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Counting of conjugacy classes in partial transformation semigroups

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Abstract

J.Koneiczny in [8] introduced the new notion $\sim_n$ notion of conjugacy in semigroups. In this paper, we count the number of conjugacy classes in Partial Transformation semigroup $\mathcal{P}(A)$ for an infinite set A with respect to $\sim_n$ notion of conjugacy.

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1. Introduction and Preliminaries

Let S is a semigroup and let $a, b \in S$. Then,

$$a \sim_n b \Leftrightarrow \exists\ u, v \in S^1 \text{ such that } au = ub, bv = va, a = ubv \text{ and } b = vau.$$

This relation is an equivalence relation in any semigroup and in a semigroup with zero it is not a universal relation. J. Konieczny in [7] introduced the $\sim_n$ notion of conjugacy in semigroups.

A *digraph* (or a *directed graph*) is a pair $\Pi = (A, R)$, where A is a non-empty set(finite or infinite) and R is a binary relation on A.

If $\sigma \in \mathcal{P}(A)$, then it can be represented by the digraph $\Pi(\sigma) = (A, R_\sigma)$, where for all $u, v \in t, (u, v) \in R_\sigma$ if and only if $u \in \text{dom}(\sigma)$ and $u\sigma = v$.

Let $\Pi_1 = (A_1, R_1)$ and $\Pi_2 = (A_2, R_2)$ be digraphs. A mapping α from A_1 to A_2 is called a *homomorphism* from Π_1 to Π_2 if for all $u, v \in A_1, (u, v) \in R_1$ implies $(u\alpha,\ v\alpha) \in R_2$. A partial

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mapping α from A_1 to A_2 is called a *partial homomorphism* from Π_1 to Π_2 if for all $u, v \in \text{dom}(\alpha)$, $(u, v) \in R_1$ implies $(u\alpha, v\alpha) \in R_2$.

A vertex $p \in A$ for which there exists no q in A such that $(p, q) \in R$ is called a *terminal vertex* of Γ. A vertex $p \in A$ is said to be initial vertex if there is no $q \in A$ for which $(q, p) \in R$ while as a vertex $p \in A$ is said to be a non initial vertex if $(q, p) \in R$ for some $q \in A$.

Let $\Pi_1 = (A_1, R_1)$ and $\Pi_2 = (A_2, R_2)$ be digraphs. A partial homomorphism α from A_1 to A_2 is called a *restrictive partial homomorphism* (or an *rp-homomorphism*) from Π_1 to Π_2 if it satisfies the following conditions:

(a) If $(u, v) \in R_1$, then $u, v \in \text{dom}(\alpha)$ and $(u\alpha, v\alpha) \in R_2$.
(b) If u is a terminal vertex in Π_1 and $u \in \text{dom}(\alpha)$, then $u\alpha$ is a terminal vertex in Π_2.

We say that Π_1 is *rp-homomorphic* to Π_2 if there is an rp-homomorphism from Π_1 to Π_2.

Throughout this paper by an rp-hom we shall mean an rp-homomorphism between any two digraphs and by hom we shall mean a homomorphism.

The next theorem provides a necessary and sufficient condition for two elements of $\mathcal{P}(A)$ to be $\sim_n$ related.

Theorem 1.1: [1, Theorem 2.1] *Let $S \leq \mathcal{P}(A)$ and $\sigma, \tau \in S$. Then $\sigma \sim_n \tau$ if and only if there are $\alpha, \beta \in S^1$ for which α is an rp-hom from $\Pi(\sigma)$ to $\Pi(\tau)$ and β is an rp-hom from $\Pi(\tau)$ to $\Pi(\sigma)$ with $q\alpha\beta = q$ for every non initial vertex q of $\Pi(\sigma)$ and $k\beta\alpha = k$ for every non initial vertex k of $\Pi(\tau)$.*

Definition 1.2: Let $\cdots, a_{-2}, a_{-1}, a_0, a_1, a_2, \cdots$ be pairwise distinct elements of A. Then,

(1) A $\sigma \in \mathcal{P}(A)$ is called a *cycle* of length k if $\sigma = (a_0 a_1 a_2 \cdots a_{k-1})$ where $(k \geq 1)$ i.e, $a_j = a_{j-1}\sigma$, $j = 1, 2, \cdots, k$ and $a_0 = a_{k-1}\sigma$ and we write it as

$$a_0 \to a_1 \to a_2 \to \cdots \to a_{k-1} \to a_0.$$

(2) A $\sigma \in \mathcal{P}(A)$ is called a *right ray* if $\sigma = [a_0\ a_1\ a_2 \cdots >$ i,e $a_j = a_{j-1}\sigma$, $j \geq 1$ and we write it as

$$a_0 \to a_1 \to a_2 \to \cdots \quad .$$

(3) A $\sigma \in \mathcal{P}(A)$ is called a *double ray* if $\sigma =< \cdots a_{-1}\ a_0\ a_1 \cdots >$ i.e, $a_j = a_{j-1}\sigma$, $j \in \mathbb{Z}$ and we write it as

$$\cdots \to a_{-1} \to a_0 \to a_1 \to a_2 \to \cdots \quad .$$

(4) A $\sigma \in \mathcal{P}(A)$ is called a *left ray*, if $\sigma =< \cdots a_2\ a_1\ a_0]$ i.e, $a_j\sigma = a_{j-1}$, $j \geq 1$ and we write it as

$$\cdots \to a_2 \to a_1 \to a_0.$$

(5) A $\sigma \in \mathcal{P}(A)$ is called a *chain* of length k if $\sigma = [a_0\ a_1\ a_2 \cdots a_k]$ i.e, $a_j = a_{j-1}\sigma$, $j = 1, 2, \cdots, k$ and we write it as

$$a_0 \to a_1 \to a_2 \to \cdots \to a_k.$$

Notation 1.3: For $\sigma \in \mathcal{P}(A)$, Span($\sigma$) denotes dom($\sigma$)$\cup$ im(σ).
For $\sigma \in \mathcal{P}(A)$ and $a \in A$, we will write $a\sigma = \diamond$ if and only if $a \notin \text{dom}(\sigma)$. We will also assume that $\diamond\sigma = \diamond$. With this notation it makes sense to write $u\sigma = v\tau$ or $u\sigma \neq v\tau$ ($\sigma, \tau \in \mathcal{P}(A)$, $u, v \in A$) even when $u \notin \text{dom}(\sigma)$ or $v \notin \text{dom}(\tau)$.

Definition 1.4: An element $\theta \in \mathcal{P}(A)$ is called *connected* if $\theta \neq 0$ and for all $a, b \in \text{span}(\theta)$, $a\theta^k = b\theta^m \neq \diamond$ for some $k, m \geq 0$.

Definition 1.5: If $(N, |)$ is a partially ordered subset of the set of positive integers with $m_1 < m_2 < m_3 \cdots$. Then

$$\text{sac}(N) = \{m_n \in N\colon \text{ for all } i < n, m_n \text{ is not a multiple of } m_i\}.$$

Let σ be in $P(A)$ and let N denotes the set of lengths of cycles in σ. The standard anti-chain of $(N, |)$ will be called the *cycle set* of σ and denoted by cs(σ).

Definition 1.6: Let $\theta \in \mathcal{P}(A)$ be connected. we will say that θ is of rro type if θ contains a maximal right ray but no cycles or double rays or left rays or maximal chains. We will say that θ is of cho type if θ contains a maximal chain but no cycles or rays.

Let $\theta \in \mathcal{P}(A)$ be connected such that θ has a maximal left ray or is of cho type. The unique terminal vertex of θ is called as the root of θ.

Definition 1.7: A unique function π defined on A with ordinals as values and R is a well founded relation such that for every $a \in A$,

$$\pi(a) = \sup\{\pi(b) + 1 : (b, a) \in R\}.$$

The ordinal $\pi(a)$ is called the rank of a in $< A, R >$ [10, Theorem 1.27].

Let $\theta \in \mathcal{P}(A)$ be connected of rro type and $\kappa = [x_0x_1x_2 \cdots >$ be a maximal right ray in θ. We denote by $< \kappa_n^\theta >_{n\geq 0}$ the sequence of ordinals such that

$$\kappa_n^\theta = \pi_\theta(u_n) \text{ for every } n \geq 0 \text{ where } \pi_\theta(u_n) \text{ is the rank of } u_n \text{ in } \theta\ .$$

Definition 1.8: Let $< p_n >_{n\geq 0}$ and $< q_n >_{n\geq 0}$ be sequences of ordinals. Then we say that $< q_n >$ *dominates* $< p_n >$ if there is $k \geq 0$ such that

$$q_{k+n} \geq p_n \text{ for every } n \geq 0.$$

Definition 1.9: Let C be a set of pairwise disjoint elements of $\mathcal{P}(A)$. The *join* of the elements of C denoted $\bigcup_{\gamma\in C} \gamma$ is an element of $\mathcal{P}(A)$ defined by

$$x(\bigcup_{\gamma\in C} \gamma) = \begin{cases} x\gamma & \text{if } x \in \text{dom}(\gamma) \text{ for some } \gamma \in C \\ \diamond & \text{otherwise.} \end{cases}$$

Proposition 1.10: [7, Proposition 3.5] *Let $\sigma \in \mathcal{P}(A)$ with $\sigma \neq 0$. Then there exists a unique set C of pairwise completely disjoint, connected transformations contained in σ such that $\sigma = \bigcup_{\gamma\in C} \gamma$.*

The elements of C in Proposition 1.10 are called as connected component of σ. Throughout this paper by c-component we shall mean connected component.

Theorem 1.11: [1, Theorem 3.23] *For any subsemigroup $S \leq \mathcal{P}(A)$, Let σ, $\tau \in \mathcal{P}(A)$. Then $\sigma \sim_n \tau$ in $\mathcal{P}(A)$ if and only if $\sigma = \tau = 0$ or σ, $\tau \neq 0$ and the following conditions are satisfied:*

(1) *$cs(\sigma) = cs(\tau)$.*

(2) *σ has a double ray but not a cycle if and only if τ has a double ray but not a cycle.*

(3) *If σ has a c-component γ which is of type rro, but no cycles or double rays,then τ has a c-component δ which is of type rro, but no cycles or double rays, and $<\zeta_n^\delta>$ dominates $<\eta_n^\gamma>$ for some maximal right rays η in γ and ζ in δ.*

(4) *If τ has a c-component δ which is of type rro, but no cycles or double rays,then σ has a c-component γ which is of type rro, but no cycles or double rays, and $<\eta_n^\gamma>$ dominates $<\zeta_n^\delta>$ for some maximal right rays ζ in δ and η in γ.*

(5) *σ has a maximal left ray if and only if τ has a maximal left ray.*

(6) *If σ has a c-component γ which is of cho type with root x_0, but no maximal left rays, then τ has a c-component δ which is of cho type with root y_0, but no maximal left rays, and $\pi_\gamma(x_0) \leq \pi_\delta(y_0)$.*

(7) *If τ has a c-component δ which is of cho type with root y_0, but no maximal left rays, then σ has a c-component γ which is of cho type with root x_0, but no maximal left rays, and $\pi_\delta(y_0) \leq \pi_\gamma(x_0)$.*

(8) *There is $\alpha, \beta \in S^1$ such that $y\alpha\beta = y$ for every non initial vertex y of $\Pi(\sigma)$ and $z\beta\alpha = z$ for every non initial vertex z of $\Pi(\tau)$*

2. Counting of Conjugacy Classes using $\sim_n$ notion of conjugacy

Using Theorem 1.11, we now count the conjugacy classes in $\mathcal{P}(A)$ for an infinite set A for the $\sim_n$ notion of conjugacy.

Notation 2.1: We use $\aleph$ notation for an infinite cardinal, that is, for an ordinal ϵ, we will write $\aleph_\epsilon$ for the cardinal indexed by ϵ. If $\aleph_\epsilon$ is viewed as an ordinal, we will consistently write ω_ϵ for it.

The next results are from [7] and shall be used to prove the main theorem.

Lemma 2.2: [7, Lemma 3.30] *Let $\sigma, \tau \in \mathcal{P}(A)$ be connected of rro type. Let μ be a maximal right ray in σ and κ be a maximal right ray in τ such that $< \kappa_p^\tau >$ dominates $< \mu_p^\sigma >$. Then for every maximal right ray μ_1 in σ and every maximal right ray κ_1 in τ $< (\kappa_1)_p^\tau >$ dominates $< (\mu_1)_p^\sigma >$.* □

Lemma 2.3: [7, Lemma 5.9] *Let $|A| = \aleph_\epsilon$ and let $\sigma \in \mathcal{P}(A)$ be of cho type with root p_0. Then $\pi(p_0) < \omega_{\epsilon+1}$.* □

Lemma 2.4: [7, Lemma 5.10] *Let $|A| = \aleph_\epsilon$. Then for every non zero ordinal $\nu < \omega_{\epsilon+1}$, there is $\sigma \in \mathcal{P}(A)$ of cho type with root p_0 such that $\pi(p_0) = \nu$.* □

Lemma 2.5: [7, Lemma 5.11] *Let $|A| = \aleph_\epsilon$ and let $< u_p >$ be an increasing sequence of ordinals $u_p < \omega_{\epsilon+1}$ such that $u_0 = 0$. Then there is $\sigma \in \mathcal{T}(A)$ of rro type with a maximal right ray μ such that $< \mu_p^\sigma >=< u_p >$.* □

Let $\aleph_{\epsilon+1}$ be a successor cardinal. Denote by $IS_{\omega_{\epsilon+1}}$ the set of increasing sequences $< p_n >$ of ordinals $p_n < \omega_{\epsilon+1}$ such that $u_0 = 0$. Define a relation $\approx$ on $IS_{\omega_{\epsilon+1}}$ is defined as

$$< p_n > \quad \approx \quad < q_n > \text{ if } < q_n > \text{ dominates } < p_n > \text{ and } < p_n > \text{ dominates } < q_n >.$$

$\approx$ is an equivalence relation on $IS_{\omega_{\epsilon+1}}$. By this $[< p_n >]_\approx$ we denote the equivalence class of $< p_n >$ and by $IS^{\approx}_{\omega_{\epsilon+1}}$ we mean the set of all equivalence classes of $\approx$.

Lemma 2.6: [7, Lemma 5.15] *For any successor cardinal* $\aleph_{\epsilon+1}$, $|IS_{\omega_{\epsilon+1}}| = \aleph_{\epsilon+1}^{\aleph_0}$ *and* $\aleph_{\epsilon+1} \leq |IS^{\approx}_{\omega_{\epsilon+1}}| \leq \aleph_{\epsilon+1}^{\aleph_0}$. □

Theorem 2.7: *Let A be an infinite set with $|A| = \aleph_\epsilon$. Then the number of conjugacy classes with respect to the $\sim_n$ notion of conjugacy in $\mathcal{P}(A)$ are*

(1) $max\{2^{\aleph_0}, \aleph_{\epsilon+1}\}$ containing a representative with a cycle, of which $\aleph_0$ have a connected representative.
(2) $2^{\aleph_\epsilon}$ containing a representative with a c-component of rro type, but no cycles, of which at least $\aleph_{\epsilon+1}$ and at most $\aleph_{\epsilon+1}^{\aleph_0}$ have a connected representative.
(3) $\aleph_{\epsilon+1}$ containing a representative with a c-component of cho type, but no cycles or c-components of rro type, of which $\aleph_{\epsilon+1}$ have a connected representative.

In total there are $2^{\aleph_\epsilon}$ conjugacy classes in $\mathcal{P}(A)$, of which at least $\aleph_{\epsilon+1}$ and at most $\aleph_{\epsilon+1}^{\aleph_0}$ have a connected representative.

Proof:

(1) Let $\sigma \in \mathcal{P}(A)$ and let $i_\sigma, j_\sigma \in \{0,1\}$ be defined by $i_\sigma = 1$ if there exists a c-component of σ of cho type with root p_0 so that $\pi_\kappa(p_0) = s(\sigma)$, otherwise $i_\sigma = 0$ and $j_\sigma = 1$ if σ has a double ray, otherwise $j_\sigma = 0$.
To prove (1), let $X = \{[\sigma]_n : \sigma \in \mathcal{P}(A)$ has a cycle $\}$. Let $X' = \{[\sigma]_n \in X : \sigma$ has no maximal left rays $\}$ and $X'' = \{[\sigma]_n \in X : \sigma$ has a maximal left rays$\}$. By part (6) of Theorem 1.11, we have $\{X', X''\}$ is a partition of X. Define $f' : X' \to \mathcal{P}(\mathbb{Z}_+) \times (\omega_{\epsilon+1}+1) \times \{0,1\}$ by $([\sigma]_n)f' = (cs(\sigma), s(\sigma), i_\sigma)$. Then by Theorem 1.11, we get f' is well defined and injectivity of f' follows from Lemma 2.3. Similar holds for the mapping $f'' : X'' \to \mathcal{P}(\mathbb{Z}_+) \times (\omega_{\epsilon+1}) \times \{0,1\}$ defined by $([\sigma]_n)f'' = (cs(\sigma), s(\sigma), i_\sigma)$. Therefore

$$|X'| \leq |\mathcal{P}(Z_+)| \cdot |\omega_{\epsilon+1}+1| \cdot 2 = 2^{\aleph_0} \cdot \aleph_{\epsilon+1} \cdot 2 = \max\{2^{\aleph_0}, \aleph_{\epsilon+1}\},$$

and the same holds for $|X''|$. Hence we have

$$\begin{aligned} |X| &= |X'| + |X''| \\ &\leq \max\{2^{\aleph^0}, \aleph_{\epsilon+1}\} + \max\{2^{\aleph^0}, \aleph_{\epsilon+1}\} \\ &= 2 \cdot \max\{2^{\aleph^0}, \aleph_{\epsilon+1}\} \\ &= \max\{2^{\aleph_0}, \aleph_{\epsilon+1}\}. \end{aligned}$$

Let U be the set of prime positive integers and $V \subseteq U$, let $\{\theta_v\}_{v\in V}$ be a collection of completely disjoint cycles θ_v having length v (since X is infinite such a collection exists.) Define $\tau_V \in \mathcal{P}(A)$ by $\tau_V = \bigcup_{v\in V} \theta_v$. For $V_1, V_2 \subseteq U$ with $V_1 \neq V_2$. Now by part (2) of Theorem 1.11, we have $(\tau_{V_1}, \tau_{V_2}) \notin \sim_n$. So, $|X| \geq |\mathcal{P}(U)| = 2^{\aleph_0}$. By Lemma 2.4, for every nonzero ordinal $\nu < \omega_{\epsilon+1}$, there is $\kappa_\nu \in \mathcal{P}(A)$ of cho type with root p_0 such that $\pi(p_0) = \nu$. For all non zero ordinals $\delta, \nu < \omega_{\epsilon+1}$ with $\delta \neq \nu$, we have $(\kappa_\delta, \kappa_\nu) \notin \sim_n$ by Theorem 1.11. It follows that $|X| \geq |\omega_{\epsilon+1}| = \aleph_{\epsilon+1}$. Hence $|X| \geq \max\{2^{\aleph_0}, \aleph_{\epsilon+1}\}$, and so $|X| = \max\{2^{\aleph_0}, \aleph_{\epsilon+1}\}$.
Let $X_1 = \{[\kappa]_n : \kappa \in \mathcal{P}(A)$ has a cycle and κ is connected $\}$. Fix a subset $A_0 = \{t_0, t_1, \cdots\}$ of A, and for every integer $p \geq 0$, define a cycle $\kappa_p = (t_0 t_1 \cdots t_{p-1}) \in \mathcal{P}(A)$. Then, by Proposition 1.1 and Theorem 1.11, we have $X_1 = \{[\kappa_0]_n, [\kappa_1]_n, [\kappa_2]_n, \cdots\}$, and so $|X_1| = \aleph_0$.

(2) Let $Q = \{[\sigma]_n : \sigma \in \mathcal{P}(A)$ contains a c-component of rro type with no cycles$\}$, and let $Q_1 \subseteq Q$ consisting of all conjugacy classes $[\kappa]_n \in Q$ such that κ is connected. Fix a double ray $\omega =< \cdots t_{-1}t_0t_1 \cdots >\in \mathcal{P}(A)$ and note that

$$Q_1 = \{[\kappa]_n : \kappa \in \mathcal{P}(A) \text{ of rro type}\} \cup\{[\omega]_n\}.$$

Let $Q_1' = \{[\kappa]_n : \kappa \in \mathcal{P}(A)$ is of rro type$\}$. For every $\kappa \in \mathcal{P}(A)$ of rro type, we fix a maximal right ray μ^κ in κ. Let $g : Q_1' \to IS^{\approx}_{\omega_{\epsilon+1}}$ be defined by $([\kappa]_n)g = [< \mu^\kappa_n >]_\approx$. Note that $< \mu^\kappa_n >\in IS_{\omega_{\epsilon+1}}$ by the Lemma 2.3. Suppose $[\kappa_1]_n, [\kappa_2] \in Q_1'$ with $[\kappa_1]_n = [\kappa_2]_n$ then by Theorem 1.11 and Lemma 2.2, the sequences $< \mu^{\kappa_1}_n >$ and $\mu^{\kappa_2}_n$ dominate each other, and so $[< \mu^{\kappa_1} >]_\approx = [< \mu^{\kappa_2}_n >]_\approx$. So, g is well defined. The injectivity of g follows from Part(4) Theorem 1.11 and by Lemma 2.5 g is surjective. Thus $|Q_1'| = IS^{\approx}_{\omega_{\epsilon+1}}$ and so by Lemma 2.6, $\aleph_{\epsilon+1} \leq |Q_1'| \leq \aleph_{\epsilon+1}^{\aleph_0}$. Then $\aleph_{\epsilon+1} \leq |Q_1'| \leq \aleph_{\epsilon+1}^{\aleph_0}$ since $|Q_1| = |Q_1'| + 1$. Clearly $|Q| \leq |\mathcal{P}(A)| = (\aleph_\epsilon + 1)^{\aleph_\epsilon} = 2^{\aleph_\epsilon}$. Let

$$Q' = \{[\sigma]_n \in Q : \sigma \text{ has no maximal left rays or double rays}\},$$
$$Q'' = \{[\sigma]_n \in Q : \sigma \text{ has a maximal left ray but no double rays}\}.$$

We will show $|Q'| \geq 2^{\aleph_\epsilon}$. Since $|Q_1'| \geq \aleph_{\epsilon+1}$, there is a collection $\{\kappa_\nu\}_{\nu<\omega_{\epsilon+1}}$ of transformations $\kappa_\nu \in \mathcal{P}(A)$ of rro type such that $(\kappa_\nu, \kappa_\delta) \notin\sim_n$ if $\nu \neq \delta$. Since $|\omega_\epsilon| = \aleph_\epsilon$ and $\aleph_\epsilon.\aleph_\epsilon = \aleph_\epsilon$, there is a partition $\{A_\nu\}_{\nu<\omega_\epsilon}$ of A such that $|A_\nu| = |A| = \aleph_\epsilon$ for every $\nu < \omega_\epsilon$. Let $\nu < \omega_\epsilon$. Since $|A_\nu| = |A|$, there is a bijection $h_\nu : A_\nu \to A$. So, we can use h_ν to find a "*copy*" of κ_ν in $\mathcal{P}(A_\nu)$. Let $\kappa'_\nu \in \mathcal{P}(A_\nu)$ be defined by

$$t\kappa'_\mu = t' \iff (th_\nu)\kappa_\nu = t'h_\nu \text{ for all } t, t' \in A_\nu.$$

Let $\nu, \delta < \omega_\epsilon$ with $\nu \neq \delta$. Then $(\kappa_\nu, \kappa_\delta) \notin\sim_n$, and so, by part (4) of Theorem 1.11 and Lemma 2.2, $(< \mu_n >, < \kappa_n >) \notin\approx$ for every maximal right ray $\mu \in \kappa_\nu$ and every maximal right ray $\kappa \in \kappa_\delta$. Therefore

$$(< \mu'_n >, < \kappa'_n >) \notin \approx \tag{1.1}$$

for every maximal right ray μ' in κ'_μ and every maximal right ray δ' in κ'_δ. Let $R \subseteq \omega_\epsilon$. Choose $\mu = \mu_R \in R$ and a maximal right ray $[t_0\ t_1\ t_2\ \cdots >$ in κ'_μ. Let $\sigma_R \in \mathcal{P}(A)$ be defined by $\sigma_R = \bigcup_{\nu\in R} \kappa'_\nu$, and note that σ_R does not have a cycle or a double ray. Let R, S be nonempty subsets of ω_ϵ such that $R \neq S$. We may assume that there is $\nu \in R$ such that $\nu \notin S$. Consider κ'_ν, which is a c-component of σ_R. Let κ'_δ be any c-component of σ_S. Then, by equation (1.1), $(< \mu'_n >, < \kappa'_n >) \notin\approx$ for every maximal right ray μ' in κ'_μ and every maximal right ray κ' in κ'_δ. (Note that, by definition of σ_R, this is also true when $\nu = \mu_k$ or $\delta = \mu_S$.) So $(\sigma_R, \sigma_S) \notin\sim_n$ by Theorem 1.11. Hence any two different transformations from the collection $\{\sigma_K\}_{\emptyset\neq K\subseteq\omega_\epsilon}$ are in different equivalence classes of $\sim_n$. Since there are $2^{\aleph_\epsilon}$ transformations in the collection, it follows that $|Q'| \geq 2^{\aleph_\epsilon}$. Hence $|Q| = |Q'| + |Q"| + |\{[\omega]_c\}| \geq |Q'| \geq 2^{\aleph_\epsilon}$, and so $|Q| = 2^{\aleph_\epsilon}$.

(3) Let Z be the set of all $[\sigma]_n$ such that $\sigma \in \mathcal{P}(A)$ has a c-component of cho typewith no cycles or c-components of rro type. Let $Z' = \{[\sigma_n] \in Z : \sigma$ has no maximal left rays$\}$ and $Z'' = \{[\sigma]_n \in Z : \sigma$ has a maximal left ray$\}$. By part (5) of Theorem 1.11, $\{Z', Z''\}$ is a partition of Z. Fix a maximal left ray $\delta =< \cdots z_2z_1z_0] \in \mathcal{P}(A)$ and note that $Z" = \{[\delta]_n\}$. Define $h : Z'' \to (\omega_{\epsilon+1} + 1 \times \{0, 1\} \times \{0, 1\}$ by $([\sigma]_n)h = (s(\sigma), i_\sigma, j_\sigma)$. Then h is well defined and injective by Theorem 1.11 and Lemma 2.3. So $|Z'| \leq \aleph_{\epsilon+1} \cdot 2 \cdot 2 = \aleph_{\epsilon+1}$. Thus $|Z| = |Z'| + |Z''| = |Z''| + 1 \leq \aleph_{\epsilon+1}$. Thus $|Z| = |Z'| + |Z''| = |Z'| + 1 \leq \aleph_{\epsilon+1} + 1 = \aleph_{\epsilon+1}$.
Let $Z_1 \subseteq Z$ consisting of all $[\kappa]_n \in Z$ such that κ is connected. Note that $Z_1 = \{[\kappa]_n : \kappa \in \mathcal{P}(A)$ is of cho type$\} \cup\{[\delta]_n\}$. As in the proof of (1), we can construct a collection $\{\kappa_\nu\}_{0<\nu<\omega_{\epsilon+1}}$ of

connected elements of $\mathcal{P}(A)$ of cho type such that $(\kappa_\delta, \kappa_\nu) \notin \sim_n$ if $\delta \neq \nu$. Thus $|Z_1| \geq \aleph_{\epsilon+1}$, and so $\aleph + 1 \leq |Z_1| = \aleph_{\epsilon+1}$ which contradicts the proof of (3).

The conjugacy classes in (1)-(3) cover all conjugacy classes in $\mathcal{P}(\mathcal{A})$. Thus there are at most $\max\{2^{\aleph_0}, \aleph_{\epsilon+1}\} + 2^{\aleph_\epsilon} + \aleph_{\epsilon+1} = 2^{\aleph_\epsilon}$ conjugacy classes in $\mathcal{P}(A)$. By (2), there are at least $2^{\aleph_\epsilon}$ conjugacy classes, so number of conjugacy classes in $\mathcal{P}(A)$ is $2^{\aleph_\epsilon}$ conjugacy classes. By (1)-(3), at least $\aleph_{\epsilon+1}$ and at most $\aleph_0 + \aleph_{\epsilon+1}^{\aleph_0} + \aleph_{\epsilon+1} = \aleph_{\epsilon+1}^{\aleph_0}$ of these conjugacy classes have a connected representative. □

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