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Convexity in G-metric spaces and approximation of fixed points by Mann iterative process

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Abstract

In this paper, we first define the concept of convexity in G-metric spaces. We then use Mann iterative process in this newly defined convex G-metric space to prove some convergence results for some classes of mappings. In this way, we can extend several existence results to those approximating fixed points. Our results are just new in the setting.

Keywords: (Convex structure, Convex G-metric space, Mann iterative process, convergence)
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1. Introduction and preliminaries

The study of metric fixed point theory has been researched extensively in the past two decades or so because fixed point theory plays a key role in mathematics and applied sciences. For example, in the areas such as optimization, mathematical models, and economic theories.

In 2005, Mustafa and Sims introduced a new class of generalized metric spaces called G-metric spaces (see [5], [6]) as a generalization of metric spaces (X, d). This was done to introduce and develop a new fixed point theory for a variety of mappings in this new setting. This helped to extend some known metric space results to this more general setting. The G-metric space is defined as follows:

Definition 1.1. *[6] Let X be a nonempty set and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:*
(i) $G(x, y, z) = 0$ if $x = y = z$
(ii) $0 < G(x, x, y)$ for all $x, y \in X$, with $x \neq y$

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(iii) $G(x,x,y) \leq G(x,y,z)$ *for all* $x,y,z \in X$, *with* $z \neq y$
(iv) $G(x,y,z) = G(x,z,y) = G(y,z,x) = ...$, *(symmetry in all three variables); and*
(v) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ *for all* $x,y,z,a \in X$ *(rectangle inequality).*
Then the function G *is called a generalized metric or more specifically, a* G*-metric on* X, *and the pair* (X,G) *is called a* G*-metric space.*

Example 1.2. *[5] Let* $X = \mathbb{R}\backslash\{0\}$. *Define* $G : X \times X \times X \to \mathbb{R}^{+}$ *by*

$$G(x,y,z) = \begin{cases} |x-y| + |y-z| + |x-z|; & \text{if } x,y,z \text{ all have the same sign} \\ 1 + |x-y| + |y-z| + |x-z|; & \text{otherwise} \end{cases}$$

Then (X,G) *is a* G*-metric space.*

Proposition 1.3. *[6] Let* (X,G) *be a* G*-metric space. Then for any* x,y,z, *and* $a \in X$, *it follows that*
(i) if $G(x,y,z) = 0$, *then* $x = y = z$,
(ii) $G(x,y,z) \leq G(x,x,y) + G(x,x,z)$,
(iii) $G(x,y,y) \leq 2G(y,x,x)$,
(iv) $G(x,y,z) \leq G(x,a,z) + G(a,y,z)$,
(v) $G(x,y,z) \leq 2/3\,(G(x,y,a) + G(x,a,z) + G(a,y,z))$,
(vi) $G(x,y,z) \leq G(x,a,a) + G(y,a,a) + G(z,a,a)$.

Definition 1.4. *[6] Let* (X,G) *be a* G*-metric space and* (x_n) *a sequence of points of* X. *A point* $x \in X$ *is said to be the limit of the sequence* (x_n) *if* $lim_{n,m\to\infty} G(x,x_n,x_m) = 0$, *and we say that the sequence* (x_n) *is* G*-convergent to* x.

Proposition 1.5. *[6] Let* (X,G) *be a* G*-metric space. Then the following are equivalent.*
(i) (x_n) *is* G*-convergent to* x.
(ii) $G(x_n,x_n,x) \to 0$ *as* $n \to \infty$.
(iii) $G(x_n,x,x) \to 0$ *as* $n \to \infty$.
(iv) $G(x_m,x_n,x) \to 0$ *as* $m,n \to \infty$.

In 2008, 2009 and 2010, Mustafa et al. ([7], [8], [9]) gave the following fixed point theorems on some classes of contractive mappings defined on a G-metric space.

Theorem 1.6. *[7] Let* (X,G) *be a complete* G*-metric space and let* $T : X \to X$ *be a mapping satisfying one of the following conditions:*

$$G(Tx,Ty,Tz) \leq aG(x,y,z) + bG(x,Tx,Tx) + cG(y,Ty,Ty) + dG(z,Tz,Tz) \tag{1.1}$$

or

$$G(Tx,Ty,Tz) \leq aG(x,y,z) + bG(x,x,Tx) + cG(y,y,Ty) + dG(z,z,Tz)$$

for all $x,y,z \in X$ *where* $0 \leq a+b+c+d < 1$. *Then* T *has a unique fixed point* (*say* u, *i.e.,* $Tu = u$), *and* T *is* G*-continuous at* u.

Theorem 1.7. *[8] Let* (X,G) *be a* G*-metric space and let* $T : X \to X$ *be a mapping such that* T *satisfies the following three conditions:*
(1) $G(Tx,Ty,Tz) \leq aG(x,Tx,Tx) + bG(y,Ty,Ty) + cG(z,Tz,Tz)$ *for all* $x,y,z \in X$ *where* $0 < a+b+c < 1$,
(2) T *is* G*-continuous at a point* $u \in X$,
(3) *there is* $x \in X$; $\{T^n x\}$ *has a subsequence* $\{T^{n_i} x\}$ G*-converges to* u.
Then u *is the unique fixed point of* T.

Theorem 1.8. *[9] Let (X, G) be a complete G-metric space and let $T : X \to X$ be a mapping satisfying the condition*

$$G(Tx, Ty, Tz) \leq \alpha G(x, y, z) + \beta \{G(y, Ty, Ty) + G(z, Tz, Tz) + G(x, Tx, Tx)\} \tag{1.2}$$

for all $x, y, z \in X$, where $0 \leq \alpha + 3\beta < 1$. Then T has unique fixed point (say u), and T is G-continuous at u.

Theorem 1.9. *[9] Let (X, G) be complete G-metric space and let $T : X \to X$ be a mapping satisfying the condition*

$$G(Tx, Ty, Tz) \leq \alpha G(x, y, z) + \beta \max \{G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\} \tag{1.3}$$

for all $x, y, z \in X$, where $0 \leq \alpha+\beta < 1$. Then T has unique fixed point $(say\ u)$, and T is G-continuous at u.

Note that all above results deal with existence of fixed points without finding or approximating them. The reason behind is the unavailablity of convex structure in G-metric spaces by that time. Note that it is imperative to have a convex structure in order to approximate fixed points using various iterative processes, for example, Mann iterative process.

Keeping the above in mind, in this paper, we first define the concept of convexity in G-metric spaces. We then use Mann iterative process in this newly defined convex G-metric space to prove some convergence results for approximating fixed points of some classes of mappings. In this way, several existence results (including above Theorems 1.1-1.4) can be extended to those approximating fixed points. Our results are just new in the setting.

In 1970, Takahashi [10] introduced the notion of convex metric spaces and studied the approximation of fixed points for nonexpansive mappings in this setting. Later on, many authors ([1, 2, 3, 11]) discussed the existence of fixed points and convergence of different iterative processes for various mappings in convex metric spaces. .

We recall some definitions as follows:

Definition 1.10. *[10] A convex structure in a metric space (X, d) is a mapping $W : X^2 \times [0, 1] \to X$ satisfying, for all $x, y, u \in X$ and all $\alpha \in [0, 1]$,*

$$d\left(W\left(x, y; \alpha\right), u\right) \leq \alpha d\left(x, u\right) + \left(1 - \alpha\right) d\left(y, u\right). \tag{1.4}$$

The triplet (X, d, W) is called a convex metric space.

Modi et al. [4] introduced convex structure in G-metric spaces as follows.

Definition 1.11. *[4] Let (X, G) be a G-metric space. A mapping $W : X^3 \times (0, 1] \to X$ is said to be a convex structure on (X, G) if for each $(x, y, z, \lambda) \in X^3 \times (0, 1]$ and for all u, $v \in X$ the condition*

$$G(u, v, W(x, y, z, \lambda)) \leq \frac{\lambda}{3} G(u, v, x) + \frac{\lambda}{3} G(u, v, y) + \frac{\lambda}{3} G(u, v, z)$$

holds. If W is convex structure on a G-metric space (X, G), then the triplet (X, G, W) is called a convex G-metric space.

Using this definition, they gave some fixed point results for weakly compatible mappings.

Continuing, we define convex structure in G-metric spaces in a differnt way as follows. Our defintion is more natural than that due to Modi et al. [4]. We do not divide λ into three equal parts but let λ and β take any values in $[0, 1]$ as far as their sum is 1. We keep part of the domain as X^2 (and hence only two terms on the right hand side) which is a better analog to the well-celebrated convexity of Takahashi [10] and simpler in calculations than taking X^3. Here is our definition of convex structure in a G-metric space.

Definition 1.12. *Let (X, G) be a G-metric space. A mapping $W : X^2 \times I^2 \to X$ is termed as a convex structure on X if $G(W(x, y; \lambda, \beta), u, v) \leq \lambda G(x, u, v) + \beta G(y, u, v)$ for real numbers λ and β in $I = [0, 1]$ satisfying $\lambda + \beta = 1$ and x, y, u and $v \in X$.*

A G-metric space (X, G) with a convex structure W is called a convex G-metric space and denoted as (X, G, W).

A nonempty subset C of a convex G-metric space (X, G, W) is said to be convex if $W(x, y; a, b) \in C$ for all $x, y \in C$ and $a, b \in I$.

Next, we transform the Mann iterative process to a convex G-metric space as follows.

Definition 1.13. *Let (X, G, W) be convex G-metric space with convex structure W and $T : X \to X$ be a mapping. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ for $n \in \mathbb{N}$. Then for any given $x_0 \in X$, the iterative process defined by the sequence $\{x_n\}$ as*

$$x_{n+1} = W\left(x_n, Tx_n; 1 - \alpha_n, \alpha_n\right), \qquad n \in \mathbb{N}, \tag{1.5}$$

is called Mann iterative process in the convex metric space (X, G, W).

It follows from the structure of convex G-metric space that

$$\begin{aligned} G(x_{n+1}, u, v) &= G(W\left(x_n, Tx_n; 1 - \alpha_n, \alpha_n\right), u, v) \\ &\leq \left(1 - \alpha_n\right) G(x_n, u, v) + a_n G(Tx_n, u, v). \end{aligned}$$

Now, having given the most needed definition of convex structure on a G-metric space and re-written the Mann iterative process (1.5) in this setting, we are able to transform the above mentioned existence results (Theorems 1.6, 1.7, 1.8 and 1.9) to those approximating fixed points through strong convergence. And this is what we are going to do in the next section.

2. Main Results

The following is our first result which approximates the fixed points of the mappings (1.1) but naturally in a convex G-metric space.

Theorem 2.1. *Let (X, G, W) be a convex G-metric space with a convex structure W and let $T : X \to X$ be a mapping with a fixed point u satisfying the condition (1.1) for all $x, y, z \in X$ where a, b, c, d are nonnegative real numbers such that $0 \leq a + 3b < 1$. Let $\{x_n\}$ be defined iteratively by (1.5) and $x_0 \in X$, with $\{\alpha_n\} \subset [0, 1]$ satisfying $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof . *Since u is a fixed point of the mapping T, we have*

$$\begin{aligned} G(x_{n+1},u,u) &= G(W(x_n,Tx_n;1-\alpha_n,\alpha_n),u,u) \\ &\leq (1-\alpha_n)G(x_n,u,u)+a_nG(Tx_n,u,u) \\ &= (1-\alpha_n)G(x_n,u,u)+a_nG(Tx_n,Tu,Tu). \end{aligned} \tag{2.1}$$

Using the inequality (1.1) for $G(Tx_n,Tu,Tu)$,

$$\begin{aligned} G(Tx_n,Tu,Tu) &\leq aG(x_n,u,u)+bG(x_n,Tx_n,Tx_n) \\ &\quad +cG(u,Tu,Tu)+dG(u,Tu,Tu) \\ &= aG(x_n,u,u)+bG(x_n,Tx_n,Tx_n) \\ &\leq aG(x_n,u,u)+b\left[G(x_n,u,u)+G(u,u,Tx_n)\right] \\ &\leq aG(x_n,u,u)+b\left[G(x_n,u,u)+2G(Tx_n,u,u)\right]. \end{aligned} \tag{2.2}$$

Thus from (2.2) we have

$$G(Tx_n,Tu,Tu)\leq \frac{a+b}{1-2b}G(x_n,u,u). \tag{2.3}$$

From the inequalities (2.1) and (2.3), we obtain

$$G(x_{n+1},u,u) \leq (1-\alpha_n)G(x_n,u,u)+a_n\frac{a+b}{1-2b}G(x_n,u,u) \tag{2.4}$$

$$= \left[1-\alpha_n\left(1-\frac{a+b}{1-2b}\right)\right]G(x_n,u,u)$$

$$= \left[1-\alpha_n(1-\delta)\right]G(x_n,u,u) \tag{2.5}$$

where

$$\delta=\frac{a+b}{1-2b}.$$

Note that $0\leq\delta<1$ and $1-2b\neq 0$. Indeed,

$$a+3b<1\Longrightarrow a+b<1-2b\Rightarrow\frac{a+b}{1-2b}<1.$$

Moreover, if $1-2b=0$, then from above calculations, $a+b<1-2b$ means $a<-b$, which is contradiction to $a\geq 0$.

Inductively we get

$$G(x_{n+1},u,u)\leq\prod_{k=0}^{n}\left[1-\alpha_k(1-\delta)\right]G(x_0,u,u). \tag{2.6}$$

As $\delta<1,\alpha_k\in[0,1]$ and $\sum\limits_{n=0}^{\infty}\alpha_n=\infty$, it results that

$$\lim_{n\to\infty}\prod_{k=0}^{n}\left[1-\alpha_k(1-\delta)\right]=0,$$

which by (2.1) implies

$$\lim_{n\to\infty}G(x_n,u,u)=0.$$

Hence the sequence $\{x_n\}$ defined iteratively by (1.5) converges strongly to the fixed point of T. □

The following corollary gives approximation result for the of mappings used inTheorem 1.3.

Corollary 2.2. *Let (X,G,W) be a convex G-metric space with a convex structure W and let $T : X \to X$ be a mapping with a fixed point u satisfying the following inequality*

$$G(Tx,Ty,Tz) \leq aG(x,y,z) + b\{G(x,Tx,Tx) + G(y,Ty,Ty) + G(z,Tz,Tz)\} \tag{2.7}$$

for all $x,y,z \in X$ where $0 < a + 3b < 1$. Let $\{x_n\}$ be defined iteratively by (1.5) and $x_0 \in X$, with $\{\alpha_n\} \subset [0,1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof . If we let $b = c = d$ in condition (1.1), then it becomes condition (2.7) and the proof follows from Theorem 2.1. □

We can also get easily the following more general result if we replace sum with max in (2.7) at a proper place.

Corollary 2.3. *Let (X,G,W) be a convex G-metric space with a convex structure W and let $T : X \to X$ be a mapping with a fixed point u satisfying the following inequality*

$$G(Tx,Ty,Tz) \leq aG(x,y,z) + b\max\{G(x,Tx,Tx), G(y,Ty,Ty), G(z,Tz,Tz)\} \tag{2.8}$$

for all $x,y,z \in X$ where $0 < a + 3b < 1$. Let $\{x_n\}$ be defined iteratively by (1.5) and $x_0 \in X$, with $\{\alpha_n\} \subset [0,1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$.Then $\{x_n\}$ converges strongly to a fixed point of T.

We also give the following result corresponding to contractive condition. (1.2).

Theorem 2.4. *Let (X,G,W) be a convex G-metric space with a convex structure W and let $T : X \to X$ be a mapping with a fixed point u satisfying the following inequality*

$$G(Tx,Ty,Tz) \leq aG(x,Tx,Tx) + bG(y,Ty,Ty) + cG(z,Tz,Tz) \tag{2.9}$$

for all $x,y,z \in X$ where where a,b,c are nonnegative real numbers $0 \leq a+b+c < 1$ and $a < \frac{1}{2}$. Let $\{x_n\}$ be defined iteratively by (1.5) and $x_0 \in X$, with $\{\alpha_n\} \subset [0,1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof . *From the proof of Theorem 2.1, we know that*

$$G(x_{n+1},u,u) \leq (1-\alpha_n)\,G(x_n,u,u) + a_n G(Tx_n,Tu,Tu). \tag{2.10}$$

Using the property of T for $G(Tx_n,Tu,Tu)$,

$$\begin{aligned} G(Tx_n,Tu,Tu) &\leq aG(x_n,Tx_n,Tx_n) + bG(u,Tu,Tu) + cG(u,Tu,Tu) \\ &\leq a[G(x_n,u,u) + G(u,u,Tx_n)] \\ &\leq a[G(x_n,u,u) + 2G(Tx_n,u,u)]. \end{aligned} \tag{2.11}$$

Thus, from (2.11) we get

$$G(Tx_n,Tu,Tu) \leq \frac{a}{1-2a} G(x_n,u,u). \tag{2.12}$$

From the inequalities (2.10) and (2.12), we have

$$\begin{aligned} G(x_{n+1},u,u) &\leq (1-\alpha_n)\,G(x_n,u,u) + a_n\frac{a}{1-2a}G(x_n,u,u) \\ &= \left[1-\alpha_n\left(1-\frac{a}{1-2a}\right)\right]G(x_n,u,u). \end{aligned}$$

If we denote

$$\delta = \frac{a}{1-2a}$$

then we have $0 \leq \delta < 1$ *and*

$$G(x_{n+1},u,u) \leq [1-\alpha_n(1-\delta)]\,G(x_n,u,u).$$

Indeed

$$a < \frac{1}{2} \Longrightarrow \frac{a}{1-2a} < 1 \quad \text{and } 1-2a \neq 0.$$

In a way similar to the proof of above the theorem, we obtain

$$\lim_{n\to\infty} G(x_n,u,u) = 0.$$

Hence the proof. □

Corollary 2.5. *Let* (X,G,W) *be a convex* G*-metric space with a convex structure* W *and let* $T : X \to X$ *be a mapping with a fixed point* u *satisfying the following inequality*

$$G(Tx,Ty,Tz) \leq k\{G(x,Tx,Tx) + G(y,Ty,Ty) + G(z,Tz,Tz)\} \tag{2.13}$$

for all $x,y,z \in X$ *where* $0 < k < \frac{1}{3}$. *Let* $\{x_n\}$ *be defined iteratively by* (1.5) *and* $x_0 \in X$, *with* $\{\alpha_n\} \subset [0,1]$ *satisfying* $\sum_{n=0}^{\infty}\alpha_n = \infty$. *Then* $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof . Take $a = b = c = k$ in condition (2.9) of Theorem 2.4. □

2.1. Conclusion

In this paper, we define a new concept about convexity in G-metric spaces. After, we prove some convergence theorems for some classes of mappings using Mann iterative process in this newly defined convex G-metric space. Then, we extend some existence results to those approximating fixed points.

We close this section with the following open questions.

2.2. Open Problems

(i) Can the fixed points of various contraction mappings in convex G-metric spaces be approached with different iteration methods such as Ishikawa iteration, Noor iteration, Picard-Mann iteration, S-iteration?

(ii) Can the convergence rates of the iteration methods mentioned above and in the paper be compared for different contractions in convex G-metric spaces?

(iii) Can the stability of an iteration method be defined in convex G-metric spaces?

(iv) If the answer to the above question is yes, is the Mann iteration in this paper T-stable?

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