class function [6], which is defined as:

Definition 2.1. *[6] A continuous function $f : [0, \infty)^2 \to \mathbb{R}$ is a function of $\mathcal{C}-$class, if subsequent axioms hold true:*
(1) $f(\rho, \omega) \leq \rho$;
(2) $f(\rho, \omega) = \rho$ implies that either $\rho = 0$ or $\omega = 0$, $\rho, \omega \in [0, \infty)$.
Noticeably, $f(0, 0) = 0$. A collection of $\mathcal{C}-$class of functions is symbolized as $\mathbf{C}$.

Example 2.2. *[6], [12] Subsequent functions $f : [0, \infty)^2 \to \mathbb{R}$ are elements of* **C***:*

(1) $f(\rho, \omega) = \rho - \omega$;

(2) $f(\rho, \omega) = \mathfrak{m}\rho$, $0 < \mathfrak{m} < 1$;

(3) $f(\rho, \omega) = \frac{\rho}{(1+\omega)^{\mathfrak{r}}}$, $\mathfrak{r} \in (0, \infty)$;

(4) $f(\rho, \omega) = = \log \frac{\omega + \mathfrak{a}^{\rho}}{1+\omega}$, $\mathfrak{a} > 1$;

(5) $f(\rho, \omega) = \ln(1 + \mathfrak{a}^{\rho})/2$, $\mathfrak{a} > e$;

(6) $f(\rho, \omega) = (\rho + \mathfrak{l})^{(1/(1+\omega))} - \mathfrak{l}$, $\mathfrak{l} > 1)$;

(7) $f(\rho, \omega) = (\rho + \mathfrak{l})^{(1/(1+\omega)^{\mathfrak{r}})} - \mathfrak{l}$, $\mathfrak{l} > 1, \mathfrak{r} \in (0, \infty)$;

(8) $f(\rho, \omega) = \rho \log_{\omega + \mathfrak{a}} \mathfrak{a}$, $\mathfrak{a} > 1$;

(9) $f(\rho, \omega) = \rho - (\frac{1+\rho}{2+\rho})(\frac{\omega}{1+\omega})$;

(10) $f(\rho, \omega) = \rho\beta(\rho)$, $\beta : [0, \infty) \to [0, 1)$ *is continuous*;

(11) $f(\rho, \omega) = \rho - \frac{\omega}{k+\omega}$;

(12) $f(\rho, \omega) = \rho - \varphi(\rho)$, *here* $\varphi : [0, \infty) \to [0, \infty)$ *is a continuous function so that* $\varphi(\omega) = 0 \Leftrightarrow \omega = 0$;

(13) $f(\rho, \omega) = \rho h(\rho, \omega)$, *here* $h : [0, \infty) \times [0, \infty) \to [0, \infty)$ *is a continuous function so thatip* $h(\rho, \omega) < 1$;

(14) $f(\rho, \omega) = \sqrt[n]{\ln(1 + \rho^n)}$;

(15) $f(\rho, \omega) = \frac{\rho - \omega}{1+\omega}$;

(16) $f(\rho, \omega) = \vartheta(\rho)$, ϑ *is a generalized Mizoguchi-Takahashi type function*;

(17) $f(\rho, \omega) = \frac{\rho}{\Gamma(\frac{1}{2})} \int_0^{\infty} \frac{e^{-\mathfrak{u}}}{\sqrt{\mathfrak{u} + \omega}} d\mathfrak{u}$, Γ *is the Euler Gamma function*.

Definition 2.3. *[6] (see also [9]) A continuous function* $h : [0, \infty) \to [0, \infty)$ *is a function of* $\mathcal{A}-$*class if* $h\mathfrak{t} \geq \mathfrak{t}$, $\mathfrak{t} \in [0, \infty)$.

Ansari [6] symbolized the set of functions of $\mathcal{A}$-class by ***A***.

Example 2.4. *[6] (see also [9]) For* $\mathfrak{t} \in [0, \infty)$, *the subsequent functions* $h : [0, \infty) \to [0, \infty)$ *are elements of* $\mathcal{A}$:

(1) $h(\mathfrak{t}) = \mathfrak{a}^{\mathfrak{t}} - 1$, $\mathfrak{a} > 1$;

(2) $h(\mathfrak{t}) = \mathfrak{m}\mathfrak{t}$, $\mathfrak{m} \geq 1$.

To establish a fixed point, Ansari [6] utilized the inequality

$$h(\psi(d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}))) \leq f(\psi(d(\mathfrak{u}, \mathfrak{v})), \varphi(d(\mathfrak{u}, \mathfrak{v}))), \quad \mathfrak{u}, \mathfrak{v} \in \mathcal{F} \subseteq \mathcal{U}, \tag{2.1}$$

ψ, ϕ, f, and h are earlier described as an altering distance function, an ultra-altering distance function, a function of $\mathcal{C}-$class, and a function of $\mathcal{A}-$class respectively.

Remark 2.5.   1. *If* $h(\omega) = \omega$, $\psi(\omega) = \omega$, $\mathcal{F} = \mathcal{U}$, *and* $f(\rho, \omega) = \frac{\rho - \omega}{1+\omega}$, *then inequality (2.1) reduces to a contractive condition utilized in Corollaries* 3.1 *and* 3.2 *[7]*.

   2. *If* $h(\omega) = \omega$, $\psi(\omega) = \omega$, $\mathcal{F} = \mathcal{U}$, *and* $f(\rho, \omega) = \frac{\omega + \mathfrak{a}^{\rho}}{1+\omega}$, *then inequality (2.1) reduces to a contractive condition utilized in Corollaries* 3.3 *and* 3.4 *[7]*.

   3. *If* $h(\omega) = \omega$, $\psi(\omega) = \omega$, $\mathcal{F} = \mathcal{U}$, *and* $f(\rho, \omega) = (\rho + \mathfrak{l})^{\frac{1}{1+\omega}}$, *then inequality (2.1) reduces to a contractive condition utilized in Corollaries* 3.5 *and* 3.6 *[7]*.

   4. *If* $h(\omega) = \omega$, $\psi(\omega) = \omega$, $\mathcal{F} = \mathcal{U}$, *and* $f(\rho, \omega) = \rho\beta(\rho)$, $\beta : [0, \infty) \longrightarrow [0, 1)$ *is a continuous function, then inequality (2.1) reduces to Geraghty contraction 1.2 [21]*.

5. *If $h(\omega) = \omega$, $\psi(\omega) = \omega$, $\mathcal{F} = \mathcal{U}$, $f(\rho, \omega) = \rho - \omega$, then inequality (2.1) reduces to weak contractive condition (1.4) [39].*

6. *If $h(\omega) = \omega$, $\mathcal{F} = \mathcal{U}$, and $f(\rho, \omega) = \frac{\rho - \omega}{1 + \omega}$, then inequality (2.1) reduces to a contractive condition utilized in Corollaries 3.7 and 3.8 [7].*

7. *If $h(\omega) = \omega$, $\mathcal{F} = \mathcal{U}$, and $f(\rho, \omega) = \frac{\omega + \mathfrak{a}\rho}{1 + \omega}$, then inequality (2.1) reduces to a contractive condition utilized in Corollaries 3.9 and 3.10 [7].*

8. *If $h(\omega) = \omega$, $\mathcal{F} = \mathcal{U}$, and $f(\rho, \omega) = (\rho + \mathfrak{l})^{\frac{1}{1+\omega}} - 1$, then inequality (2.1) reduces to a contractive condition utilized in Corollaries 3.11 and 3.12 of [7].*

9. *If $h(\omega) = \omega$, $\mathcal{F} = \mathcal{U}$, and $f(\rho, \omega) = \rho \log_{\omega + \mathfrak{a}} \mathfrak{a}, \mathfrak{a} > 1$, then inequality (2.1) reduces to a contractive condition utilized in Corollaries 3.13 and 3.14 [7].*

10. *If $h(\omega) = \omega$, $\psi(\omega) = \omega$, $\mathcal{F} = \mathcal{U}$, and $f(\rho, \omega) = \mathfrak{m}\rho, \mathfrak{m} \in (0, 1)$, then inequality (2.1) reduces to a contractive condition utilized in Corollary 3.15 [7].*

11. *If $h(\omega) = \omega, \mathcal{F} = \mathcal{U}$, $f(\rho, \omega) = \rho - \omega$, then inequality (2.1) reduces to an inequality (1.5) [19].*

Here it merits noticing that on altering the elements of $\mathcal{C}-$class rationally, extended/ improved versions of contractions present in the literature may originate (see, Corollaries, 1-15 Ansari [7]).

Remark 2.6. *Let $h, g : [0, \infty) \to [0, \infty)$ so that $\quad \mathfrak{t} \le g(\omega) \le h(\omega) \quad$ and if*

$$h(\psi(d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}))) \le f(\psi(d(\mathfrak{u}, \mathfrak{v})), \varphi(d(\mathfrak{u}, \mathfrak{v}))) \tag{2.2}$$

and

$$g(\psi(d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v})) \le f(\psi(d(\mathfrak{u}, \mathfrak{v})), \varphi(d(\mathfrak{u}, \mathfrak{v}))), \tag{2.3}$$

so inequality (2.3) is more general than inequality (2.2), therefore,

$$\psi(d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}) \le f(\psi(d(\mathfrak{u}, \mathfrak{v})), \varphi(d(\mathfrak{u}, \mathfrak{v}))) \tag{2.4}$$

is the best case.

Remark 2.7. *Let $h, g : [0, \infty) \to [0, \infty)$ with $\quad g(\mathfrak{t}) \le h(\mathfrak{t}) \quad$ and if*

$$\psi(d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v})) \le g(\psi(d(\mathfrak{u}, \mathfrak{v}))) \tag{2.5}$$

and

$$\psi(d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v})) \le h(\psi(d(\mathfrak{u}, \mathfrak{v}))), \tag{2.6}$$

so inequality (2.6) is more general than inequality (2.5).
Therefore,

$$\psi(d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v})) \le f(g(\psi(d(\mathfrak{u}, \mathfrak{v}))), \varphi(d(\mathfrak{u}, \mathfrak{v}))),$$

where, $f \in \mathbf{C}$, is not new, because

$$\psi(d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v})) \le f(g(\psi(d(\mathfrak{u}, \mathfrak{v}))), \varphi(d(\mathfrak{u}, \mathfrak{v}))) \le g(\psi(d(\mathfrak{u}, \mathfrak{v}))).$$

3. Upper-class functions

Definition 3.1. *[44] Let $\alpha : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$. A self-mapping $\mathcal{A}$ is an α-admissible if $\alpha(\mathfrak{u}, \mathfrak{v}) \geq 1$ implies $\alpha(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}) \geq 1$.*

Let a continuous self-mapping $\mathcal{A}$ defined on $\mathcal{U}$ be an $\alpha-$admissible and there exists a function $\beta \in S$ (Greghaty [21]) so that $\{\mathfrak{t}_n\}$ is a bounded sequence.
Hussain et al. [23] considered the subsequent conditions to establish the fixed point:

1. $(d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}) + l)^{\alpha(\mathfrak{u}, \mathcal{A}\mathfrak{u})\alpha(\mathfrak{v}, \mathcal{A}\mathfrak{v})} \leq \beta\left(d\left(\mathfrak{u}, \mathfrak{v}\right)\right) d\left(\mathfrak{u}, \mathfrak{v}\right) + l, \ l > 1;$
2. $\left(\alpha\left(\mathfrak{u}, \mathcal{A}\mathfrak{u}\right)\alpha\left(\mathfrak{v}, \mathcal{A}\mathfrak{v}\right) + 1\right)^{d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v})} \leq 2^{\beta(d(\mathfrak{u}, \mathfrak{v}))d(\mathfrak{u}, \mathfrak{v})};$
3. $\alpha\left(\mathfrak{u}, \mathcal{A}\mathfrak{u}\right)\alpha\left(\mathfrak{v}, \mathcal{A}\mathfrak{v}\right)d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}) \leq \beta(d(\mathfrak{u}, \mathfrak{v}))d(\mathfrak{u}, \mathfrak{v}).$

In 2014, these contractive conditions guided the introduction of an upper-class function [7].

Definition 3.2. *[10] The function $h \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function of a subclass of type I, if $\mathfrak{u} \geq 1 \implies h(1, \mathfrak{v}) \leq h(\mathfrak{u}, \mathfrak{v}), \ \mathfrak{u}, \mathfrak{v} \in \mathbb{R}^+.$*

Example 3.3. *[10] For $l > 1, n \in \mathbb{N}$ and $\mathfrak{u}, \mathfrak{v} \in \mathbb{R}$, subsequent functions $h \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ are functions of a subclass of type I:*

(a) $h(\mathfrak{u}, \mathfrak{v}) = (\mathfrak{v} + l)^{\mathfrak{u}};$
(b) $h(\mathfrak{u}, \mathfrak{v}) = (\mathfrak{u} + l)^{\mathfrak{v}};$
(c) $h(\mathfrak{u}, \mathfrak{v}) = \mathfrak{u}^{\mathfrak{n}}\mathfrak{v};$
(d) $h(\mathfrak{u}, \mathfrak{v}) = \mathfrak{v};$
(e) $h(\mathfrak{u}, \mathfrak{v}) = \frac{1}{n+1}\left(\sum_{i=0}^{\mathfrak{n}} \mathfrak{u}^{i}\right)\mathfrak{v};$
(f) $h(\mathfrak{u}, \mathfrak{v}) = \left[\frac{1}{n+1}\left(\sum_{i=0}^{\mathfrak{n}} \mathfrak{u}^{i}\right) + l\right]^{\mathfrak{v}}.$

Definition 3.4. *[10] Let $h, \mathcal{F} \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, then the pair $(\mathcal{F}, h)$ is an upper-class of type I, if h is a function of a subclass of type I and*
(i) $0 \leq \rho \leq 1 \implies \mathcal{F}(\rho, \omega) \leq \mathcal{F}(1, \omega),$
(ii) $h(1, \mathfrak{v}) \leq \mathcal{F}(1, \omega) \implies \mathfrak{v} \leq \rho, \ \omega, \mathfrak{v} \in \mathbb{R}^+.$

Example 3.5. *[10] For $\mathfrak{u}, \mathfrak{v}, \rho, \omega \in \mathbb{R}^+, \ l > 1, \mathfrak{m}, \mathfrak{n} \in \mathbb{N}$, subsequent pairs $(\mathcal{F}, h)$ are functions of a upper-class of type I:*

(a) $h(\mathfrak{u}, \mathfrak{v}) = (\mathfrak{v} + l)^{\mathfrak{u}}$ and $\mathcal{F}(\rho, \omega) = \rho\omega + l;$
(b) $h(\mathfrak{u}, \mathfrak{v}) = (\mathfrak{u} + l)^{\mathfrak{v}}$ and $\mathcal{F}(\rho, \omega) = (1 + l)^{\rho\omega};$
(c) $h(\mathfrak{u}, \mathfrak{v}) = \mathfrak{u}^{\mathfrak{m}}\mathfrak{v}$ and $\mathcal{F}(\rho, \omega) = \rho\omega;$
(d) $h(\mathfrak{u}, \mathfrak{v}) = \mathfrak{v}$ and $\mathcal{F}(\rho, \omega) = \omega;$
(e) $h(\mathfrak{u}, \mathfrak{v}) = \frac{1}{n+1}\left(\sum_{i=0}^{\mathfrak{n}} \mathfrak{u}^{i}\right)\mathfrak{v}$ and $\mathcal{F}(\rho, \omega) = \rho\omega;$
(f) $h(\mathfrak{u}, \mathfrak{v}) = \left[\frac{1}{n+1}\left(\sum_{i=0}^{\mathfrak{n}} \mathfrak{u}^{i}\right) + l\right]^{\mathfrak{v}}$ and $\mathcal{F}(\rho, \omega) = (1 + l)^{\rho\omega}.$

Definition 3.6. *[7] The continuous function $h_1 : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function of $1-1-subclass$, if $\mathfrak{u}, \mathfrak{v} \geq 1 \implies h_1(1, 1, \mathfrak{w}) \leq h_1(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}), \ \mathfrak{w} \in \mathbb{R}^+.$*
A function of $1 - 1$ subclass is reintroduced in Ansari and Shukla [10] as a function of subclass of type II .

Example 3.7. *[7], [10] For* $\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}, k \in \mathbb{N},\ \mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in \mathbb{R}^+,\ \text{subsequent functions } h \colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *are functions of* $1 - 1-\text{subclass (subclass of type II)}:$

(a) $h(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = (\mathfrak{w}+l)^{\mathfrak{u}\mathfrak{v}};$
(b) $h(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = (\mathfrak{u}\mathfrak{v}+l)^{\mathfrak{w}};$
(c) $h(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = \mathfrak{w};$
(d) $h(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = \mathfrak{u}^{\mathfrak{m}}\mathfrak{v}^{\mathfrak{n}}\mathfrak{w}^{\mathfrak{p}};$
(e) $h(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = \frac{\mathfrak{u}^{\mathfrak{m}}+\mathfrak{u}^{\mathfrak{n}}\mathfrak{v}^{\mathfrak{p}}+\mathfrak{v}^{\mathfrak{q}}}{3}\mathfrak{w}^{k};$
(f) $h(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = \mathfrak{u}\mathfrak{v}\mathfrak{w};$
(g) $h(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = (\frac{\mathfrak{u}+\mathfrak{v}}{2})\mathfrak{w}.$

Definition 3.8. *[7](see also [10]) Let* $h_1, \mathcal{F}_1 : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R},$ *then* $\mathcal{F}_1$ *is a* $1-\text{up-class, if } h_1$ *is a function of* $1 - 1$ *subclass (subclass of class II) and:*

1. $0 \leq \rho \leq 1 \Longrightarrow \mathcal{F}_1(\rho, \omega) \leq \mathcal{F}_1(1, \omega);$
2. $h_1(1, 1, \mathfrak{w}) \leq \mathcal{F}_1(\rho, \omega) \Longrightarrow \mathfrak{w} \leq \rho\omega,\ \rho, \omega, \mathfrak{w} \in \mathbb{R}^+.$

Ansari and Shukla [10] called a pair $(\mathcal{F}_1, h_1)$ *to be an upper-class of type II.*

Example 3.9. *[7], [10] For* $\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}, l, k \in \mathbb{N},\ \mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in \mathbb{R}^+,\ l > 1,\ \text{subsequent functions } h \colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *are functions of* $1 - 1$ *subclass (subclass of type II):*

(a) $h_1(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = (\mathfrak{w} + l)^{\mathfrak{u}\mathfrak{v}},\ \mathcal{F}_1(\mathfrak{u}, \mathfrak{v}) = \mathfrak{u}\mathfrak{v} + l;$
(b) $h_1(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = (\mathfrak{u}\mathfrak{v} + l)^{\mathfrak{w}},\ \mathcal{F}_1(\mathfrak{u}, \mathfrak{v}) = (1 + l)^{\mathfrak{u}\mathfrak{v}};$
(c) $h_1(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = \mathfrak{w},\ \mathcal{F}_1(\mathfrak{u}, \mathfrak{v}) = \mathfrak{u}\mathfrak{v};$
(d) $h_1(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = \mathfrak{u}^{\mathfrak{m}}\mathfrak{v}^{\mathfrak{n}}\mathfrak{w}^{\mathfrak{p}},\ \mathcal{F}_1(\mathfrak{u}, \mathfrak{v}) = \mathfrak{u}^{\mathfrak{p}}\mathfrak{v}^{\mathfrak{p}};$
(e) $h_1(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = \frac{\mathfrak{u}^{\mathfrak{m}}+\mathfrak{u}^{\mathfrak{n}}\mathfrak{v}^{\mathfrak{p}}+\mathfrak{v}^{\mathfrak{q}}}{3}\mathfrak{w}^{k},\ \mathcal{F}_1(\mathfrak{u}, \mathfrak{v}) = \mathfrak{u}^{k}\mathfrak{v}^{k};$
(f) $h_1(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = \mathfrak{u}\mathfrak{v}\mathfrak{w},\ \mathcal{F}_1(\mathfrak{u}, \mathfrak{v}) = \mathfrak{u}\mathfrak{v};$
(g) $h_1(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = (\frac{\mathfrak{u}+\mathfrak{v}}{2})\mathfrak{w},\ \mathcal{F}_1(\mathfrak{u}, \mathfrak{v}) = \mathfrak{u}\mathfrak{v}.$

Definition 3.10. *[7] Let* $\alpha : \boldsymbol{F} \times \boldsymbol{F} \to \mathbb{R}^+$ *and* $\boldsymbol{F} \subseteq \mathcal{U}.$ *A self-mapping* $\mathcal{A} : \mathcal{U} \to \mathcal{U}$ *is an* $\alpha_{\boldsymbol{F}}-\text{admissible if } \alpha(\mathfrak{u}, \mathfrak{v}) \geq 1 \Longrightarrow \alpha(\mathcal{A}\mathfrak{u},\ \mathcal{A}\mathfrak{v}) \geq 1,\ \mathfrak{u}, \mathfrak{v} \in\ \boldsymbol{F}.$

Remark 3.11. $\mathcal{A}$ *is* α*-admissible (see [44]) if* $\boldsymbol{F} = \mathcal{U}.$

Definition 3.12. *[7] Let* $\boldsymbol{F} \subseteq \mathcal{U}$ *and* $\alpha : \boldsymbol{F} \times \boldsymbol{F} \to \mathbb{R}^+.$ *A self -mapping* $\mathcal{A}$ *is an* α_{β}*-contractive mapping if*

$$h_1(\alpha(\mathfrak{u}, \mathcal{A}\mathfrak{u}), \alpha(\mathfrak{v}, \mathcal{A}\mathfrak{v}), \psi(d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v})) \leq \mathcal{F}_1\big(\beta(d(\mathfrak{u}, \mathfrak{v})), \psi(d(\mathfrak{u}, \mathfrak{v}))\big),\ \mathfrak{u}, \mathfrak{v} \in \boldsymbol{F}, \tag{3.1}$$

where, pair $(\mathcal{F}_1, h_1)$ *is an upper-class of type II (*h_1 *is a* $1 - 1$ *subclass,* $\mathcal{F}_1$ *is a* $1-\text{up-class of } h_1$*),* $\mathcal{A}$ *is* $\alpha_{\boldsymbol{F}}-\text{admissible, } \beta \in S$ *is as utilized by Husain et al. [23] and* $\psi \in \boldsymbol{\Psi}.$

Remark 3.13. 1. *If* $h_1(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = (\mathfrak{w} + l)^{\mathfrak{u}\mathfrak{v}}, l > 1,\ \psi(\mathfrak{t}) = \mathfrak{t},\ \boldsymbol{F} = \mathcal{U},$ *and* $\mathcal{F}_1(\mathfrak{u}, \mathfrak{v}) = \mathfrak{u}\mathfrak{v} + l,$ *then inequality (3.1) reduces to a contractive condition utilized in Theorem 4 of Husain et al. [23].*
 2. *If* $h_1(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = (\mathfrak{u}\mathfrak{v} + l)^{\mathfrak{w}},\ \psi(\mathfrak{t}) = \mathfrak{t},\ \boldsymbol{F} = \mathcal{U},$ *and* $\mathcal{F}_1(\mathfrak{u}, \mathfrak{v}) = (1 + l)^{\mathfrak{u}\mathfrak{v}}, l > 1,$ *then inequality (3.1) reduces to a contractive condition utilized in Theorem 6 of Husain et al. [23].*
 3. *If* $h_1(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = \mathfrak{u}\mathfrak{v}\mathfrak{w},\ \psi(\rho) = \rho,\ \alpha_{\boldsymbol{F}} = \mathcal{U},$ *and* $\mathcal{F}_1(\mathfrak{u}, \mathfrak{v}) = \mathfrak{u}\mathfrak{v},$ *then inequality (3.1) reduces to a contractive condition utilized in Theorem 8 of Husain et al. [23].*

4. *Ansari and Shukla [10] familiarized $F - (\mathcal{F}, h)-$contraction ((3.1) [10]), $F - (\mathcal{F}, h)-$weak contraction ((3.2) [10]), $F - (\mathcal{F}, h)-$generalized contraction ((3.3) [10]). On taking $h(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = \mathfrak{w}$ and $\mathcal{F}(\mathfrak{u}, \mathfrak{v}) = \mathfrak{u}\mathfrak{v}$, each $F-$generalized contraction (Shukla et al. [45] and [46]) is an $F - (\mathcal{F}, h)-$contraction.*

5. *Ansari and Shukla [10] familiarized $F - (\mathcal{F}, h)-$sub-contraction ((3.1) [10]), $F - (\mathcal{F}, h)-$sub-weak contraction ((3.2) [10]), $F - (\mathcal{F}, h)-$sub-generalized contraction ((3.3) [10]). On taking $h(\mathfrak{u}, \mathfrak{v}) = \mathfrak{v}$ and $\mathcal{F}(\mathfrak{u}, \mathfrak{v}) = \mathfrak{v}$, each $\mathcal{F}-$generalized contraction (Shukla et al. [45] and [46]) is an $F - (\mathcal{F}, h)-$sub-contraction.*

4. 1-1-up-class functions

Definition 4.1. *[9],[16] A pair of functions (ψ, ϕ), $\psi, \phi : [0, \infty) \to [0, \infty)$, is a pair of generalized altering distance if subsequent postulates hold true:*

(a1) ψ is continuous and non-decreasing ;

(a2) $\lim_{n \to \infty} \phi(\rho_n) = 0 \Rightarrow \lim_{\mathfrak{n} \to \infty} \rho_\mathfrak{n} = 0.$

Definition 4.2. *[16] A self-mapping $\mathcal{A} : \mathcal{U} \to \mathcal{U}$ is an $(\alpha\psi, \beta\phi)$-contractive mapping if there exists a pair of generalized altering distance (ψ, ϕ) satisfying*

$$\psi(d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v})) \leq \alpha(\mathfrak{u}, \mathfrak{v})\psi(d(\mathfrak{u}, \mathfrak{v})) - \beta(\mathfrak{u}, \mathfrak{v})\phi(d(\mathfrak{u}, \mathfrak{v})), \tag{4.1}$$

where, $\alpha, \beta : \mathcal{U} \times \mathcal{U} \to [0, \infty)$.

Remark 4.3. *For $\alpha(\mathfrak{u}, \mathfrak{v}) = \beta(\mathfrak{u}, \mathfrak{v}) = 1$, inequality 4.1 reduces to inequality 1.5 of Dutta and Choudhry [19].*

Definition 4.4. *[9] A mapping $f : [0, \infty)^4 \to \mathbb{R}$ is a 1-1-up-class function if the subsequent conditions hold for $\mathfrak{u}, \mathfrak{v}, \rho, \omega \in [0, \infty) :$*

1. *$f(1, 1, \rho, \omega)$ is continuous;*
2. *$0 \leq \mathfrak{u} \leq 1, \mathfrak{v} \geq 1 \Rightarrow f(\mathfrak{u}, \mathfrak{v}, \rho, \omega) \leq f(1, 1, \rho, \omega) \leq \rho;$*
3. *$f(1, 1, \rho, \omega) = \rho \Rightarrow \rho = 0 \text{ or } \mathfrak{t} = 0.$*

We symbolize the set of all functions of 1-1-up-class by $\mathcal{C}_1$.
Noticeably, $f(1, 1, 0, 0) = 0$.

Example 4.5. *[9] The subsequent functions $f : [0, \infty)^4 \to \mathbb{R}$ are elements of $\mathcal{C}_1$ for, $\mathfrak{u}, \mathfrak{v}, \rho, \mathfrak{t} \in [0, \infty)$, $\mathfrak{a} > 1$:*

1. *$f(\mathfrak{u}, \mathfrak{v}, \rho, \omega) = \mathfrak{u}\rho - \mathfrak{v}\omega;$*
2. *$f(\mathfrak{u}, \mathfrak{v}, \rho, \omega) = \dfrac{\mathfrak{u}\rho - \mathfrak{v}\omega}{1 + \mathfrak{v}\omega};$*
3. *$f(\mathfrak{u}, \mathfrak{v}, \rho, \omega) = \dfrac{\mathfrak{u}\rho}{1 + \mathfrak{v}\omega};$*
4. *$f_\mathfrak{a}(\mathfrak{u}, \mathfrak{v}, \rho, \omega) = \log_a \dfrac{\mathfrak{u}\omega + \mathfrak{a}^\mathfrak{u}\rho}{1 + \mathfrak{v}\omega}, \ \mathfrak{a} > 1;$*
5. *$f(\mathfrak{u}, \mathfrak{v}, \rho, \omega) = \ln \dfrac{\mathfrak{u} + e^{\mathfrak{u}\rho}}{1 + \mathfrak{v}};$*

6. $f_{\mathfrak{a}}(\mathfrak{u}, \mathfrak{v}, \rho, \omega) = (\mathfrak{u}\rho + \mathfrak{a})^{\frac{1}{1+\mathfrak{v}\omega}} - \mathfrak{a}, \ \mathfrak{a} > 1;$

7. $f_{\mathfrak{a}}(\mathfrak{u}, \mathfrak{v}, \rho, \omega) = \mathfrak{u}\rho \log_{\mathfrak{a}+\mathfrak{v}\omega} \mathfrak{a}, \ \mathfrak{a} > 1.$

Definition 4.6. *[9] A self-mapping $\mathcal{A}: \mathcal{U} \to \mathcal{U}$ is (CAB)-contractive mapping if there exists a pair of generalized altering function (ψ, ϕ), $\alpha, \beta: \mathcal{U} \times \mathcal{U} \to [0, \infty)$, $h \in \mathcal{A}$ and $f \in \mathcal{C}_1$ satisfying*

$$h(\psi(d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}))) \leq f(\alpha(\mathfrak{u}, \mathfrak{v}), \beta(\mathfrak{u}, \mathfrak{v}), \psi(d(\mathfrak{u}, \mathfrak{v})), \phi(d(\mathfrak{u}, \mathfrak{v}))), \mathfrak{u}, \mathfrak{v} \in \mathcal{U}. \tag{4.2}$$

Remark 4.7. *If $f(\mathfrak{u}, \mathfrak{v}, \rho, \omega) = \mathfrak{u}\rho - \mathfrak{v}\omega$ and $\mathfrak{h}(\omega) = \omega$, we get $(\alpha\psi, \beta\phi)-$contractive mapping [16].*

5. Cone $\mathcal{C}-$class functions

Huang and Zhang [22] familiarized cone metric spaces by utilizing ordered Banach space instead of the real numbers in metric spaces. Let $\mathcal{E}$ be a real Banach space, θ be the zero vector, and $\mathcal{P}$ a nonempty subset of $\mathcal{E}$. $\mathcal{P}$ is a cone if and only if:

(i) $\mathcal{P}$ is closed and $\mathcal{P} \neq \{\theta\}$;

(ii) $\mathfrak{a}\mathfrak{u} + \mathfrak{b}\mathfrak{v} \in \mathcal{P}$, $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}_0^+$ and $\mathfrak{u}, \mathfrak{v} \in \mathcal{P}$;

(iii) $\mathcal{P} \cap (-\mathcal{P}) = \{\theta\}$.

A partial ordering $\preceq$ with respect to $\mathcal{P} \subseteq \mathcal{E}$ is defined as $\mathfrak{u} \preceq \mathfrak{v}$ if and only if $\mathfrak{v} - \mathfrak{u} \in \mathcal{P}$. Here, $\mathfrak{u} \prec \mathfrak{v}$ is defined as $\mathfrak{u} \preceq \mathfrak{v}$ and $\mathfrak{u} \neq \mathfrak{v}$ and $\mathfrak{u} \ll \mathfrak{v}$ if $\mathfrak{v} - \mathfrak{u} \in int\mathcal{P}$, the interior of $\mathcal{P}$.

Definition 5.1. *[11] Let $\psi, \phi: int\mathcal{P} \cup \{\theta\} \to int\mathcal{P} \cup \{\theta\}$ be two monotone increasing and continuous functions satisfying:*
(a) $\psi(\omega) = \phi(\omega) = \theta$ if and only if $\omega = \theta$;
(b) $\omega - \psi(\omega) \in \mathcal{P} \cup \{\theta\}, \phi(\omega) \ll \omega, \omega \in int\mathcal{P}$.

Definition 5.2. *[11] A continuous mapping $\mathcal{F}: \mathcal{P}^2 \to \mathcal{P}$ is a function of cone C-class if subsequent axioms hold true:*

1. $\mathcal{F}(\rho, \omega) \preceq \rho$;
2. $\mathcal{F}(\rho, \omega) = \rho$ *implies that either $\rho = \theta$ or $\omega = \theta$, $\rho, \omega \in \mathcal{P}$.*

Ansari et al. [11] symbolized functions of cone $\mathcal{C}-$class as $\mathcal{C}_{co}$.

Example 5.3. *[11] The subsequent functions $\mathcal{F}: \mathcal{P}^2 \to \mathcal{P}$ are elements of $\mathcal{C}_{co}$, for $\rho, \omega \in \mathcal{P}$:*
(1) $\mathcal{F}(\rho, \omega) = \rho - \omega$;
(2) $\mathcal{F}(\rho, \omega) = k\rho, \ 0<k<1$;
(3) $\mathcal{F}(\rho, \omega) = \rho\beta(\rho), \ \beta: \mathcal{P} \to [0, 1)$;
(4) $\mathcal{F}(\rho, \omega) = \rho - \varphi(\rho)$, here $\varphi: \mathcal{P} \to \mathcal{P}$ is a continuous function so that $\varphi(\omega) = \theta \Leftrightarrow \omega = \theta$;
(5) $\mathcal{F}(\rho, \omega) = \rho - h(\rho, \omega)$, here $h: \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ is a continuous function and $h(\rho, \omega) = \theta \Leftrightarrow \omega = \theta, \omega, \rho \succ \theta$;
(6) $\mathcal{F}(\rho, \omega) = \psi(\rho), \ \psi: \mathcal{P} \to \mathcal{P}, \ \psi(0) = 0, \ \psi(\rho) > 0, \rho \in \mathcal{P}, \ \rho \neq 0$ and $\psi(\rho) \leq \rho$;
(7) $\mathcal{F}(\rho, \omega) = \phi(\rho)$, here $\phi: [0, \infty) \to [0, \infty)$ is a upper semi continuous function so that $\phi(0) = 0$ and $\phi(\omega) < \omega, \omega > 0$.

Ansari et al. [11] utilized the condition

$$\psi(d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v})) \preceq \mathcal{F}(\psi(d(\mathfrak{u}, \mathfrak{v})), \phi(d(\mathfrak{u}, \mathfrak{v}))), \tag{5.1}$$

where $\mathcal{F}$ is an element of $\mathcal{C}_{co}$; ψ and ϕ are as in Definition 5.1, $d(\mathfrak{u}, \mathfrak{v}) \in int\mathcal{P}$, and satisfy
(i) ψ is continuous and strongly monotone increasing $(\psi(\mathfrak{u}) \preceq \psi(\mathfrak{v}) \Leftrightarrow \mathfrak{u} \preceq \mathfrak{v})$,
(ii) either $d(\mathfrak{u}, \mathfrak{v}) \ll \phi(\mathfrak{t})$ or $\phi(\mathfrak{t}) \preceq d(\mathfrak{u}, \mathfrak{v})$, $\mathfrak{t} \in int\mathcal{P} \cup \{\theta\}$.

Remark 5.4.   1. *If $\mathcal{F}(\rho, \omega) = \varphi(\rho)$, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ so that $\varphi(\omega) < \omega, \omega > 0$, then inequality 5.1 reduces to contractive condition utilized in Theorem 3(iii) of Jachymski [24].*

2. *If $\psi(\omega) = \omega$ and $\mathcal{F}(\rho, \omega) = \rho - \omega$, then inequality 5.1 reduces to contractive condition utilized in Theorem 2.1 of Aage and Salunke [1].*

3. *If $\psi(\omega) = \omega$ and $\mathcal{F}(\rho, \omega) = \rho\beta(\rho)$, $\beta : \mathcal{P} \to [0, 1)$, then inequality 5.1 reduces to contractive condition utilized in Theorem 2.2 of Amini-Harandi and Fakhar [4].*

4. *If $\psi(\omega) = \omega$ and $\mathcal{F}(\rho, \omega) = k\rho$, $k \in [0, 1)$, then inequality 5.1 reduces to contractive condition utilized in Theorem 1 of Huang and Zhang [22] (Corollary 2.3 of Amini-Harandi and Fakhar [4]).*

Remark 5.5. *Because operators exponential, rational, logarithmic, etc. have no meaning in cone, so we can not utilize functions of cone $\mathcal{C}-$class freely in a cone metric space.*

6. Multiplicative $\mathcal{C}-$class functions

Definition 6.1. *[5] Let $\mathcal{H}$ symbolize the class of non-decreasing and continuous mappings $\psi : [1, \infty) \to [1, \infty)$ with $\psi(1) = 1$ and $\psi(\mathcal{F} : \mathcal{P}^2 \to \mathcal{P}) > 1$, $\omega > 1$.*

Definition 6.2. *Let $\mathcal{Q}$ symbolize the class of non-decreasing and lower semi-continuous from the right mapping $\varphi : [1, \infty) \to [1, \infty)$ with $\varphi(1) = 1$ and $\varphi(\omega) > 1$, $\omega > 1$.*

Definition 6.3. *[5] A continuous mapping $\mathcal{F} : [1, \infty)^2 \to \mathbb{R}$ is a function of multiplicative C-class if subsequent axioms hold true:*
   (a) $\mathcal{F}(\mathfrak{u}, \mathfrak{v}) \leq \mathfrak{u}$;
   (b) $\mathcal{F}(\mathfrak{u}, \mathfrak{v}) = \mathfrak{u}$ implies that either $\mathfrak{u} = 1$ or $\mathfrak{v} = 1$; $\mathfrak{u}, \mathfrak{v} \in [1, \infty)$.
   Ampadu and Ansari [5] symbolized the function of multiplicative $\mathcal{C}$-class as $\mathcal{C}_m$.

For simplicity, we write, $\mathcal{F}(\mathfrak{u}, \mathfrak{v}) := \mathcal{F}(\frac{\mathfrak{u}}{\mathfrak{v}})$.

Remark 6.4. *Noticeably, $\mathcal{F}(1, 1) = 1$. Also, regarding axiom (b) of Definition 6.3, the condition on $\mathfrak{v}$ is sufficient unless $\mathcal{F}(\mathfrak{u}, \mathfrak{v})$, is defined explicitly by means of $\mathfrak{u}$. As $\mathcal{F}(\mathfrak{u}, \mathfrak{v}) = \frac{\mathfrak{u}}{\mathfrak{v}}$ is a multiplicative function, not defined explicitly by means of $\mathfrak{u}$ and $\mathcal{F}(\mathfrak{u}, \mathfrak{v}) = u^k, k \in (0, 1)$ is a multiplicative function defined explicitly by means of $\mathfrak{u}$.*

Based on the recent work [2] we state the following proposition.

Proposition 6.5. *There is a bijective mapping between $\mathcal{C}_m$ and $\mathcal{C}$.*

Proof . For each $f \in \mathcal{C}$, consider, $\mathcal{F}(\mathfrak{u}, \mathfrak{v}) = e^{f(\ln \mathfrak{u}, \ln \mathfrak{v})}$, where $\mathfrak{u}, \mathfrak{v} \geq 1$, $\mathcal{F} \in \mathcal{C}_m$. Consider $f(\rho, \omega) = \ln[\mathcal{F}(e^\rho, e^\omega)]$, where $\rho, \omega \geq 0$, these show that there is a bijection between the $\mathcal{C}-$class function and the multiplicative $\mathcal{C}-$class function. $\square$

Example 6.6. *The following examples show a relation between $\mathcal{C}$ and $\mathcal{C}_\mathfrak{m}$:*

1. $f(\rho, \omega) = \rho - \omega \iff \mathcal{F}(\rho, \omega) = \frac{\rho}{\omega}$;
2. $f(\rho, \omega) = \mathfrak{m}\rho, \mathfrak{m} \in (0, 1) \iff \mathcal{F}(\rho, \omega) = \rho^\mathfrak{m}, \mathfrak{m} \in (0, 1)$;
3. $f(\rho, \omega) = \frac{\rho}{(1+\omega)^\mathfrak{r}}, \mathfrak{r} \in (0, \infty) \iff \mathcal{F}(\rho, \omega) = \rho^{e^{\frac{1}{(1+\ln \omega)^\mathfrak{r}}}}, \mathfrak{r} \in (0, \infty)$;
4. $f(\rho, \omega) = \ln(\frac{1+\mathfrak{a}^\rho}{2}), e > \mathfrak{a} > 1 \iff \mathcal{F}(\rho, \omega) = \frac{1+\mathfrak{a}^{\ln \rho}}{2}, e > \mathfrak{a} > 1$;

5. $f(\rho, \omega) = \rho - \frac{\omega}{k + \omega} \iff \mathcal{F}(\rho, \omega) = \frac{\rho}{\omega^{\frac{1}{1 + \ln \omega}}}$;

6. $f(\rho, \omega) = \frac{\rho}{(1 + \rho)^{\mathfrak{r}}}, \mathfrak{r} \in (0, \infty) \iff \mathcal{F}(\rho, \omega) = \rho^{e^{\frac{1}{(1 + \ln \rho)^{\mathfrak{r}}}}}$;

7. $f(\rho, \omega) = \ln(1 + \rho) \iff \mathcal{F}(\rho, \omega) = 1 + \ln \rho$.

Definition 6.7. *[5] Let f and g be self-mappings of a multiplicative metric space $(\mathcal{U}, m)$ and $\mathcal{F}$ be a multiplicative $C-$class function. The pair (f, g) is a $(\psi, \phi, \mathcal{F})-$weak multiplicative contraction if there exist $\psi \in \mathcal{H}$, $\phi \in \mathcal{Q}$ satisfying*

$$\psi(m(f\mathfrak{u}, f\mathfrak{v})) \leq \mathcal{F}(\frac{\psi(m(g\mathfrak{u}, g\mathfrak{v}))}{\phi(m(g\mathfrak{u}, g\mathfrak{v}))}), \ \mathfrak{u}, \mathfrak{v} \in \mathcal{U}.$$

Definition 6.8. *[5] Let f and g be self-mappings of a multiplicative metric space $(\mathcal{U}, m)$ and $\mathcal{F}$ be a multiplicative $C-$class function. The pair (f, g) is a generalized $(\psi, \phi, \mathcal{F})-$weak multiplicative contraction if there exist $\psi \in \mathcal{H}$, $\phi \in \mathcal{Q}$ satisfying*

$$\psi(m(f\mathfrak{u}, f\mathfrak{v})) \leq \mathcal{F}(\frac{\psi(\mathcal{M}(g\mathfrak{u}, g\mathfrak{v}))}{\phi(\mathcal{M}(g\mathfrak{u}, g\mathfrak{v}))}), \ \mathfrak{u}, \mathfrak{v} \in \mathcal{U},$$

where, $\mathcal{M}(\mathfrak{u}, \mathfrak{v}) = \max\{m(g\mathfrak{u}, g\mathfrak{v}), m(g\mathfrak{u}, f\mathfrak{u}), m(g\mathfrak{v}, f\mathfrak{v}), (m(g\mathfrak{u}, f\mathfrak{v})m(g\mathfrak{v}, f\mathfrak{u}))^{\frac{1}{2}}\}$.

Noticeably, if $\mathcal{F}(\rho, \omega) := \mathcal{F}(\frac{\rho}{\omega}) = \rho - \omega$ is a $C-$class function [6], Theorems 3.1 and 3.3, Corollaries 3.2 and 3.4 contained in Moradi and Analoei [35] may be written in terms of this function.

7. Inverse-$\mathcal{C}$-class functions

Definition 7.1. *[42] A continuous mapping $F : [0, \infty)^2 \to \mathbb{R}$ is a function of inverse$-\mathcal{C}-$class if subsequent axioms hold true:*

(1) $\mathcal{F}(\rho, \omega) \geq \rho$;

(2) $\mathcal{F}(\rho, \omega) = \rho$ *implies that either* $\rho = 0$ *or* $\omega = 0$; $\rho, \omega \in [0, \infty)$.

Noticeably, $\mathcal{F}(0, 0) = 0$.

Saleem et al. [42] symbolized the collection of functions of inverse $\mathcal{C}-$class as $\mathcal{C}_{inv}$.

Example 7.2. *[42] The subsequent functions $\mathcal{F} : [0, \infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}_{inv}$, for $l < \mathfrak{m} < \infty$, $\mathfrak{a} > 1, \omega > 0, \mathfrak{r}, \rho, \omega \in [0, \infty)$:*

1. $\mathcal{F}(\rho, \omega) = \rho + \omega$;
2. $\mathcal{F}(\rho, \omega) = \mathfrak{m}\rho$;
3. $\mathcal{F}(\rho, \omega) = \rho(1 + \omega)^{\mathfrak{r}}$;
4. $\mathcal{F}(\rho, \omega) = \log_a(\omega + \mathfrak{a}^\rho)(1 + \omega)$;
5. $\mathcal{F}(\rho, \omega) = \phi(\rho)$, *here* $\phi : [0, \infty) \to [0, \infty)$ *is a upper semi-continuous function so that* $\phi(0) = 0$, *and* $\phi(\omega) > \omega$;
6. $f(\rho, \omega) = \vartheta(\rho)$; $\vartheta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *is a generalized Mizoguchi-Takahashi type function.*

Saleem et al. [42] utilized the subsequent condition to establish fixed point

$$\psi(d(f\mathfrak{u}, f\mathfrak{v})) \geq F(\psi(m(\mathfrak{u}, \mathfrak{v})), \varphi(m(\mathfrak{u}, \mathfrak{v}))), \tag{7.1}$$

$m(\mathfrak{u}, \mathfrak{v}) = \min_0\{d(g\mathfrak{u}, g\mathfrak{v}), d(f\mathfrak{u}, g\mathfrak{u}), d(f\mathfrak{v}, g\mathfrak{v})\}$.

8. $\mathcal{C}_{\mathcal{F}}-$simulation functions

Definition 8.1. *[32] A simulation function is a mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ so that the subsequent axioms hold true:*

    (ζ_a) $\zeta(0, 0) = 0$;
    (ζ_b) $\zeta(\omega, \rho) < \rho - \omega$, $\omega, \rho > 0$;
    (ζ_c) if $\{\omega_n\}$, $\{\rho_n\}$ are sequences in $(0, \infty)$ so that $\lim_{n\to\infty} \omega_n = \lim_{n\to\infty} \rho_n > 0$, then
$$\limsup_{n\to\infty} \zeta(\omega_n, \rho_n) < 0.$$

Remark 8.2. *Noticeably, $\zeta(\mathfrak{r}, \mathfrak{r}) < 0$, $\mathfrak{r} > 0$.*

Example 8.3. *[23] If $\phi, \psi : [0, \infty) \to [0, \infty)$ are two continuous functions so that $\psi(\omega) = \phi(\omega) = 0 \iff \omega = 0$. Also, if $\psi(\omega) < \omega \le \phi(\omega)$ and $f, g : [0, \infty) \to (0, \infty)$ are two continuous functions so that $f(\omega, \rho) > g(\omega, \rho), \omega, \rho > 0$ then $\zeta_i : [0, \infty) \times [0, \infty) \to \mathbb{R}$, $i = 1, 2, 3$ are elements of a class of simulation functions*

    1. $\zeta_1(\omega, \rho) = \psi(\rho) - \phi(\omega)$;
    2. $\zeta_2(\omega, \rho) = \rho - \mathfrak{f}(\omega, \rho)\omega$;
    3. $\zeta_3(\omega, \rho) = \rho - \phi(\rho) - \omega$.

Khojasteh et al. [32] symbolized the set of simulation functions by means of $\mathcal{Z}$.

(ζ_c) necessitate the symmetry in both the arguments of ζ, however, this is not essentially required while proving a result. Moreover, the arguments of ζ have distinct meanings and perform a distinct role in practical problems. Liu et al. [33] improved the definition of simulation function to generalize and extend the class of simulation functions, familiarized by Khojasteh et al. [32] utilizing a function of $\mathcal{C}-$class. Roldán-Loópez-de Hierro et al. [40] included $\omega_n < \rho_n$ in the definition of Khojasteh et al. [32].

Definition 8.4. *[33] A mapping $\mathcal{F} : [0, \infty)^2 \to \mathbb{R}$ have a property $\mathcal{C}_{\mathcal{F}}$, if there exists a $\mathcal{C}_{\mathcal{F}} \ge 0$ satisfying*

    (1) $\mathcal{F}(\rho, \omega) > \mathcal{C}_{\mathcal{F}} \implies \rho > \omega$;
    (2) $\mathcal{F}(\rho, \omega) \le \mathcal{C}_{\mathcal{F}}$, $\omega \in [0, \infty)$.

Example 8.5. *[33] The subsequent functions $\mathcal{F} : [0, \infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$ having property $\mathcal{C}_{\mathcal{F}}$, $\rho, \omega \in [0, \infty)$:*

$$
\begin{aligned}
(1)\ \mathcal{F}(\rho, \omega) &= \rho - \omega, C_F = \mathfrak{r},\ \mathfrak{r} \in [0, \infty); \\
(2)\ \mathcal{F}(\rho, \omega) &= \frac{\rho}{(1 + \omega)^{\mathfrak{r}}},\ \mathfrak{r} \in (0, \infty),\ \mathcal{C}_{\mathcal{F}} = 1; \\
(3)\ \mathcal{F}(\rho, \omega) &= \frac{\rho}{1 + \mathfrak{m}\omega},\ \mathfrak{m} \ge 1,\ \mathcal{C}_{\mathcal{F}} = \frac{\mathfrak{r}}{1 + \mathfrak{m}},\ \mathfrak{r} \in [2, \infty); \\
(4)\ \mathcal{F}(\rho, \omega) &= (\rho + \mathfrak{l})^{\frac{1}{1 + A}} - \mathfrak{l},\ \mathfrak{l} > 1,\ \mathcal{C}_{\mathcal{F}} = 0, 1; \\
(5)\ \mathcal{F}(\rho, \omega) &= \rho - (\frac{2 + \omega}{1 + \omega})\omega;\ \mathcal{C}_{\mathcal{F}} = 0; \\
(6)\ \mathcal{F}(\rho, \omega) &= \frac{\mathfrak{m}\rho}{1 + \omega}; 0 < \mathfrak{m} < 1,\ \mathcal{C}_{\mathcal{F}} = \mathfrak{m}, 1; \\
(7)\ \mathcal{F}(\rho, \omega) &= \frac{\mathfrak{m}\rho}{1 + \mathfrak{m}\omega},\ \mathfrak{m} > 0,\ \mathcal{C}_{\mathcal{F}} = \frac{\mathfrak{m} + 1}{\mathfrak{m}}, 1; \\
(8)\ \mathcal{F}(\rho, \omega) &= \frac{\rho}{1 + \omega},\ \mathcal{C}_{\mathcal{F}} = 1, 2.
\end{aligned}
$$

Definition 8.6. *[33] A $\mathcal{C}_{\mathcal{F}}-$simulation function is a mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ so that the subsequent conditions hold true.*

(ζ_a) $\zeta(0, 0) = 0$;

(ζ_b) $\zeta(\omega, \rho) < \mathcal{F}(\rho, \omega)$, $\rho, \omega > 0$, function $\mathcal{F} : [0, \infty)^2 \longrightarrow \mathbb{R}$ is an element of $C-$class having property $\mathcal{C}_{\mathcal{F}}$;

(ζ_c) if $\{\omega_n\}, \{\rho_n\}$ are sequences in $(0, \infty)$ so that $\lim_{n\to\infty} \omega_n = \lim_{n\to\infty} \rho_n > 0$, and $\omega_n < \rho_n$, then $\limsup\limits_{n\to\infty} \zeta(\omega_n, \rho_n) < \mathcal{C}_{\mathcal{F}}$.

Let $\mathcal{Z}_{\mathcal{F}}$ symbolizes the class of $\mathcal{C}_{\mathcal{F}}-$simulation functions.

Example 8.7. *[33] Let $l, \mathfrak{m} \in \mathbb{R}$ and $l, \mathfrak{m}, k < 1$, then the subsequent functions $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ are elements of $\mathcal{Z}_{\mathcal{F}}$:*

1. $\zeta(\omega, \rho) = k\mathcal{F}(\rho, \omega) - \omega$;
2. $\zeta(\omega, \rho) = k\mathfrak{m}\rho - \omega$;
3. $\zeta(\omega, \rho) = k\mathcal{F}(\rho, \omega)$;
4. $\zeta(\omega, \rho) = \mathcal{F}(\psi(\rho), \phi(\omega)) - \omega$.

One may validate that these simulation functions have property $\mathcal{C}_{\mathcal{F}}$.

Example 8.8. *[40] Let $k \in \mathbb{R}$, $k < 1$ and $\zeta : [0, \infty) \times [0, \infty) \to R$ be a function defined as*

$$\zeta(\omega, \rho) \Longrightarrow \begin{cases} 2(\rho - \omega), & if \quad \rho < \omega \\ k\rho - \omega, & otherwise. \end{cases}$$

Since,

$$\omega, \rho > 0 \quad , \quad \left\{ \begin{array}{l} 0 < \rho < \omega \Longrightarrow \zeta(\omega, \rho) = 2(\rho - \omega) < \rho - \omega \\ 0 < \omega < \rho \Longrightarrow \zeta(\omega, \rho) = k\rho - \omega < \rho - \omega \end{array} \right\}, \quad \zeta \ validates(\zeta_a) \ and \ (\zeta_b).$$

Let $\{\omega_n\}$ and $\{\rho_n\}$ be sequences in $(0, \infty)$ so that $\lim_{n\to\infty} \omega_n = \lim_{n\to\infty} \rho_n = \delta > 0$ and $\omega_n < \rho_n$, $n \in \mathbb{N}$, then

$$\limsup_{n\to\infty} \zeta(\omega_n, \rho_n) = \limsup_{n\to\infty} (k\rho_n - \omega_n) = (k - 1)\delta < 0.$$

Therefore, ζ is a simulation function [40]. If we assume $\omega_n = 2$ and $\rho_n = 2 - \frac{1}{n}$, $n \geq 1$, then we have

$$\limsup_{n\to\infty} \zeta(\omega_n, \rho_n) = \limsup_{n\to\infty} 2[(2 - \frac{1}{n}) - 2] = \limsup_{n\to\infty} \frac{-2}{n} = 0,$$

that is, ζ does not verify axiom (ζ_c) in Definition 8.1.

Each simulation function is also a $\mathcal{C}_{\mathcal{F}}-$simulation function but the reverse is not essentially true.

Definition 8.9. *[33] Let $\mathcal{A}$ and g be self-mappings of a metric space $(\mathcal{U}, d)$. A mapping $\mathcal{A}$ is $(\mathcal{Z}_{\mathcal{F}}, g)$-contraction if there exists $\zeta \in \mathbf{Z}_{\mathcal{F}}$ satisfying*

$$\zeta(d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}), d(g\mathfrak{u}, g\mathfrak{v})) \geq \mathcal{C}_{\mathcal{F}}, \tag{8.1}$$

$g\mathfrak{u} \neq g\mathfrak{v}$, $\mathfrak{u}, \mathfrak{v} \in \mathcal{U}$.

$(\mathcal{Z}_{\mathcal{F}}, g)-$contraction is also symbolized as $(\mathcal{Z}_{\mathcal{F},d}, g)-$contraction with respect to ζ, to highlight the involvement of a metric d.

Remark 8.10. *If g is the identity mapping on $\mathcal{U}$, $\mathcal{A}$ is a $\mathcal{Z}_{\mathcal{F},d}-$contraction with respect to ζ.*

Now if $\mathcal{C}_{\mathcal{F}} = 0$, we have the subsequent Definition.

Definition 8.11. *[40] A mapping $\mathcal{A}$ is $(\mathcal{Z}_d, g)-$contraction if there exists $\zeta \in \mathbf{Z}$ satisfying*

$$\zeta(d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}), d(g\mathfrak{u}, g\mathfrak{v})) \geq 0,$$

so that $g\mathfrak{u} \neq g\mathfrak{v}$, $\mathfrak{u}, \mathfrak{v} \in \mathcal{U}$.

Remark 8.12.   1. *[33] Noticeably, $\zeta(\mathfrak{r}, \mathfrak{r}) < \mathcal{C}_{\mathcal{F}}, \mathfrak{r} > 0$ in Definition 8.6.*
   2. *[33] If $\mathcal{A}$ is $(\mathcal{Z}_{\mathcal{F},d}, g)-$contraction with respect to $\zeta \in \mathcal{Z}_{\mathcal{F}}$, then*

$$d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}) < d(g\mathfrak{u}, g\mathfrak{v})), \ \ g\mathfrak{u} \neq g\mathfrak{v}.$$

   3. *[40] If $\mathcal{A}$ is $(\mathcal{Z}_d, g)-$contraction with respect to $\zeta \in \mathcal{Z}$, then*

$$d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}) < d(g\mathfrak{u}, g\mathfrak{v})), \ \ g\mathfrak{u} \neq g\mathfrak{v}.$$

   4. *$\mathcal{A}$ is $\mathcal{Z}_d-$contraction with respect to $\zeta \in \mathbf{Z}$ and $g = I$, then*

$$d(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}) < d(\mathfrak{u}, \mathfrak{v})), \ \ \mathfrak{u} \neq \mathfrak{v}.$$

9. $\mathcal{C}^*-$class function

In 2019, Chandok et al. [17] familiarized $\mathcal{C}^*-$class function to establish a fixed point. Let $\mathbb{A}$ denote an unital $\mathcal{C}^*-$algebra with a unit I. A partial ordering on the elements of $\mathbb{A}$ is defined as, $\mathfrak{b} \succcurlyeq \mathfrak{a} \implies \mathfrak{b} - \mathfrak{a} \in \mathbb{A}^+$, the cone of positive elements in $\mathbb{A} \implies \mathfrak{b} - \mathfrak{a} \succcurlyeq \theta$, the zero element in $\mathbb{A}$. If $\mathfrak{u} \in \mathbb{A}$ is of the form $\mathfrak{v}\mathfrak{v}^*$, $\mathfrak{v} \in \mathbb{A}$, then $\mathfrak{u}$ is a positive element in $\mathbb{A}$ [20].

Definition 9.1. *A continuous function $F^* : \mathbb{A}^+ \times \mathbb{A}^+ \to \mathbb{A}$ is a function of $\mathcal{C}^*-$class, if subsequent axioms hold true:*
   (1) $\mathcal{F}^*(\mathcal{A}, \mathcal{B}) \preccurlyeq \mathcal{A}$;
   (2) $\mathcal{F}^*(\mathcal{A}, \mathcal{B}) = \mathcal{A}$ implies that either $\mathcal{A} = \theta$ or $\mathcal{B} = \theta$, $\mathcal{A}, \mathcal{B} \in \mathbb{A}^+$.
Noticeably, $\mathcal{F}^*(\theta, \theta) = \theta$. A $\mathcal{C}^*-$class of functions is symbolized as $\mathbf{C}^*$.

Example 9.2. *[48] Subsequent functions $\mathcal{F}^* : \mathbb{A}^+ \times \mathbb{A}^+ \to \mathbb{R}$ are elements of $\mathcal{C}^*-$class*

   1. $\mathcal{F}^*(\mathcal{A}, \mathcal{B}) = \mathcal{A} - \mathcal{B}$;
   2. $\mathcal{F}^*(\mathcal{A}, \mathcal{B}) = \frac{\mathcal{B} - \mathcal{A}}{\mathcal{I} + \mathcal{A}}$;
   3. $\mathcal{F}^*(\mathcal{A}, \mathcal{B}) = \frac{\mathcal{A}}{\mathcal{I} + \mathcal{B}}$;
   4. $\mathcal{F}^*(\mathcal{A}, \mathcal{B}) = \log(\frac{\mathcal{B} + \mathcal{M}^{\mathcal{A}}}{\mathcal{I} + \mathcal{B}})$, $\|\mathcal{M}\| > 1$;
   5. $\mathcal{F}^*(\mathcal{A}, \mathcal{B}) = (\mathcal{A} + \mathcal{I})^{\frac{\mathcal{I}}{\mathcal{I} + \mathcal{B}}}$;
   6. $\mathcal{F}^*(\mathcal{A}, \mathcal{B}) = \mathcal{A} \log_{\mathcal{M} + \mathcal{B}} \mathcal{M}$, $\|\mathcal{M}\| > 1$;
   7. $\mathcal{F}^*(\mathcal{A}, \mathcal{B}) = \mathcal{K}\mathcal{A}$, $\theta < \|\mathcal{K}\| < 1$.

Remark 9.3. *[17] If $\mathbb{A} = \mathbb{C}$ in above Definition 9.1, then it symbolizes the set of complex $\mathcal{C}-$class functions familiarized by Ansari [6].*

Definition 9.4. *[17] Let $\boldsymbol{\Psi}$ be the set of continuous functions $\psi : \mathbb{A}^+ \to \mathbb{A}^+$ satisfying:*

1. ψ is nondecreasing and continuous;
2. $\psi(\omega) = \theta \iff \omega = \theta$.

Definition 9.5. *[17] A self-mapping $\mathcal{A}$ of a $\mathcal{C}^*-$algebra valued partial metric space $(\mathcal{U}, \mathbb{A}, p)$ is a contractive mapping if*

$$\psi(p(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v})) \preccurlyeq \mathcal{F}^*(\psi(p(\mathfrak{u}, \mathfrak{v}), \phi(p(\mathfrak{u}, \mathfrak{v}))), \qquad (9.1)$$

$\psi, \phi \in \boldsymbol{\Psi}$ and $\mathcal{F}^* \in \mathbf{C}^*$.

Remark 9.6. *If $\mathcal{F}^*(\mathcal{A}, \mathcal{B}) = \mathcal{A} - \mathcal{B}$, then inequality 9.1 reduces to a contractive condition utilized in Corollary 3.2 of Chandok et al. [17]. For $\mathcal{C}^*-$algebra valued metric space, refer to Ma et al. [34] (see also, Tomar and Joshi [47]).*

Definition 9.7. *[48] A self-mapping $\mathcal{A}$ of a $\mathcal{C}^*-$algebra valued partial metric space $(\mathcal{U}, \mathbb{A}, p)$ is a $\mathcal{C}^*-$algebra valued Chatterjea-type contractive mapping if*

$$\psi(p(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v})) \preccurlyeq \mathcal{F}^*(\psi(p(\mathfrak{u}, \mathcal{A}\mathfrak{v}) + p(\mathfrak{v}, \mathcal{A}\mathfrak{u})), \phi(p(\mathfrak{u}, \mathcal{A}\mathfrak{v}) + p(\mathfrak{v}, \mathcal{A}\mathfrak{u}))), \qquad (9.2)$$

$\|\mathcal{K}\| < \frac{1}{2}$, $\psi, \phi \in \boldsymbol{\Psi}$ and $\mathcal{F}^* \in \mathbf{C}^*$.

Remark 9.8. *On taking the different elements of $\mathcal{C}^*-$class function extended and improved versions of diverse contractions present in the literature may originate. For details, one may refer to Tomar et al. [48].*

1. $\psi(p(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}) \preceq \psi(\mathcal{K}[p(\mathfrak{u}, \mathcal{A}\mathfrak{v}) + p(\mathfrak{v}, \mathcal{A}\mathfrak{u})] - \phi(K[p(\mathfrak{u}, \mathcal{A}\mathfrak{v}) + p(\mathfrak{v}, \mathcal{A}\mathfrak{u})]$, taking $\mathcal{F}^*(\mathcal{A}, \mathcal{B}) = \mathcal{A} - \mathcal{B}$;
2. $\psi(p(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}) \preceq \frac{\phi(\mathcal{K}[p(\mathfrak{u},\mathcal{A}\mathfrak{v})+p(\mathfrak{v},\mathcal{A}\mathfrak{u})] - \psi(\mathcal{K}[p(\mathfrak{u},\mathcal{A}\mathfrak{v})+p(\mathfrak{v},\mathcal{A}\mathfrak{u})]}{\mathcal{I} + \psi(\mathcal{K}[p(\mathfrak{u},\mathcal{A}\mathfrak{v})+p(\mathfrak{v},\mathcal{A}\mathfrak{u})]}$, taking $\mathcal{F}^*(\mathcal{A}, \mathcal{B}) = \frac{\mathcal{B} - \mathcal{A}}{\mathcal{I} + \mathcal{A}}$;
3. $\psi(p(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}) \preceq \frac{\psi(\mathcal{K}[p(\mathfrak{u},\mathcal{A}\mathfrak{v})+p(\mathfrak{v},\mathcal{A}\mathfrak{u})]}{\mathcal{I} + \phi(\mathcal{K}[p(\mathfrak{u},\mathcal{A}\mathfrak{v})+p(\mathfrak{v},\mathcal{A}\mathfrak{u})]}$, taking $\mathcal{F}^*(\mathcal{A}, \mathcal{B}) = \frac{\mathcal{A}}{\mathcal{I} + \mathcal{B}}$;
4. $\psi(p(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}) \preceq \log \frac{\phi(\mathcal{K}[p(\mathfrak{u},\mathcal{A}\mathfrak{v})+p(\mathfrak{v},\mathcal{A}\mathfrak{u})] + \mathcal{M}^{\psi(\mathcal{K}[p(\mathfrak{u},\mathcal{A}\mathfrak{v})+p(\mathfrak{v},\mathcal{A}\mathfrak{u})]}}{\mathcal{I} + \psi(\mathcal{K}[p(\mathfrak{u},\mathcal{A}\mathfrak{v})+p(\mathfrak{v},\mathcal{A}\mathfrak{u})]}$, taking $\mathcal{F}^*(\mathcal{A}, \mathcal{B}) = \log \frac{\mathcal{B} + \mathcal{M}^{\mathcal{A}}}{\mathcal{I} + \mathcal{B}}$, $\|\mathcal{M}\| > 1$;
5. $\psi(p(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v})) \preceq (\psi(\mathcal{K}[p(\mathfrak{u}, \mathcal{A}\mathfrak{v}) + p(\mathfrak{v}, \mathcal{A}\mathfrak{u})] + \mathcal{I})^{\frac{\mathcal{I}}{\mathcal{I}+\phi(K[p(\mathfrak{u},\mathcal{A}\mathfrak{v})+p(\mathfrak{v},\mathcal{A}\mathfrak{u})]}}$, taking $\mathcal{F}^*(\mathcal{A}, \mathcal{B}) = (\mathcal{A} + \mathcal{I})^{\frac{\mathcal{I}}{\mathcal{I}+\mathcal{B}}}$;
6. $\psi(p(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v})) \preceq \psi(\mathcal{K}[p(\mathfrak{u}, \mathcal{A}\mathfrak{v}) + p(\mathfrak{v}, \mathcal{A}\mathfrak{u})] \log_{\mathcal{M}+\phi(K[p(\mathfrak{u},\mathcal{A}\mathfrak{v})+p(\mathfrak{v},\mathcal{A}\mathfrak{u})]} \mathcal{M}$, taking $\mathcal{F}^*(\mathcal{A}, \mathcal{B}) = \mathcal{A} \log_{\mathcal{M}+\mathcal{B}} \mathcal{M}$, $\quad \|\mathcal{M}\| > 1$;
7. $\psi(p(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v})) \preceq \mathcal{K}\psi(\mathcal{K}[p(\mathfrak{u}, \mathcal{A}\mathfrak{v}) + p(\mathfrak{v}, \mathcal{A}\mathfrak{u})])$, taking $\mathcal{F}^*(\mathcal{A}, \mathcal{B}) = \mathcal{K}\mathcal{A}$, $0 < \|\mathcal{K}\| < 1$.

10. Conclusion and future Work

We have discussed functions of $\mathcal{C}-$class, upper-class and their variants and demonstrated that the functions of $\mathcal{C}-$class, pair (h, f) upper-class, cone $\mathcal{C}-$class, $1 - 1-$up-class, multiplicative $\mathcal{C}-$class, inverse$-\mathcal{C}-$class, $\mathcal{C}_{\mathcal{F}}-$simulation, $\mathcal{C}^*-$class, that envelope enormous classes of contractive conditions, are powerful and fascinating weapons for the generalization, improvement, and extension of outstanding results (for instance, [14], [15],[18], [30], [41], [43], [51], and so on]) in the metric fixed point theory. It is illustrated that these functions subsume the majority of the celebrated and contemporary contractive conditions and perform a magnificent role in the expansion of fixed point theory. Noticeably, in most of the results, discussed in this paper, the fixed point is unique. However, if the mapping has non-unique fixed points, then the exploration of the geometry of fixed points is very natural and fascinating (see, [25]-[29], [36]-[38], [49]-[50], and so on). The examination of novel contractive conditions which ensure some closed figure to be fixed by self-mapping may be regarded as an upcoming problem which has significance in both theory and application. There are a lot of examples wherein

the set of fixed points of the self-mapping incorporate some geometric shape or in other words, a self mapping fixes some geometrical shape. For example, let us consider the usual metric space and the self-mapping $\mathcal{A} : \mathcal{U} \to \mathcal{U}$. Let $F(\mathcal{A})$ denotes the set of fixed points of $\mathcal{A}$, $\mathbb{C}$ and $\mathbb{R}$ denote the set of complex numbers and real numbers respectively.

1. If $\mathcal{U} = \mathbb{C}$ and $\mathcal{A}\mathfrak{u} = \frac{4}{\mathfrak{u}}$, $\mathfrak{u} \neq 0$, then the circle $\mathcal{C}(0, 2)$, centered at 0 and having a radius 2 unit is a fixed circle for the self-mapping $\mathcal{A}$. Here $F(\mathcal{A}) = \{\mathfrak{u} \in \mathbb{C} : |\mathfrak{u}| = 2\}$.

2. If $\mathcal{U} = \mathbb{C}$ and $\mathcal{A}\mathfrak{u} = \begin{cases} \mathfrak{u}, & \mathfrak{u} \in \mathcal{D}(0, 1) \\ 0, & \text{otherwise} \end{cases}$, then the disc $\mathcal{D}(0, 1)$, centered at 0 and having a radius 1 unit is a fixed disc for the self-mapping $\mathcal{A}$. Here $F(\mathcal{A}) = \{\mathfrak{u} \in \mathbb{C} : |\mathfrak{u}| \leq 1\}$.

3. If $\mathcal{U} = \mathbb{R}$ and $\mathcal{A}\mathfrak{u} = \begin{cases} -6, & \mathfrak{u} = -6 \\ 4, & \text{otherwise} \end{cases}$, then the ellipse having foci at -5 and 3 and length of semi major axis as 5 is $\mathcal{E}(-5, 3, 5) = \{-6, 4\}$, which is a fixed ellipse for the self-mapping $\mathcal{A}$. Here $F(\mathcal{A}) = \{-6, 4\}$.

4. Let $\mathcal{U} = \mathbb{R}$ and $\mathcal{A}\mathfrak{u} = \begin{cases} [-6, 4], & \mathfrak{u} \in \mathcal{E}_{\mathcal{D}}(-5, 3, 5) \\ 0, & \text{otherwise} \end{cases}$, then the elliptic disc having foci at -5 and 3 and length of semi major axis as 5 is $\mathcal{E}_{\mathcal{D}}(-5, 3, 5) = [-6, 4]$, which is a fixed elliptic disc for the self-mapping $\mathcal{A}$. Here $F(\mathcal{A}) = [-6, 4]$.

However, it is interesting to point out that this may not always be true, that is, there may exist mappings that map some geometric figure to itself but do not fix all the points of that particular figure. In view of the above discussion, we include some open problems in reference to $\mathcal{C}-$class and pair upper-class functions that may be of interest.

Open Problems:

1. Does the equation $\mathcal{A}\omega = \omega$ always has a unique solution via $\mathcal{C}-$class functions or pair upper-class functions (or any of its variants)?

2. What are the conditions under which a self-mapping via $\mathcal{C}-$class functions or pair upper-class functions (or any of its variants) has more than one fixed point?

3. What are the conditions on self-mapping via $\mathcal{C}-$class functions or pair upper-class functions that make a given figure to be the fixed figure?

4. What type of contraction via $\mathcal{C}-$class functions or pair upper-class functions have the set of non-unique fixed points, including some geometric figure (for instance, circle, disc, ellipse, elliptic disc)?

5. What are the applications of techniques utilized to establish unique and non-unique fixed point via $\mathcal{C}-$class functions or pair upper-class in real-world problems?

6. Can we extend $\mathcal{C}-$class functions or pair upper-class to recently developed variants of metric spaces?

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