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The bifurcation diagram of an elliptic P-Kirchhoff-type problem with respect to the stiffness of the material

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Abstract

We study a superlinear and subcritical p-Kirchhoff-type problem which is variational and depends upon a real parameter λ. The nonlocal term forces some of the fiber maps associated with the energy functional to have two critical points. This suggests multiplicity of solutions, and indeed, we show the existence of a local minimum and a mountain pass-type solution. We characterize the first parameter λ_0^* for which the local minimum has nonnegative energy when $\lambda \geq \lambda_0^*$. Moreover, we characterize the extremal parameter λ^* for which if $\lambda > \lambda^*$; then, the only solution to the p-Kirchhoff problem is the zero function. In fact, λ^* can be characterized in terms of the best constant of Sobolev embeddings. We also study the asymptotic behavior of the solutions when $\lambda \downarrow 0$

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1. Introduction

In this work, we study the following p-Kirchhoff-type problem

$$\begin{cases} -\Big(a+\lambda \displaystyle\int_\Omega |\nabla u|^p dx\Big)\Delta_p u = |u|^{\gamma-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

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where $a > 0$, $\lambda > 0$ is a parameter, Δ_p is the p-Laplacian operator and $\Omega \subset \mathbb{R}^3$ is a bounded regular domain. Kirchhoff-type equations have been extensively studied in the literature. It was proposed by Kirchhoff in [5] as a model to study some physical problems related to elastic string vibrations, and then it has been studied by many author, see for example the works of Lions [6], Alves et al. [1], Wu et al. [2], Zhang and Perera [15] and the references therein. Physically speaking, if one wants to study string or membrane vibrations, one is led to problem (1.1) when $p = 2$, where u represents the displacement of the membrane, $|u|^{q-2}u$ is an external force and a and λ are related to some intrinsic properties of the membrane. In particular, λ is related to the Young's modulus of the material and it measures its stiffness. Our main purpose here is to analyze problem (1.1) with respect to the parameter λ (stiffness) and provide a description of the bifurcation diagram that the type $p = 2$ have been studied by Kaye Silva in [12]. To this end, we will use the fibering method of Pohozaev [9] to analyze how the Nehari set (see Nehari [7, 8]) changes with respect to the parameter λ and then apply this analysis to study bifurcation properties of the problem (1.1) (see Chen et al. [2] and Zhang et al. [14]).

Let $W_0^{1,p}(\Omega)$, $p > 1$ denoted the standard Sobolev space, equipped with the norm

$$\|u\| = \Big(\int_\Omega |\nabla u|^p dx\Big)^{\frac{1}{p}},$$

and $\phi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional associated with (1.1), that is

$$\phi_\lambda(u) = \frac{a}{p}\Big(\int_\Omega |\nabla u|^p dx\Big) + \frac{\lambda}{2p}\Big(\int_\Omega |\nabla u|^p dx\Big)^2 - \frac{1}{\gamma}\Big(\int_\Omega |u|^\gamma dx\Big). \tag{1.2}$$

We observe that ϕ_λ is a C^1 functional. By definition, a solution to problem (1.1) is a critical point of ϕ_λ. Our main result is the following

Theorem 1.1. *Suppose $\gamma \in (p, 2p)$. Then, there exist parameters $0 < \lambda_0^* < \lambda^*$ and $\varepsilon > 0$ such that :*

(1) For each $\lambda \in (0, \lambda^*]$, problem (1.1) has a positive solution u_λ which is a global minimizer for ϕ_λ when $\lambda \in (0, \lambda_0^*]$, while u_λ is a local minimizer for ϕ_λ when $\lambda \in (\lambda_0^*, \lambda^*)$. Moreover, $\phi''(u_\lambda)(u_\lambda, u_\lambda) > 0$ for $\lambda \in (0, \lambda^*)$ and $\phi''(u_{\lambda^*})(u_{\lambda^*}, u_{\lambda^*}) = 0$.

(2) For each $\lambda \in (0, \lambda_0^* + \varepsilon)$, problem (1.1) has a positive solution w_λ which is a mountain pass critical point for ϕ_λ.

(3) If $\lambda \in (0, \lambda_0^*)$, then $\phi_\lambda(u_\lambda) < 0$ while $\phi_{\lambda_0^*}(u_{\lambda_0^*}) = 0$ and if $\lambda \in (\lambda_0^*, \lambda^*]$ then $\phi_\lambda(u_\lambda) > 0$.

(4) $\phi_\lambda(w_\lambda) > 0$ and $\phi_\lambda(w_\lambda) > \phi_\lambda(u_\lambda)$ for each $\lambda \in (0, \lambda_0^* + \varepsilon)$.

(5) If $\lambda > \lambda^*$, then the only solution $u \in W_0^{1,p}(\Omega)$ for the problem (1.1) is the zero function $u = 0$.

The extremal parameter λ^* (see Il'yasov [3]) which appears in Theorem 1.1 can be characterized variationally by

$$\lambda^* = S_{a,\gamma} \sup\left\{\left(\frac{(\int |u|^\gamma)^{\frac{1}{\gamma}}}{(\int |\nabla u|^p)^{\frac{1}{p}}}\right)^{\frac{p\gamma}{\gamma-p}} : u \in W_0^{1,p}(\Omega)\backslash\{0\}\right\},$$

where $S_{a,\gamma}$ is some positive constant. One can easily see from the last expression that $\lambda^* = S_{a,\gamma} C_\gamma^{\frac{p\gamma}{p-\gamma}}$, where C_γ is best Sobolev constant for the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^\gamma(\Omega)$.

In this paper, the extremal parameter λ^* has an important role that if $\lambda > \lambda^*$, then the Nehari set is empty while if $\lambda \in (0, \lambda^*]$, then the Nehari set is not empty. Another interesting parameter is $\lambda^* > \lambda_0^*$ which is characterized by the property that if $\lambda \in (0, \lambda_0^*)$, then $\inf_{u \in W_0^{1,p}(\Omega)} \phi_\lambda(u) < 0$ while if $\lambda \geq \lambda_0^*$, the infimum is zero. When $\lambda \in (0, \lambda_0^*]$, one can easily provide a mountain pass geometry and a global minimizer for the functional ϕ_λ. Although here we characterize λ_0^* variationally, one can see that the parameter a^* defined in Theorem 1.3 (ii) of Sun and Wu [13] serves to the same purpose as λ_0^*, and hence, our result for $(0, \lambda_0^*)$ is not new; however, when $\lambda > \lambda_0^*$, we could not find this result in the literature and in this case, we need to provide some finer estimates on the Nehari sets in order to solve some technical issues to obtain again a mountain pass geometry and a local minimizer for the functional ϕ_λ.

The hypothesis $\gamma \in (p, 2p)$ has the fundamental role that it forces the problem to be superlinear, subcritical and it allows the existence of fiber maps with two critical points. The existence of these kinds of fiber maps implies multiplicity of solutions (at least two solutions), and hence, for $\lambda > \lambda^*$ there is no solution at all, the parameter λ^* is a bifurcation point where these solutions collapse. We refer the reader to the recent works of Silva [12], Siciliano and Silva [10], Il'yasov and Silva [4], Silva and Macedo [11], where the extremal parameters of some indefinite nonlinear elliptic problems were analyzed.

Concerning the asymptotic behavior of the solutions when $\lambda \downarrow 0$, we state the following theorem.

Theorem 1.2. *There holds :*

(i) $\phi_\lambda(u_\lambda) \to -\infty$ *and* $\|u_\lambda\| \to \infty$ *as* $\lambda \downarrow 0$.

(ii) $w_\lambda \to w_0$ *in* $W_0^{1,p}(\Omega)$ *where* $w_0 \in W_0^{1,p}(\Omega)$ *is a mountain pass critical point associated with the equation* $-a\Delta_p w = |w|^{q-2}w$.

The plan of the paper is as follows: In Sect. 2, we provide some definitions and prove technical results which will be used in the next section. In Sect. 3, we show the existence of local minimizers for the functional ϕ_λ. In Sect. 4, we prove the existence of mountain pass critical point for the functional ϕ_λ. In Sect. 5, we prove Theorem 1.1. In Sect. 6, we prove Theorem 1.2. In Sect. 7, we provide a picture detailing the bifurcation diagram with respect to the energy and make some conjectures and in "Appendix", we prove some auxiliary results.

2. Technical results

We denote by $\|u\|$ the standard Sobolev norm on $W_0^{1,p}(\Omega)$ and $\|u\|_\gamma$ the $L^\gamma(\Omega)$ Norm. It follows from (1.2) that

$$\phi_\lambda(u) = \frac{a}{p}\Big(\int_\Omega |\nabla u|^p dx\Big) + \frac{\lambda}{2p}\Big(\int_\Omega |\nabla u|^p dx\Big)^2 - \frac{1}{\gamma}\Big(\int_\Omega |u|^\gamma dx\Big), \forall u \in W_0^{1,p}(\Omega).$$

For each $\lambda > 0$, consider the Nehari set

$$\mathcal{N}_\lambda = \{u \in W_0^{1,p}(\Omega)\backslash\{0\} : \phi'_\lambda(u)u = 0\}.$$

To study the Nehari set, we will make use of the fiber maps: for each $\lambda > 0$ and $u \in W_0^{1,p}(\Omega)\backslash\{0\}$ define $\psi_{\lambda,u} : (0, \infty) \to \mathbb{R}$ by

$$\psi_{\lambda,u}(t) = \phi_\lambda(tu).$$

We divide the Nehari set into three disjoint sets as follows:

$$\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0 \cup \mathcal{N}_\lambda^-$$

where

$$\begin{aligned} \mathcal{N}_\lambda^+ &= \{u \in W_0^{1,p}(\Omega)\backslash\{0\} : \psi'_{\lambda,u}(1) = 0,\, \psi''_{\lambda,u}(1) > 0\}, \\ \mathcal{N}_\lambda^0 &= \{u \in W_0^{1,p}(\Omega)\backslash\{0\} : \psi'_{\lambda,u}(1) = 0,\, \psi''_{\lambda,u}(1) = 0\}, \\ \mathcal{N}_\lambda^- &= \{u \in W_0^{1,p}(\Omega)\backslash\{0\} : \psi'_{\lambda,u}(1) = 0,\, \psi''_{\lambda,u}(1) < 0\}. \end{aligned}$$

By using the Implicit Function Theorem, one can prove the following lemma.

Lemma 2.1. *[7] If $\mathcal{N}_\lambda^+$, $\mathcal{N}_\lambda^-$ are non-empty, then $\mathcal{N}_\lambda^+$, $\mathcal{N}_\lambda^-$ are C^1 manifolds of codimension 1 in $W_0^{1,p}(\Omega)$. Moreover, if $u \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$ is a critical point of $(\phi_\lambda)_{|\mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-}$, then u is a critical point of ϕ_λ.*

In order to understand the Nehari set $\mathcal{N}_\lambda$, we study the fiber maps $\psi_{\lambda,u}$. Simple analysis arguments show that

Proposition 2.2. *For each $\lambda > 0$ and $u \in W_0^{1,p}(\Omega)\backslash\{0\}$, there are only three possibilities for the graph of $\psi_{\lambda,u}$*

(I) The function $\psi_{\lambda,u}$ has only two critical points, namely, $0 < t_\lambda^-(u) < t_\lambda^+(u)$. Moreover, $t_\lambda^-(u)$ is a local maximum with $\psi''_{\lambda,u}(t_\lambda^-(u)) < 0$ and $t_\lambda^+(u)$ is a local minimum with $\psi''_{\lambda,u}(t_\lambda^+(u)) > 0$;

(II) The function $\psi_{\lambda,u}$ has only one critical point when $t > 0$ at the value $t_\lambda(u)$. Moreover, $\psi''_{\lambda,u}(t_\lambda(u)) = 0$ and $\psi_{\lambda,u}$ is increasing;

(III) The function $\psi_{\lambda,u}$ is increasing and has no critical points.

It follows from Proposition 2.2 that $\mathcal{N}_\lambda^+$, $\mathcal{N}_\lambda^-$ are non-empty if and only if the item (I) is satisfied. Therefore, it remains to show whether (I) is satisfied or not. For this purpose, we study for what values of λ there holds $\mathcal{N}_\lambda^0 \neq \emptyset$. Note that $tu \in \mathcal{N}_\lambda^0$ for $t > 0$ and $u \in W_0^{1,p}(\Omega)\backslash\{0\}$ if and only if

$$\begin{cases} \psi'_{\lambda,u}(t) = 0, \\ \psi''_{\lambda,u}(t) = 0, \end{cases}$$

or equivalently

$$\begin{cases} a\|u\|^p\, t^p + \lambda\|u\|^{2p}\, t^{2p} - \|u\|_\gamma^\gamma\, t^\gamma = 0, \\ pa\|u\|^p\, t^p + 2p\lambda\|u\|^{2p}\, t^{2p} - \gamma\|u\|_\gamma^\gamma\, t^\gamma = 0. \end{cases}$$

By dividing on t^p, we have :

$$a\|u\|^p + \lambda\|u\|^{2p}\, t^p - \|u\|_\gamma^\gamma\, t^{\gamma-p} = 0, \text{ and } \; pa\|u\|^p + 2p\lambda\|u\|^{2p}\, t^p - \gamma\|u\|_\gamma^\gamma\, t^{\gamma-p} = 0.$$

Now, by difference between second and first term of the system, we have

$$\begin{cases} a\|u\|^p + \lambda\|u\|^{2p}\, t^p - \|u\|_\gamma^\gamma\, t^{\gamma-p} = 0, \\ a(p-1)\|u\|^p + (2p-1)\lambda\|u\|^{2p}\, t^p - (\gamma-1)\|u\|_\gamma^\gamma\, t^{\gamma-p} = 0. \end{cases} \tag{2.1}$$

We solve the system (2.1) with respect to the variable (t, λ) to obtain for each $u \in W_0^{1,p}(\Omega)\backslash\{0\}$, a unique pair $(t(u), \lambda(u))$ which is obtained as follow :

$$a\|u\|^p + \lambda\|u\|^{2p}\ t^p - \|u\|_\gamma^\gamma\ t^{\gamma-p} = 0 \ \Rightarrow\ \lambda\|u\|^{2p}\ t^p = \|u\|_\gamma^\gamma\ t^{\gamma-p} - a\|u\|^p,$$

and by replacing it in second equation we obtain that

$$\begin{aligned} & a(p-1)\|u\|^p - (2p-1)(a\|u\|^p - \|u\|_\gamma^\gamma\ t^{\gamma-p}) - (\gamma-1)\|u\|_\gamma^\gamma\ t^{\gamma-p} = 0, \\ \Rightarrow & -pa\|u\|^p + (2p-\gamma)\|u\|_\gamma^\gamma\ t^{\gamma-p} = 0, \\ & pa\|u\|^p = (2p-\gamma)\|u\|_\gamma^\gamma\ t^{\gamma-p}, \end{aligned}$$

$$\Rightarrow t(u) = \left(\frac{pa}{(2p-\gamma)}\frac{\|u\|^p}{\|u\|_\gamma^\gamma}\right)^{\frac{1}{\gamma-p}}.$$

Now, by resolving the system (2.1) we have : $a\|u\|^p + \lambda\|u\|^{2p}\ t^p = \|u\|_\gamma^\gamma\ t^{\gamma-p}$, and so

$$a(p-1)\|u\|^p + (2p-1)\lambda\|u\|^{2p}\ t^p - a(\gamma-1)\|u\|^p - \lambda(\gamma-1)\|u\|^{2p}\ t^p = 0,$$

and then

$$\begin{aligned} & a(p-\gamma)\|u\|^p - \lambda(2p-\gamma)\|u\|^{2p}\ t^p = 0, \\ \Rightarrow & a(p-\gamma)\|u\|^p = \lambda(2p-\gamma)\|u\|^{2p}\ t^p, \\ \Rightarrow & \lambda(u) = a\left(\frac{\gamma-p}{2p-\gamma}\right)\frac{1}{\|u\|^p\ t^p}\ . \end{aligned}$$

Now put $t(u) = \left(\frac{pa}{(2p-\gamma)}\frac{\|u\|^p}{\|u\|_\gamma^\gamma}\right)^{\frac{1}{\gamma-p}}$, then have :

$$\begin{aligned} \lambda(u) &= a\left(\frac{\gamma-p}{2p-\gamma}\right)\left(\frac{2p-\gamma}{pa}\frac{\|u\|_\gamma^\gamma}{\|u\|^p}\right)^{\frac{p}{\gamma-p}}\frac{1}{\|u\|^p}, \\ \Rightarrow &= a\left(\frac{\gamma-p}{2p-\gamma}\right)\left(\frac{2p-\gamma}{pa}\right)^{\frac{p}{\gamma-p}}\|u\|_\gamma^{\frac{p\gamma}{\gamma-p}}\|u\|^{\frac{\gamma-p}{p\gamma}}, \\ \Rightarrow &= a\left(\frac{\gamma-p}{2p-\gamma}\right)\left(\frac{2p-\gamma}{pa}\right)^{\frac{p}{\gamma-p}}\left(\frac{\|u\|_\gamma}{\|u\|}\right)^{\frac{p\gamma}{\gamma-p}}. \end{aligned}$$

Then we obtain for each $u \in W_0^{1,p}(\Omega)\backslash\{0\}$, a unique pair $(t(u), \lambda(u))$ such that :

$$t(u) = \left(\frac{pa}{(2p-\gamma)}\frac{\|u\|^p}{\|u\|_\gamma^\gamma}\right)^{\frac{1}{\gamma-p}}, \tag{2.2}$$

$$\lambda(u) = S_{a,\gamma}\left(\frac{\|u\|_\gamma}{\|u\|}\right)^{\frac{p\gamma}{\gamma-p}}, \tag{2.3}$$

where

$$S_{a,\gamma} = a\left(\frac{\gamma-p}{2p-\gamma}\right)\left(\frac{2p-\gamma}{pa}\right)^{\frac{p}{\gamma-p}}.$$

We define the extremal parameter (see Il'Yasov [3])

$$\lambda^* = \sup_{u\in W_0^{1,p}(\Omega)\backslash\{0\}} \lambda(u). \tag{2.4}$$

We also consider another parameter which is defined as a solution of the system

$$\begin{cases} \psi_{\lambda,u}(t) = 0, \\ \psi'_{\lambda,u}(t) = 0, \end{cases}$$

or equivalently

$$\begin{cases} \dfrac{a}{p}\|u\|^p + \dfrac{\lambda}{2p}\|u\|^{2p}\, t^p - \dfrac{1}{\gamma}\|u\|_\gamma^\gamma\, t^{\gamma-p} = 0, \\ a\|u\|^p + \lambda\|u\|^{2p}\, t^p - \|u\|_\gamma^\gamma\, t^{\gamma-p} = 0. \end{cases} \tag{2.5}$$

Similar to the system (2.1), we can solve the system (2.5) with respect to the variable (t, λ) to find a unique pair $(t_0(u), \lambda_0(u))$ which is obtained as follows :

$$\frac{\lambda}{2p}\|u\|^{2p}\, t^p = \frac{1}{\gamma}\|u\|_\gamma^\gamma\, t^{\gamma-p} + \frac{a}{p}\|u\|^p,$$

then we have :

$$\begin{aligned} & a\|u\|^p - \|u\|_\gamma^\gamma\, t^{\gamma-p} - 2a\|u\|^p + \frac{2p}{\gamma}\|u\|_\gamma^\gamma\, t^{\gamma-p} = 0, \\ \Rightarrow & -a\|u\|^p + (\frac{2p}{\gamma} - 1)\|u\|_\gamma^\gamma\, t^{\gamma-p} = 0, \\ \Rightarrow & \, t_0(u) = \left(a\Big(\frac{\gamma}{2p-\gamma}\Big)\frac{\|u\|^p}{\|u\|_\gamma^\gamma}\right)^{\frac{1}{\gamma-p}}. \end{aligned}$$

Now, by resolving the system (2.1), we have again : $\dfrac{1}{\gamma}\|u\|_\gamma^\gamma\, t^{\gamma-p} = -\dfrac{a}{p}\|u\|^p - \dfrac{\lambda}{2p}\|u\|^{2p}\, t^p$, hence:

$$\begin{aligned} & a\|u\|^p + \lambda\|u\|^{2p}\, t^p - \frac{a\gamma}{p}\|u\|^p - \frac{\lambda\gamma}{2p}\|u\|^{2p}\, t^p = 0, \\ \Rightarrow & \, a(1 - \frac{\gamma}{p})\|u\|^p + \lambda(1 - \frac{\gamma}{2p})\|u\|^{2p}\, t^p = 0, \\ \Rightarrow & \, \lambda_0(u) = \frac{\gamma - p}{2p-\gamma}\frac{2a}{\|u\|^p}(t^{-p}). \end{aligned}$$

Now put $t_0(u) = \left(a\Big(\dfrac{\gamma}{2p-\gamma}\Big)\dfrac{\|u\|^p}{\|u\|_\gamma^\gamma}\right)^{\dfrac{1}{\gamma-p}}$, then have :

$$\begin{aligned} & \lambda_0(u) = 2a\Big(\frac{\gamma-p}{2p-\gamma}\Big)\Big(\frac{2p-\gamma}{pa}\Big)^{\frac{p}{\gamma-p}}(pa)^{\frac{p}{\gamma-p}}(a\gamma)^{\frac{-p}{\gamma-p}}(\frac{\|u\|_\gamma}{\|u\|})^{\frac{p\gamma}{\gamma-p}}, \\ \Rightarrow & \, 2\Big(\frac{p}{\gamma}\Big)^{\frac{p}{\gamma-p}}\lambda(u) = \lambda_0(u). \end{aligned}$$

Then we obtain a unique pair $(t_0(u), \lambda_0(u))$ such that :

$$\lambda_0(u) = S_{0,a,\gamma}\,\lambda(u), \;\; \forall u \in W_0^{1,p}(\Omega),$$

where

$$S_{0,a,\gamma} = 2\Big(\frac{p}{\gamma}\Big)^{\frac{p}{\gamma-p}}.$$

Observe that $S_{0,a,\gamma} < 1$. We define

$$\lambda_0^*(u) = \sup_{u \in W_0^{1,p}(\Omega)\backslash\{0\}} \lambda_0(u). \tag{2.6}$$

The function $\lambda(u)$ and $\lambda_0(u)$ have the following geometrical interpretation.

Proposition 2.3. *For each $u \in W_0^{1,p}(\Omega)\backslash\{0\}$, there holds :*

(i) $\lambda(u)$ is the unique parameter $\lambda > 0$ for which the fiber map $\psi_{\lambda,u}$ has a critical point with second derivative zero at $t(u)$. Moreover, if $0 < \lambda < \lambda(u)$, then $\psi_{\lambda,u}$ satisfies (I) of proposition 2.2 while if $\lambda > \lambda(u)$, $\psi_{\lambda,u}$ satisfies (III) of proposition 2.2.

(ii) $\lambda_0(u)$ is the unique parameter $\lambda > 0$ for which the fiber map $\psi_{\lambda,u}$ has a critical point with zero energy at $t_0(u)$. Moreover, if $0 < \lambda < \lambda_0(u)$, then $\inf_{t>0} \psi_{\lambda,u}(t) < 0$ while if $\lambda > \lambda_0(u)$, then $\inf_{t>0} \psi_{\lambda,u}(t) = 0$.

Proof . (i) The uniqueness of $\lambda(u)$ comes from problem (2.1). Assume that $\lambda \in (0, \lambda(u))$, then $\psi_{\lambda,u}$ must satisfy (I) or (II) of proposition 2.2. We claim that it must satisfy (I). Indeed, suppose on the contrary that it satisfies (III). Once

$$\psi'_{\lambda(u),u}(t) > \psi'_{\lambda,u}(t) > 0, \, \forall t > 0,$$

since $\psi'_{\lambda(u),u}(t(u)) = 0$ where $t(u)$ is given by (2.5), hence we reach a contradiction; therefore, $\psi_{\lambda,u}$ must satisfy (I). Now suppose that $\lambda > \lambda(u)$, then

$$\psi'_{\lambda,u}(t) > \psi'_{\lambda(u),u}(t) \geq 0, \, \forall t > 0,$$

and hence, $\psi_{\lambda,u}(t)$ must satisfy (III).
(ii) The uniqueness of $\lambda_0(u)$ comes from problem (7). If $0 < \lambda < \lambda_0(u)$, then from the definition we have

$$\psi_{\lambda,u}(t_0(u)) < \psi_{\lambda_0(u),u}(t_0(u)) = 0,$$

which implies that, $\inf_{t>0} \ \psi_{\lambda,u}(t) < 0$. If $\lambda > \lambda_0(u)$, then

$$\psi_{\lambda,u}(t) > \psi_{\lambda_0(u),u}(t) \geq 0, \, \forall t > 0,$$

and therefor, $\inf_{t>0} \ \psi_{\lambda,u}(t) = \psi_{\lambda,u}(0) = 0$. $\square$
Now, we turn our attention to the parameters λ^* and λ_0^*.

Proposition 2.4. *There holds $\lambda_0^* < \lambda^* < \infty$. Moreover, there exists $u \in W_0^{1,p}(\Omega)\backslash\{0\}$ such that $\lambda(u) = \lambda^*$ and $\lambda_0(u) = \lambda_0^*$.*

Proof . Indeed, from the Sobolev embedding it follows that :

$$\lambda(u) = S_{a,\gamma}\left(\frac{\|u\|_\gamma}{\|u\|}\right)^{\frac{p\gamma}{\gamma-p}} \leq S_{a,\gamma}\ C_1\left(\frac{\|u\|}{\|u\|}\right)^{\frac{p\gamma}{\gamma-p}} = S_{a,\gamma}\ C_1 < \infty, \quad \forall u \in W_0^{1,p}(\Omega)\backslash\{0\}$$
$$\Rightarrow \sup_{u \in W_0^{1,p}(\Omega)\backslash\{0\}} \lambda(u) = \lambda^* < \infty,$$

On the other hand, if $\lambda_0(u) = S_{0,a,\gamma}\, \lambda(u)$, then $\lambda_0(u) < \infty, \quad \forall u \in W_0^{1,p}(\Omega)\backslash\{0\}$, hence $\lambda_0^* < \infty$. Now observe that $\lambda(t\, u) = \lambda(u)$ for each $t > 0$, that is $\lambda(u)$ is homogeneous. It follows that there exists a sequence $u_n \in W_0^{1,p}(\Omega)\backslash\{0\}$ such that $\|u\| = 1$ and $\lambda(u_n) \to \lambda^*$ as $n \to \infty$. We can assume that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $u_n \to u$ in $L^\gamma(\Omega)$. Moreover, from (5) it follows that $u \neq 0$. By the weakly lower semi-continuity of norms, we conclude that

$$\lambda\Big(\frac{u}{\|u\|}\Big) = \lambda(u) = S_{a,\gamma}\Big(\frac{\|u\|_\gamma}{\|u\|}\Big)^{\frac{p\gamma}{\gamma-p}} \geq S_{a,\gamma}\left(\frac{\lim_{n\to\infty}\|u_n\|_\gamma}{\liminf_{n\to\infty}\|u_n\|}\right) \geq \limsup_{n\to\infty} \lambda(u_n) = \lambda^*,$$

and hence, $u_n \to u$ in $W_0^{1,p}(\Omega)$ and u satisfies $\lambda(u) = \lambda^*$. Once $\lambda_0(u)$ is a multiple of $\lambda(u)$, it follows also that $\lambda_0(u) = \lambda_0^*$ and from $S_{0,a,\gamma} < 1$, we conclude that $\lambda_0^* < \lambda^*$. $\square$
As a consequence of Proposition 2.4, we have the following

Proposition 2.5. *There holds*

(i) For each $\lambda \in (0, \lambda^*)$, we have that $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ are non-empty. Moreover, if $\lambda > \lambda^*$, then $\mathcal{N}_\lambda = \emptyset$.

(ii) For each $\lambda \in (0, \lambda_0^*)$, there exists $u \in W_0^{1,p}(\Omega)\backslash\{0\}$ such that $\phi_\lambda(tu) < 0$. Moreover, if $\lambda \geq \lambda_0^*$, then $\inf_{t>0} \psi_{\lambda,u}(t) = 0$ for all $u \in W_0^{1,p}(\Omega)\backslash\{0\}$.

Proof . (i) From Proposition 2.4, there exists $u \in W_0^{1,p}(\Omega)\backslash\{0\}$ such that $\lambda(u) = \lambda^*$. It follows from Proposition 2.3 that for each $\lambda \in (0, \lambda^*)$, the fiber map $\psi_{\lambda,u}$ satisfies (I) of Proposition 2.2, and hence, $t_\lambda^-(u)u \in \mathcal{N}_\lambda^-$ and $t_\lambda^+(u)u \in \mathcal{N}_\lambda^+$. Now suppose that $\lambda > \lambda^*$, then it follows that $\lambda > \lambda^* = \lambda(u)$ for each $u \in W_0^{1,p}(\Omega)\backslash\{0\}$, which implies from Proposition 2.3 that $\psi_{\lambda,u}$ satisfies (III) of Proposition 2.2, and hence, $\mathcal{N}_\lambda = \emptyset$.
(ii) From Proposition 2.4, there exists $u \in W_0^{1,p}(\Omega)\backslash\{0\}$ such that $\lambda_0(u) = \lambda_0^*$. It follows from Proposition 2.3 that for each $\lambda \in (0, \lambda_0^*)$, $\inf_{t>0} \phi_\lambda(tu) < 0$, then, there exists $t > 0$, such that $\phi_\lambda(tu) < 0$. Now assume that $\lambda \geq \lambda_0^*$. Therefore, $\lambda \geq \lambda_0^* \geq \lambda_0(u)$ for all $u \in W_0^{1,p}(\Omega)\backslash\{0\}$, which implies from Proposition 2.3 that $\inf_{t>0} \psi_{\lambda,u}(t) = 0$. $\square$
From Proposition 2.5, we obtain the following nonexistence result.

Corollary 2.6. *For each $\lambda > \lambda^*$, the functional ϕ_λ does not have critical points other than $u = 0$.*

Proof . Indeed, observe that for each $\lambda > \lambda^*$, there holds $\mathcal{N}_\lambda = \emptyset$. $\square$
Now, we turn our attention to some estimates which will prove to be useful on the next section. We start with :

Corollary 2.7. *Suppose that $\lambda \in (0, \lambda^*]$, then there exists $r_\lambda > 0$ such that $\|u\| \geq r_\lambda$ for each $u \in \mathcal{N}_\lambda$.*

Proof . The existence of $r_\lambda > 0$ is straightforward from

$$a\|u\|^p + \lambda\|u\|^{2p} - C\|u\|^\gamma \leq a\|u\|^p + \lambda\|u\|^{2p} - \|u\|_\gamma^\gamma = 0 \ \ \forall u \in \mathcal{N}_\lambda,$$

where $C > 0$ comes from the Sobolev embedding. Now we have :

$$\begin{aligned} & a\|u\|^p + \lambda\|u\|^{2p} - C\|u\|^\gamma \leq 0, \\ \Rightarrow\ & a + \lambda\|u\|^p - C\|u\|^{\gamma-p} \leq 0, \\ \Rightarrow\ & a + \lambda\|u\|^p \leq C\|u\|^{\gamma-p}, \\ \Rightarrow\ & r_\lambda = \Big(\frac{a+\lambda\|u\|^p}{C}\Big)^{\frac{1}{\gamma-p}} \leq \|u\|. \end{aligned}$$

$\square$

Proposition 2.8. *For each $\lambda \in (0, \lambda^*]$, there holds*

$$\phi_\lambda(u) = \frac{(\gamma - p)^2}{2p\gamma} \frac{a^2}{\lambda} \ \forall\, u \in \mathcal{N}_\lambda^0.$$

Proof . In fact, if $u \in \mathcal{N}_\lambda^0$, then

$$\begin{cases} a\|u\|^p + \lambda\|u\|^{2p} - \|u\|_\gamma^\gamma = 0, \\ p\, a\|u\|^p + 2p\,\lambda\|u\|^{2p} - \gamma\,\|u\|_\gamma^\gamma = 0. \end{cases} \tag{2.7}$$

It follows from (2.5) that $a\|u\|^p + \lambda\|u\|^{2p} = \|u\|_\gamma^\gamma$, then

$$\begin{aligned} & p\,a\|u\|^p + 2p\,\lambda\|u\|^{2p} - a\,\gamma\|u\|^p - \lambda\,\gamma\|u\|^{2p} = 0 \\ \Rightarrow\ & (p-\gamma)\,a\|u\|^p + (2p-\gamma)\,\lambda\|u\|^{2p} = 0 \\ \Rightarrow\ & (p-\gamma)\,a + (2p-\gamma)\,\lambda\|u\|^p = 0 \\ \Rightarrow\ & (2p-\gamma)\,\lambda\|u\|^p = a\,(p-\gamma), \end{aligned}$$

and hence

$$\|u\| = \left(\frac{a}{\lambda}\left(\frac{\gamma - p}{2p - \gamma}\right)\right)^{\frac{1}{p}}. \tag{2.8}$$

Moreover, from (2.5) we also have that

$$\begin{aligned} \phi_\lambda(u) &= \frac{a}{p}\|u\|^p + \frac{\lambda}{2p}\,\lambda\|u\|^{2p} - \frac{1}{\gamma}\|u\|_\gamma^\gamma \\ &= \frac{a}{p}\|u\|^p + \frac{\lambda}{2p}\,\lambda\|u\|^{2p} - \frac{a}{\gamma}\|u\|^p - \frac{\lambda}{\gamma}\|u\|^{2p} \\ &= a\,(\frac{\gamma - p}{p\gamma})\|u\|^p - \lambda\,(\frac{2p-\gamma}{2p\gamma})\|u\|^{2p} \end{aligned}$$

hence

$$\phi_\lambda(u) = a\,(\frac{\gamma - p}{p\gamma})\|u\|^p - \lambda\,(\frac{2p-\gamma}{2p\gamma})\|u\|^{2p},\ \forall\, u \in \mathcal{N}_\lambda^0. \tag{2.9}$$

We combine (2.6) with (2.7) and have

$$\begin{aligned} \phi_\lambda(u) &= a\,(\frac{\gamma - p}{p\gamma})\left(\frac{a}{\lambda}\left(\frac{\gamma - p}{2p-\gamma}\right)\right) - \lambda\,(\frac{2p-\gamma}{2p\gamma})\left(\frac{a}{\lambda}\left(\frac{\gamma - p}{2p-\gamma}\right)\right)^2 \\ &= \frac{(\gamma-p)^2\,a^2}{(p\gamma)\,(2p-\gamma)\,\lambda} - \frac{a^2\,(\gamma-p)^2}{(2p\gamma)\,(2p-\gamma)\,\lambda} \\ &= \frac{(\gamma-p)^2\,a^2}{(2p\gamma)\,(2p-\gamma)\,\lambda}. \end{aligned}$$

Then the proof is completed. □

We conclude this section with some variational properties related to the functional ϕ_λ.

Lemma 2.9. *For each $\lambda \in (0, \lambda^*)$, there holds*

(i) The functional ϕ_λ is weakly lower semi-continuous and coercive.

(ii) *Suppose that u_n is a Palais-Smale sequence at the level $c \in \mathbb{R}$, that is $\phi_\lambda(u_n) \to c$ and $\phi'_\lambda(u_n) \to 0$ as $n \to \infty$, then u_n converge strongly to some u in $W_0^{1,p}(\Omega)$.*

(iii) *There exist $C_\lambda > 0$ and $\rho_\lambda > 0$ satisfying*

$$\phi_\lambda(u) \geq C_\lambda, \ \forall\, u \in W_0^{1,p}(\Omega), \ \|u\| = \rho_\lambda$$

and

$$\lim_{C_\lambda \to 0} \rho_\lambda = 0.$$

Proof . (i) The proof of (i) is obvious. To prove (ii), observe from (i) that u_n is bounded, and therefore, we can assume that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $u_n \to u$ in $L^\gamma(\Omega)$. From the limit $\phi'_\lambda(u_n) \to 0$ as $n \to \infty$, we have

$$\phi'_\lambda(u_n) = a\Big(\int_\Omega |\nabla u|^p dx\Big) + \lambda\Big(\int_\Omega |\nabla u|^p dx\Big)^2 - \int_\Omega |u|^\gamma dx \to 0, \ \ \textbf{as} \ \ n \to \infty.$$

Since $u_n \to u$ in $L^\gamma(\Omega)$, we infer that

$$\limsup_{n\to\infty} \ \Big[- \Big(a + \lambda \, \|u_n\|^p\Big) \Delta_p \, u_n(u_u - u)\Big] = \limsup_{n\to\infty} \ \ |u_u|^{\gamma-2} \, u_u \, (u_n - u) = 0,$$

which easily implies that $u_n \to u$ in $W_0^{1,p}(\Omega)$.
(iii) It follows from the inequality

$$\phi_\lambda(u) = \frac{a}{p}\|u\|^p + \frac{\lambda}{2p}|u\|^{2p} - \frac{1}{\gamma}\|u\|_\gamma^\gamma \geq \frac{a}{p}\|u\|^p + \frac{\lambda}{2p}\|u\|^{2p} - \frac{C}{\gamma}\|u\|^\gamma \ \forall\, u \in W_0^{1,p}(\Omega),$$

where $C > 0$ is Sobolev embedding constant. □

3. Local minimizers for ϕ_λ

In this section, we prove the following

Proposition 3.1. *For each $\lambda \in (0, \lambda^*)$, the functional ϕ_λ has a local minimizer $u_\lambda \in W_0^{1,p}(\Omega)\backslash\{0\}$. Moreover, if $\lambda \in (0, \lambda_0^*)$, then $\phi_\lambda(u_\lambda) < 0$ while $\phi_{\lambda_0^*}(u_{\lambda_0^*}) = 0$ and if $\lambda \in (\lambda_0^*, \lambda^*)$, then $\phi_\lambda(u_\lambda) > 0$.*

Remark 3.2. *In fact, if $\lambda \in (0, \lambda_0^*]$, then the local minimizer given by Lemma 3.3 is a global minimizer.*

We divide the proof of Proposition 3.1 in some Lemmas.

Lemma 3.3. *For each $\lambda \in (0, \lambda_0^*)$, the functional ϕ_λ has a local minimizer u_λ with negative energy.*

Proof . It is a consequence of Lemma 2.9 and Proposition 2.5. □

Lemma 3.4. *The functional $\phi_{\lambda_0^*}$ has a global minimizer $u_{\lambda_0^*} \neq 0$ with zero energy.*

Proof . Suppose that $\lambda_n \uparrow \lambda_0^*$ as $n \to \infty$ and for each n choose $u_n \equiv u_{\lambda_n}$, where u_{λ_n} is given by Lemma 3.3. From the inequality $\phi_{\lambda_n}(u_n) < 0$ for each n and Lemma 2.9, we obtain that u_n is bounded. Therefore, we can assume that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $u_n \to u$ in $L^\gamma(\Omega)$. From Lemma 2.9, we have that

$$\phi_{\lambda_0^*}(u) \leq \liminf_{n\to\infty} \phi_{\lambda_n}(u_n) \leq 0.$$

From Proposition 2.5, we conclude that $\phi_{\lambda_0^*}(u) = 0$, and hence, $\phi_{\lambda_0^*}(u) = lim_{n\to\infty} \phi_{\lambda_n}(u_n)$. Therefore, $u_n \to u$ in $W_0^{1,p}(\Omega)$ and from Proposition 2.7, we obtain that $u \neq 0$. If $u_{\lambda_0^*} \equiv u$, the proof is complete. □

Remark 3.5. *Observe that* $\lambda_0^*(u_{\lambda_0^*}) = \lambda_0^*$, *and hence,* $\lambda^*(u_{\lambda_0^*}) = \lambda^*$.

In order to show the existence of local minimizers when $\lambda > \lambda_0^*$, we need the following definition : For $\lambda \in (0, \lambda^*)$, define

$$\widehat{\phi}_\lambda = \inf\Big\{\phi_\lambda(u) \,:\, u \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-\Big\}. \tag{3.1}$$

Remark 3.6. *From the Propositions 2.2 and 2.5, we conclude that*

$$\widehat{\phi}_\lambda = \inf_{u\in W_0^{1,p}(\Omega)} \phi_\lambda(u),\ \forall\lambda \in (0, \lambda_0^*].$$

Proposition 3.7. *For each* $\lambda \in (\lambda_0^*, \lambda^*)$, *there holds*

$$\widehat{\phi}_\lambda < \frac{(\gamma-p)^2}{2p\,\gamma\,(2p-\gamma)}\frac{a^2}{\lambda}.$$

Proof . Indeed, first observe from Remark 2 that $t_\lambda^+(u_{\lambda_0^*})$ is defined for each $\lambda \in (\lambda_0^*, \lambda^*)$. From Proposition A.2 in "Appendix" , we know that $t_\lambda^-(u_{\lambda_0^*}) < t_{\lambda_0^*}(u_{\lambda_0^*}) < t_\lambda^+(u_{\lambda_0^*})$ for each $\lambda \in (\lambda_0^*, \lambda^*)$, and therefore,

$$\begin{aligned}\widehat{\phi}_\lambda &\leq \phi_\lambda(t_\lambda^+(u_{\lambda_0^*})\,u_{\lambda_0^*})\\ &< \phi_\lambda(t_{\lambda^*}(u_{\lambda_0^*})\,u_{\lambda_0^*})\\ &< \phi_{\lambda^*}(t_{\lambda^*}(u_{\lambda_0^*})\,u_{\lambda_0^*})\\ &= \frac{(\gamma-p)^2}{2p\,\gamma\,(2p-\gamma)}\frac{a^2}{\lambda^*},\quad \forall\,\lambda \in (\lambda_0^*, \lambda^*),\end{aligned} \tag{3.2}$$

where the equality comes from Proposition 2.8. We combine (3.2) with $\lambda < \lambda^*$ have

$$\widehat{\phi}_\lambda < \frac{(\gamma-p)^2}{2p\,\gamma\,(2p-\gamma)}\frac{a^2}{\lambda^*} < \frac{(\gamma-p)^2}{2p\,\gamma\,(2p-\gamma)}\frac{a^2}{\lambda}.$$

that complete the proof. □

Lemma 3.8. *For each* $\lambda \in (\lambda_0^*, \lambda^*)$, *there exists* $u_\lambda \in \mathcal{N}_\lambda^+$ *such that* $\phi_\lambda(u_\lambda) = \widehat{\phi}_\lambda$.

Proof . Indeed, suppose that $u_n \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$ satisfies $\phi_\lambda(u_n) \to \widehat{\phi}_\lambda$. From Lemma 2.9, we have that u_n is bounded, and therefore, we can assume that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $u_n \to u$ in $L^\gamma(\Omega)$. From $a\|u\|^p + \lambda|u\|^{2p} - \|u\|_\gamma^\gamma = 0$ for all n and Proposition 2.7, we conclude that $u \neq 0$. We claim that $u_n \to u$ in $W_0^{1,p}(\Omega)$. In fact, suppose on the contrary that this is false. It follows that

$$\psi'_{\lambda,u}(1) = a\|u\|^p + \lambda|u\|^{2p} - \|u\|_\gamma^\gamma < \liminf_{n\to\infty}(a\|u_n\|^p + \lambda\|u_n\|^{2p} - \|u_n\|_\gamma^\gamma) = 0,$$

and hence, we conclude that the fiber map $\psi_{\lambda,u}$ satisfies (I) of Proposition 2.2 and $t^-_\lambda(u) < 1 < t^+_\lambda(u)$. It follows that

$$\phi_\lambda(t^-_\lambda(u)u) < \phi_\lambda(u) \le \liminf_{n\to\infty} \phi_\lambda(u_n) = \widehat{\phi}_\lambda,$$

which is a contradiction since $t^+_\lambda(u)u \in \mathcal{N}^+_\lambda$. We conclude that $u_n \to u$ in $W^{1,p}_0(\Omega)$, and hence $\phi_\lambda(u) = \widehat{\phi}_\lambda$. From Proposition 2.8 and 3.7, $\phi_\lambda(u) = \widehat{\phi}_\lambda < \dfrac{(\gamma-p)^2}{2p\,\gamma\,(2p-\gamma)}\dfrac{a^2}{\lambda}$, then $\|u\| > \Big(\dfrac{a}{\lambda}\,(\dfrac{\gamma-p}{2p-\gamma})\Big)^{\frac{1}{p}}$, hence we obtain that $u \in \mathcal{N}^+_\lambda$. □

Now, we want to prove the proposition (1.3).

Proof . (of Proposition 3.1.) Lemmas 3.3 and 3.4 guarantee the existence of a local minimizer u_λ for the functional ϕ_λ satisfying : if $\lambda \in (0, \lambda^*_0)$, then $\phi_\lambda(u_\lambda) < 0$ while $\phi_{\lambda^*_0}(u_{\lambda^*_0}) = 0$. For $\lambda \in (\lambda^*_0, \lambda^*)$, we use Lemma 3.5 in order to obtain a Local minimizer for the functional ϕ_λ. It remains to show that $\phi_\lambda(u_\lambda) > 0$ for $\lambda \in (\lambda^*_0, \lambda^*)$; however, once $\widehat{\phi}_{\lambda^*_0} = 0$, this is a consequence of Proposition A.1. that is :

$$0 = \widehat{\phi}_{\lambda^*_0} < \widehat{\phi}_\lambda = \phi_\lambda(u_\lambda),$$

hence the proof is completed. □

4. Mountain pass solution for ϕ_λ

In this section, we show the existence of a mountain pass-type solution to problem (1). In order to formulate our result, we need to introduce some notation. For each $\lambda \in (0, \lambda^*)$, define

$$c_\lambda = \inf_{\varphi\in\Gamma_\lambda} \max_{t\in[0.1]} \phi_\lambda(\varphi(t)), \tag{4.1}$$

where $\Gamma_\lambda = \Big\{\varphi \in C([0,1], W^{1,p}_0(\Omega)) \,:\, \varphi(0) = 0,\ \varphi(1) = \overline{u}_\lambda\Big\}$ with $\overline{u}_\lambda = u_{\lambda^*_0}$ if $\lambda \in (0, \lambda^*_0]$ and $\overline{u}_\lambda = u_\lambda$ for $\lambda \in (\lambda^*_0, \lambda^*)$.

Proposition 4.1. *There exists $\varepsilon > 0$ such that for each $\lambda \in (0, \lambda^*_0 + \varepsilon)$, one can find $w_\lambda \in W^{1,p}_0(\Omega)$ satisfying $\phi_\lambda(w_\lambda) = c_\lambda$ and $\phi'_\lambda(w_\lambda) = 0$. Moreover, $c_\lambda > 0$ and $c_\lambda > \widehat{\phi}_\lambda$.*

To proof the Proposition 4.1, we need some auxiliary results.

Lemma 4.2. *Given $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that*

$$0 < \widehat{\phi}_\lambda \le \delta,\ \forall\lambda \in (\lambda^*_0, \lambda^*_0 + \varepsilon_\delta).$$

Proof . The inequality $\widehat{\phi}_\lambda > 0$ follows from Proposition 3.1. Let $u_{\lambda^*_0}$ be given as in Lemma 3.4. Observe that if $\lambda \downarrow \lambda^*_0$, then $\phi_\lambda(u_{\lambda^*_0}) \to \phi_{\lambda^*_0}(u_{\lambda^*_0}) = 0$. Moreover, since from Remark 3.5 the fiber map $\psi_{\lambda^*_0, u_{\lambda^*_0}}$ satisfies (I) of Proposition 2.2, we have from Proposition 2.3 that $\lambda^*_0 < \lambda(u_{\lambda^*_0})$. It follows that there exists $\varepsilon_1 > 0$ such that $\lambda^*_0 + \varepsilon_1 < \lambda(u_{\lambda^*_0})$. From Proposition 2.2 and 2.3, for each $\lambda \in (\lambda^*_0, \lambda^*_0 + \varepsilon_1)$, there exists $t^+_\lambda(u_{\lambda^*_0}) > 0$ such that $t^+_\lambda(u_{\lambda^*_0})u_{\lambda^*_0} \in \mathcal{N}^+_\lambda$. Note that $t^+_\lambda(u_{\lambda^*_0}) \to 1$ as $\lambda \downarrow \lambda^*_0$, and therefore,

$$\widehat{\phi}_\lambda \le \phi_\lambda(t^+_\lambda(u_{\lambda^*_0})u_{\lambda^*_0}) \to \phi_{\lambda^*_0}(u_{\lambda^*_0}) = 0,\quad \lambda \downarrow \lambda^*_0.$$

If $\varepsilon_{2,\delta} > 0$ is chosen in such a way that $\phi_\lambda(t^+_\lambda(u_{\lambda^*_0})u_{\lambda^*_0}) < \delta$ for each $\lambda \in (\lambda^*_0, \lambda^*_0 + \varepsilon_{2,\delta})$, then we set $\varepsilon_\delta = \min\{\varepsilon_1, \varepsilon_{2,\delta}\}$, we have $\phi_\lambda(t^+_\lambda(u_{\lambda^*_0})u_{\lambda^*_0}) < \delta$ for each $\lambda \in (\lambda^*_0, \lambda^*_0 + \varepsilon_\delta)$, that the proof is completed.
□

Definition 4.3. *For $\lambda \in (0, \lambda^*)$, denote*

$$\mathcal{M}_\lambda = \min \Big\{ C_\lambda, \frac{(\gamma - p)^2}{2p\gamma\,(2p-\gamma)} \frac{a^2}{\lambda} \Big\}, \tag{4.2}$$

where C_λ is given by Lemma 2.9 and $\frac{(\gamma-p)^2}{2p\gamma\,(2p-\gamma)}\frac{a^2}{\lambda}$ is given by Proposition 2.8. We assume that $\rho_\lambda > r_\lambda$ where both numbers are given by Lemma 2.9 and Proposition 2.7, respectively. choose $0 < \delta < \mathcal{M}_\lambda$, and from Proposition 3.7 we take the corresponding ε_δ.

Now, we are in position to prove Proposition 4.1.
Proof . (of Proposition 4.1.) The proof will be done once we show that the functional ϕ_λ has a mountain pass geometry(remember that u_λ is a local minimizer for ϕ_λ); however, one can see from Definition 4.3 that :

$$\inf_{\|u\|=\rho_\lambda} \phi_\lambda(u) \geq \mathcal{M}_\lambda > \max\{\phi_\lambda(0), \phi_\lambda(\overline{u}_\lambda)\}, \tag{4.3}$$

which is the desired mountain pass geometry. It follows that $c_\lambda \geq \mathcal{M}_\lambda > \phi_\lambda(\overline{u}_\lambda)$ and if $\lambda \in (0, \lambda_0^*]$, then $\phi_\lambda(\overline{u}_\lambda) = \phi_\lambda(u_{\lambda_0^*}) \geq \widehat{\phi}_\lambda$ and $\phi_\lambda(\overline{u}_\lambda) = \phi_\lambda(u_\lambda) = \widehat{\phi}_\lambda$ otherwise.

We infer that there exists a Palais-Smale sequence for the functional ϕ_λ at the level c_λ, that is, there exists $w_n \in W_0^{1,p}(\Omega)$ such that $\phi_\lambda(w_n) \to c_\lambda$ and $\phi'_\lambda(w_n) = 0$. From Lemma 2.9, we have that $w_n \to w_\lambda$ in $W_0^{1,p}(\Omega)$, and hence, $\phi_\lambda(w_n) \to \phi_\lambda(w_\lambda)$ then $c_\lambda = \phi_\lambda(w_\lambda)$ and $\phi'_\lambda(w_\lambda) = 0$. □

5. Proof of Theorem 1.1

In this section, we prove our main result.
Proof . (of Theorem 1.1.) The existence of u_λ and w_λ is given by Proposition 3.1 and Proposition 4.1. Observe that u_λ being a global minimizer for ϕ_λ when $\lambda \in (0, \lambda_0^*]$, it is obviously a critical point for ϕ_λ and hence a solution to (1). If $\lambda \in (\lambda_0^*, \lambda^*)$, we saw in Lemma 3.8 that $u_\lambda \in \mathcal{N}_\lambda^+$, and hance, from Lemma 2.1 it is a critical point for the functional ϕ_λ. The case of $\lambda = \lambda^*$ goes as following. Choose a sequence $\lambda_n \uparrow \lambda^*$ and a corresponding sequence $u_n = u_{\lambda_n}$ such that $\phi_{\lambda_n}(u_n) = \widehat{\phi}_{\lambda_n}$ and $\phi'_{\lambda_n}(u_n) = 0$ for each $n \in \mathbb{N}$. Observe from the proof of Proposition 3.7 that :

$$\widehat{\phi}_{\lambda_n} < \frac{(\gamma-p)^2}{2p\gamma\,(2p-\gamma)} \frac{a^2}{\lambda^*}, \ \ \forall n \in \mathbb{N},$$

and therefore, from Lemma 2.9 we conclude that $u_n \to u$ in $W_0^{1,p}(\Omega)$. From Proposition A.1, in Appendix, we obtain that:

$$\phi_{\lambda^*}(u) = \lim_{n\to\infty} \phi_{\lambda_n}(u_n) = \lim_{n\to\infty} \widehat{\phi}_{\lambda_n} > 0,$$

and hence, $u \neq 0$. By passing the limit, it follows that $\phi'(\lambda^*)(u) = 0$. Moreover, from the definition of λ^* we also obtain that $\phi''_{\lambda^*}(u)(u,u) = 0$. If we set $u_{\lambda^*} \equiv u$, the proof of Theorem 1.1 item (1), (2) and (3) is complete.

The item (4) is a consequence of Proposition 4.1. Item (5) is proved by using the fact that every critical point of ϕ_λ lies in $\mathcal{N}_\lambda$ and Proposition 2.5. To conclude, we observe that standard arguments using the fact that $\phi_\lambda(u) = \phi_\lambda(|u|)$ provide positive solutions. □

6. Asymptotic behavior of u_λ and w_λ as $\lambda \downarrow 0$

Define $\phi_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$\phi_0(u) = \frac{a}{p}\|u\|^p - \frac{1}{\gamma}\|u\|_\gamma^\gamma,$$

and observe that $\phi_0(u_{\lambda_0^*}) < \phi_{\lambda_0^*}(u_{\lambda_0^*}) = 0$, where $u_{\lambda_0^*}$ is given by Theorem 1.1. Define

$$c_0 = \inf_{\varphi\in\Gamma} \max_{t\in[0,1]} \phi_0(\varphi(t)),$$

where $\Gamma = \{\varphi \in C([0,1] : W_0^{1,p}(\Omega)) : \varphi(0) = 0, \varphi(1) = u_{\lambda_0^*}\}$. Standard arguments provide a function $w_0 \in W_0^{1,p}(\Omega)$ such that $\phi_0(w_0) = c_0 > 0$ and $\phi_0'(w_0) = 0$. For $\lambda \in (0, \lambda_0^*)$, let us assume that u_λ and w_λ are given by Theorem 1.1. In this section, we prove the following :

Proposition 6.1. *There holds*

(i) $\phi_\lambda(u_\lambda) \to -\infty$ *and* $\|u_\lambda\| \to \infty$ *as* $\lambda \downarrow 0$.

(ii) $w_\lambda \to w_0$ *in* $W_0^{1,p}(\Omega)$ *where* $w_0 \in W_0^{1,p}(\Omega)$ *satisfies* $\phi_0(w_0) = c_0$ *and* $\phi_0'(w_0) = 0$.

Proof . (i) Indeed, choose any $u \in W_0^{1,p}(\Omega)$ and suppose without loss of generality that $\lambda \in (0, \lambda(u))$. It follows from Proposition 2.2 that

$$\psi_{\lambda,u}(t) \ge \psi_{\lambda,u}(t_\lambda^+(u)) = \phi_\lambda(t_\lambda^+(u)u) \ge \inf_{u\in W_0^{1,p}(\Omega)} \phi_\lambda(u) = \widehat{\phi}_\lambda$$

Now, observe that for fixed $t > 0$, there holds :

$$\psi_{\lambda,u}(t) \to \frac{a}{p}\|u\|^p - \frac{1}{\gamma}\|u\|_\gamma^\gamma, \ \ as\, \lambda \downarrow 0. \tag{6.1}$$

Once

$$\lim_{t\to\infty}\left(\frac{a}{p}\|u\|^p - \frac{1}{\gamma}\|u\|_\gamma^\gamma\right) = -\infty,$$

it follows from (17) that given $M < 0$, there exists $t > 0$ and $\delta > 0$ such that if $\lambda \in (0, \delta)$, then $\psi_{\lambda,u}(t) < M$ and hence $\widehat{\phi}_\lambda < M$, which proves that $\phi_\lambda(u_\lambda) \to -\infty$ as $\lambda \downarrow 0$. One can easily infer from the last convergence that $\|u\| \to \infty$ as $\lambda \downarrow 0$. □
To prove the item (ii) of Proposition 6.1, we need to establish some results.

Lemma 6.2. *The function* $(0, \lambda_0^*] \ni \lambda \mapsto c_\lambda = \phi_\lambda(w_\lambda)$ *is non-decreasing. Moreover,* $c_\lambda \to c_0$ *as* $\lambda \downarrow 0$.

Proof . First, observe that $\Gamma_\lambda = \Gamma$ for each $\lambda \in (0, \lambda_0^*]$. Suppose that $0 \le \lambda < \lambda' < \lambda_0^*$ and fix any $\varphi \in \Gamma$. It follows that $\max_{t\in[0,1]} \phi_\lambda(\varphi(t)) < \max_{t\in[0,1]} \phi_{\lambda'}(\varphi(t))$ and by taking the infimum in both sides, we conclude that $c_\lambda \le c_{\lambda'}$.
Once c_λ is non-decreasing, we can assume that $c_\lambda \to c \ge c_0$ as $\lambda \downarrow 0$. Suppose on the contrary that $c > c_0$. Given $\delta > 0$ such that $c_0 + \delta < c$ choose $\varphi \in \Gamma$ such that $c_0 \le \max_{t\in[0,1]} \phi_0(\varphi(t)) < c_0 + \delta$. If λ is sufficiently close to 0, then :

$$c_0 \le \max_{t\in[0,1]} \phi_0(\varphi(t)) < \max_{t\in[0,1]} \phi_\lambda(\varphi(t)) < c_0 + \delta,$$

and consequently $c_0 \leq c_\lambda < c_0 + \delta < c < c_\lambda$ which is clearly a contradiction, and therefore, $c_\lambda \to c_0$ as $\lambda \downarrow 0$. □

Now, we may finish the proof of Proposition 6.1;

Proof . (of (ii) of Proposition 6.1.) Suppose that $\lambda_n \downarrow 0$ and for each $n \in \mathbb{N}$, choose $w_n \equiv w_{\lambda_n}$ such that $\phi_{\lambda_n}(w_n) = c_{\lambda_n}$ and $\phi'_{\lambda_n}(w_n) = 0$. We claim that $\lambda_n \|w_n\|^{2p} \to 0$ as $n \to \infty$. In fact, for each n we can find a path $\varphi_n \in \Gamma_n = \Gamma$ and a function v_n such that $\phi_{\lambda_n}(v_n) = \max_{t\in[0,1]} \phi_{\lambda_n}(\varphi_n(t))$ and

$$0 < \phi_{\lambda_n}(v_n) - c_{\lambda_n} \to 0, \;\; \|v_n - w_n\| \to 0, \;\; \|v_n - w_n\|_\gamma \to 0, \; as \; n \to \infty. \tag{6.2}$$

Now, observe from the definition of c_0, Lemma 6.2 and (6.2) that

$$0 < \lim_{n\to\infty} \phi_0(v_n) - c_0 \leq \lim_{n\to\infty} \phi_{\lambda_n}(v_n) - c_0 = \lim_{n\to\infty} (\phi_{\lambda_n}(v_n) - c_{\lambda_n}). \tag{6.3}$$

It follows from (6.2) and (6.3) that :

$$\frac{a}{p}\|v_n\|^p - \frac{1}{\gamma}\|v_n\|_\gamma^\gamma \to 0, \quad and \;\; \frac{a}{p}\|v_n\|^p + \frac{\lambda_n}{2p}\|v_n\|^{2p} - \frac{1}{\gamma}\|v_n\|_\gamma^\gamma, \; as \, n \to \infty,$$

which implies that $\lambda_n \|v_n\|^{2p} \to 0$ as $n \to \infty$. From (6.2), we conclude that :

$$\Big|\lambda_n \|w_n\|^{2p} - \lambda_n \|v_n\|^{2p}\Big| \to 0, \; as \, n \to \infty,$$

and hence, $\lambda_n \|w_n\|^{2p} \to 0$ as $n \to \infty$ as we desired. Now, from the equations $\phi_{\lambda_n}(w_n) = c_{\lambda_n}$ and $\phi'_{\lambda_n}(w_n) = 0$, $n \in \mathbb{N}$, we have

$$\begin{cases} \frac{a}{p}\|w_n\|^p + \frac{\lambda_n}{2p}\|w_n\|^{2p} - \frac{1}{\gamma}\|w_n\|_\gamma^\gamma = c_{\lambda_n}, \\ a\|w_n\|^p + \lambda_n\|w_n\|^{2p} - \|w_n\|_\gamma^\gamma = 0, \end{cases} \tag{6.4}$$

which combined with the limit $\lambda_n \|w_n\|^{2p} \to 0$ as $n \to \infty$ and Lemma 6.2 implies that

$$\begin{cases} \frac{a}{p}\lambda_n\|w_n\|^p - \frac{\lambda_n}{\gamma}\|w_n\|_\gamma^\gamma = o(1), \\ a\lambda_n\|w_n\|^p - \lambda_n\|w_n\|_\gamma^\gamma = 0. \end{cases}$$

We multiply the first equation by $-\gamma$ and sum it with the second equation to obtain that

$$-\gamma\frac{a}{p}\lambda_n\|w_n\|^p + \lambda_n\|w_n\|_\gamma^\gamma + a\lambda_n\|w_n\|^p - \lambda_n\|w_n\|_\gamma^\gamma = o(1),$$

then

$$(-\frac{\gamma}{p} + 1)a\lambda_n\|w_n\|^p = o(1),$$

which implies that $\lambda_n \|w_n\|^p \to 0$ as $n \to \infty$. Now, we claim that $\|w_n\|$ is bounded. In fact, suppose on the contrary that up to a subsequence $\|w_n\|^p \to \infty$ as $n \to \infty$. From (6.4) we obtain that

$$\begin{cases} \frac{a}{p} + \frac{\lambda_n}{2p}\|w_n\|^p - \frac{1}{\gamma}\frac{\|w_n\|_\gamma^\gamma}{|w_n\|^p} = o(1), \\ \\ a + \lambda_n\|w_n\|^p - \frac{\|w_n\|_\gamma^\gamma}{|w_n\|^p} = 0. \end{cases}$$

Once $\lambda_n \|w_n\|^p \to 0$ as $n \to \infty$, we have $a - \frac{\|w_n\|_\gamma^\gamma}{|w_n\|^p} = 0$, hence $a = \frac{\|w_n\|_\gamma^\gamma}{|w_n\|^p}$, then $\frac{a}{p} - \frac{a}{\gamma} = o(1)$ and while $n \to \infty$ we conclude that $a(\frac{1}{p} - \frac{1}{\gamma}) = 0$, and $\gamma = p$, which is a contradiction. Since $\|w_n\|$ is bounded we obtain that $\phi_0(w_n) \to c_0$ and $\phi'_0(w_n) \to 0$ as $n \to \infty$, and hence, $w_n \to w_0$ as $n \to \infty$, where w_0 satisfies $\phi_0(w_0) = c_0$ and $\phi'_0(w_0) = 0$. □

Now, by using the Proposition 6.1, the Theorem 1.2 is proved.

Proof . (of Theorem 1.2.) It is a consequence of Proposition 6.1. □

7. Some conclusions and remarks

Appendix A

Proposition A.1. The function $(0,\lambda^*) \ni \lambda \mapsto \widehat{\phi}_\lambda$ is continuous and increasing.

Proof . First we prove that $(0,\lambda^*) \ni \lambda \mapsto \widehat{\phi}_\lambda$ is increasing. Indeed, suppose that $\lambda < \lambda'$. From Lemmas 3.3, 3.4 and 3.8, there exists $u_{\lambda'} = \phi_{\lambda'}(u_{\lambda'})$. Since the fiber map $\psi_{\lambda',u_{\lambda'}}$ obviously satisfies (I) of Proposition 2.2 it follows from Proposition 2.3 that $\psi_{\lambda,u_{\lambda'}}$ also satisfies (I) of Proposition 2.2 and then :

$$\widehat{\phi}_\lambda \leq \phi_\lambda(t^+_\lambda(u_{\lambda'})u_{\lambda'}) < \phi_\lambda(t^+_{\lambda'}(u_{\lambda'})u_{\lambda'}) = \phi_{\lambda'}(u_{\lambda'}) = \widehat{\phi}_{\lambda'}.$$

Now we prove that $(0,\lambda^*) \ni \lambda \mapsto \widehat{\phi}_\lambda$ is continuous. In fact, suppose that $\lambda_n \uparrow \lambda \in (0,\lambda^*)$ and choose $u_n \equiv u_{\lambda_n}$ such that $\widehat{\phi}_{\lambda_n} = \phi_{\lambda_n}(u_n)$ for all n. Similar to the proof of Lemma 3.8 we may assume that $u_n \to u \in \mathcal{N}^+_\lambda$. We claim that $\widehat{\phi}_{\lambda_n} \to \widehat{\phi}_\lambda$ as $n \to \infty$. Indeed, once $(0,\lambda^*) \ni \lambda \mapsto \widehat{\phi}_\lambda$ is increasing, we can assume that $\widehat{\phi}_{\lambda_n} < \widehat{\phi}_\lambda$ for each n and $\widehat{\phi}_{\lambda_n} \to \phi_\lambda(u) \leq \widehat{\phi}_\lambda$ as $n \to \infty$, which implies that $\phi_\lambda(u) = \widehat{\phi}_\lambda$.

Now suppose that $\lambda_n \downarrow \lambda \in (0,\lambda^*)$. Once $(0,\lambda^*) \ni \lambda \mapsto \widehat{\phi}_\lambda$ is increasing, we can assume that $\widehat{\phi}_{\lambda_n} > \widehat{\phi}_\lambda$ for each n and $\lim_{n\to\infty} \widehat{\phi}_{\lambda_n} \geq \widehat{\phi}_\lambda$. choose u_λ such that $\widehat{\phi}_\lambda = \phi_\lambda(u_\lambda)$ and observe that $\widehat{\phi}_\lambda \leq \lim_{n\to\infty} \widehat{\phi}_{\lambda_n}(t^+_{\lambda_n}(u_\lambda)u_\lambda) = \widehat{\phi}_\lambda$. □ For the next proposition we assume that $u_{\lambda_0^*}$ is given as in Lemma 3.4 and $t(u_{\lambda_0^*})$ is defined in (2.1).

Observe from Remark 3.5 that $t^+_\lambda(u_{\lambda_0^*})$ is well defined for each $\lambda \in (0,\lambda^*)$.

Proposition A.2. There holds

(i) The function $(0,\lambda^*) \ni \lambda \mapsto t^+_\lambda(u_{\lambda_0^*})$ is decreasing and continuous.

(i) The function $(0,\lambda^*) \ni \lambda \mapsto t^-_\lambda(u_{\lambda_0^*})$ is increasing and continuous.
Moreover,

$$\lim_{\lambda\uparrow\lambda^*}(t^+_\lambda(u_{\lambda_0^*})) = \lim_{\lambda\uparrow\lambda^*}(t^-_\lambda(u_{\lambda_0^*})) = t(u_{\lambda_0^*}).$$

Proof . Indeed, let $t_\lambda = t^+_\lambda(u_{\lambda_0^*})$ and note t_λ satisfies $\psi'_\lambda(t_\lambda) = 0$ for each $\lambda \in (0,\lambda^*)$. By implicit differentiation and the fact that $\psi''_\lambda(t_\lambda) > 0$, we conclude that $(0,\lambda^*) \ni \lambda \mapsto t^+_\lambda(u_{\lambda_0^*})$ is decreasing and continuous, which proves (i). The proof of (ii) is done by a similar way and the limits

$$\lim_{\lambda\uparrow\lambda^*}(t^+_\lambda(u_{\lambda_0^*})) = \lim_{\lambda\uparrow\lambda^*}(t^-_\lambda(u_{\lambda_0^*})) = t(u_{\lambda_0^*}),$$

are straightforward from the definitions. □

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