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A general solution of some linear partial differential equations via two integral transforms

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Abstract

In this paper, a new analytical method is introduced to find the general solution of linear partial differential equations. In this method, each Laplace transform (LT) and Sumudu transform (ST) is used independently along with canonical coordinates. The strength of this method is that it is easy to implement and does not require initial conditions.

Keywords: Sumudu transform, Laplace transform, Linear partial differential equations, canonical coordinates.
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1. Introduction

Most of the physical phenomena, whether they are in the field of mechanics, optics, electricity, fluid flow and heat flow or other fields such as health field e.g. modeling drug concentrations in the body, computer field, industrial mathematics and other applied sciences can be described as partial or ordinary differential equations. Because of the amplitude of this applied field that it occupies mathematicians have paid attention to differential equations and finding their solutions through several methods, some are analytical and others are numerical to find the existing or approximate solutions. One of these methods for solving differential equations is integral transforms such as Laplace transform [1], which remained in control of solutions to the problems of boundary and initial values despite the appearance of many transforms after it, such as Fourier transform [7], Laplace-Carson transform [2], Elzaki transform [4], and others, however, the Laplace transform remain the unique transform since it is characterized by the following:

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It is distinguished from other transforms such as the Fourier transform by having a decay factor e^{-st}, that allows the transformation of wider rows of functions, easy to use even more than the transforms that appeared later on and most of the transforms that appeared later is derived from it, such as Sumudu, Elzaki, Natural transforms and others.

As for the Sumudu transforms, although it is similar to the Laplace transform, it is characterized by: the scaling of the function $f(t)$ differentiation, integration and unit-step function in the t-domain are equivalent to scaling of the function $F(u)$ division, multiplication and unity in u- domain.

Integral transforms are usually used to solve problems with boundary and initial conditions, but in [5] they are used to solve linear ordinary differential equations with constant coefficients with the help of the residues and without the use of boundary and initial conditions.

In our research, we use two transforms is (LT and ST), in addition to the canonical coordinates and some theorems, to find the solution of linear partial differential equations (LPDEs) without any boundary or initial conditions, as we shall explain in the description of the method. The solution obtained here is a general solution.

2. Definitions and theorems

In this section, we will list some basic definitions and theorems that we used in our work.

Definition 2.1. [9] *LT is an integral transform method which is particularly useful to solve LODE. In order for any function of time* g(t) *the LT is* $\mathcal{L}[z(t)] = Z(s) = \int_0^\infty e^{-st} z(t)dt$, *for* $s > 0$.

Definition 2.2. [9] *Let* B *is a set of functions in exponential order. Then for a given function* g(t) *in* B, *the ST is* $H(u) = S[h(t)] = \int_0^\infty e^{-t} h(\ ut\ )dt, u \in (-a, b)$ *where* $a, b >$ *0.*

Now, we give some theorems that we needed in the prove of our results.

Theorem 2.3. *[8] If* g *is continuous and* $\acute{g}$ *piecewise continuous on* $[0, \infty]$, *with* g *of exponential order* β *on* $[0, \infty]$. *If* $H(s) = \mathcal{L}[g(t)]$ *for* $\mathrm{Re}(s) = \mathrm{Re}(y + iz) = y > \beta$, *also* $|\mathrm{H(s)}| \leq \frac{N}{|s|^k}, k > 0$, *for all* $|s|$ *sufficiently large and some* $\mathrm{k}, N > 0$.

Theorem 2.4. *[3] Given a function* $\mathrm{g(x)} = e^{ix} f(x)$, *if there exist a positive integer* n *such that the function* $\varphi_k(\mathrm{x}) = (x - x_k)^n e^{ix} f(x)$, *is analytic at* x_k *and* $\varphi_k(x_k) \neq 0, \mathrm{t}$ *is a constant, then* g *has a pole of order* n *at* x_k. *Its residue there is given by*

$$\mathrm{Res}\,(g, x_k) = \frac{1}{(n-1)!} \lim_{x \to x_k} \frac{d^{n-1}}{dx^{n-1}} e^{ix} \phi_k(\mathrm{x}) \tag{2.1}$$

if $n > 1$, *and*

$$\mathrm{Res}\,(g, x_k) = Be^{ix_k} \tag{2.2}$$

if $n = 1$, *where* $\phi_k(\mathrm{x})$ *is analytic function at* x_k *and* B *is constant.*

Theorem 2.5. *[3] Let Z(s) and* $\frac{d(s)}{r(s)}$ *denote LT of the functions* z(t) *and* g(t) *respectively and* deg(r(s)) = m. *Also let* $v, \acute{v}$ *and* V *satisfy the conditions in theorem (* 2.3 *). Then the general solution of LODE of order* h *with constant coefficients*

$$\mathrm{a_h z^{(h)} + a_{h-1} z^{(h-1)} + \cdots + a_o z = g(t)} \tag{2.3}$$

where a_j, s *are constants and* $a_h \neq 0$, *is given by*

$$z(t) = \sum_{k=1}^{N} \text{Res}\left(\frac{e^{ts}p(s)}{r(s)\sum_{j=1}^{h} a_j s^j} s_k\right) \tag{2.4}$$

where $p(s)$ *is a polynomial of* s *with degree is less than or equal to* $\mathrm{m} + \mathrm{h} -$.

Theorem 2.6. *[6] Let* $Z_1(u)$ *and* $Z(u)$ *are the Sumudu and Laplace transform of the function* $z(t)$ *respectively and* $L[g(t)] = \frac{d(s)}{r(s)}$. *Also let* $\frac{1}{u}Z_1\left(\frac{1}{u}\right)$ *satisfies the relation* $\left|\frac{1}{u}Z_1\left(\frac{1}{u}\right)\right| < \frac{m}{R^k}$ *then the general solution* $Z(t)$ *of the ODE (3) at the point s=u is*

$$\begin{aligned} Z(t) &= \sum_{k=1}^{N}\left(\frac{1}{u}Z_1\left(\frac{1}{u}\right)e^{tu}, u_k\right) \\ &= \sum_{k=1}^{N} \text{Res}\left(\frac{e^{tu}p(u)}{r(u)\sum_{j=1}^{h} a_j u_j}, u_k\right) \end{aligned}$$

Note: the proof of this theorem is from the duality relation of LT and ST (see [9]).

3. Description of the method

In this section, we introduce a method for finding the general solution of the LPDEs using two integral transforms (LT and ST) without any initial condition.

First, suppose that the general LPDEs is

$$\mathrm{f}\left(\mathrm{x}, \mathrm{y}, \mathrm{t}, \mathrm{u}, \mathrm{u_t}, \mathrm{u_x}, \mathrm{u_y}, \mathrm{u_{xt}}, \mathrm{u_{yt}}, \mathrm{u_{xy}}, \mathrm{u_{tt}}, \mathrm{u_{xx}}, \mathrm{u_{yy}}, \ldots\right) = 0 \tag{3.1}$$

where $u = u(x, y, t)$ is an unknown function, f is a polynomial in $u(x, y, t)$ and its partial derivatives. The main steps of this method are as follows:

Step 1: Using the canonical coordinates $\varpi = Ax + By + Ct$, where A, B, C are not all equal to zero to convert the equation (2.1) into an ordinary differential equation for $U(\varpi)$:

$$\mathrm{F}(\varpi, \mathrm{U}, \mathrm{U}', \mathrm{U}'', \ldots) = 0 \ldots\ldots (6), \quad \text{where} \ \ \mathrm{U}(\varpi) = \mathrm{u}(\mathrm{x}, \mathrm{y}, \mathrm{t})$$

Step 2: Determine the coefficients $\mathrm{a_0}, \mathrm{a_1}, \ldots, \mathrm{a_m}$ of equation (2.3) and using the relation $V(s) = \left(\frac{p(s)}{r(s)\sum_{j=1}^{h} a_j s^j}, s_k\right)$, see [5] to find the poles of the equation.

Step 3: Substitute the poles of the equation in the residues equations (2.1), (2.2).

Step 4: Substitute the residues in (2.4) to obtain $U(\varpi)$.

Step 5: Return the variable ϖ to the original variables from (3.1) to get the general solution.

Note: By using the same steps above and the duality of the Sumudu and Laplace transforms, we can find the general solution of the equation.

4. Applications of the method using (LT)

In this section, we provide some practical examples to illustrate the method

Example 1: Consider the Klein- Gordon equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \mu^2 u = 0 \tag{4.1}$$

of the second order, where $u = u(x,t)$ and $\mu^2 = \frac{mc^2}{h^2}$, to find the general solution of the above equation.
Sol: let $\zeta = \mathrm{kx} + \mathrm{wt}$, where k, w are constant, then, we have

$$u(x,t) = U(\zeta), \frac{\partial^2 u}{\partial t^2} = w^2 \frac{d^2 U}{d\zeta^2}, \frac{\partial^2 u}{\partial x^2} = k^2 \frac{d^2 U}{d\zeta^2} \tag{4.2}$$

Substituting (4.2) in (4.1), we get:

$$\frac{1}{c^2} w^2 \frac{d^2 U}{d\zeta^2} - k^2 \frac{d^2 U}{d\zeta^2} + \mu^2 U(\zeta) = 0$$

$$\left(\frac{w^2}{c^2} - k^2\right) U(\zeta) + \mu^2 U(\zeta) = 0$$

$$a_o = \mu^2, a_1 = 0, a_2 = \frac{w^2}{c^2} - k^2$$

$$L[U(\zeta)] = \frac{P(s)}{\sum_{i=0}^{2} a_i s^i} = \frac{P(s)}{a_o + a_2 s^2} = \frac{P(s)}{\mu^2 \left(\frac{w^2}{c^2} - k^2\right) s^2}, \text{ then } s_1 = \frac{\mu c}{\sqrt{w^2 - k^2 c^2}} i, s_2 = \frac{-\mu c}{\sqrt{w^2 - k^2 c^2}} i$$

$$\operatorname{Ref}(f, s_1) = A_1 e^{\frac{\mu c \zeta i}{\sqrt{w^2 - k^2 c^2}}} \text{ and } \operatorname{Ref}(f, s_2) = A_2 e^{\frac{-\mu c \zeta i}{\sqrt{w^2 - k^2 c^2}}}$$

$$U(\zeta) = A_1 e^{\frac{\mu c \zeta i}{\sqrt{w^2 - k^2 c^2}}} + A_2 e^{\frac{-\mu c \zeta i}{\sqrt{w^2 - k^2 c^2}}} = c_1 \cos \frac{\mu c \zeta}{\sqrt{w^2 - k^2 c^2}} + c_2 \sin \frac{\mu c \zeta}{\sqrt{w^2 - k^2 c^2}},$$

where $c_1 = A_1 + A_2,$ and $c_2 = A_1 i - A_2 i.$

Then, the general solution is $u(x,t) = c_1 \cos \frac{\mu c}{\sqrt{w^2 - k^2 c^2}} (\mathrm{kx} + \mathrm{wt}) + c_2 \sin \frac{\mu c}{\sqrt{w^2 - k^2 c^2}} (\mathrm{kx} + \mathrm{wt})$
Second we explain the ST with the same example to find the general solution let $\zeta = ex + dt$, where e, c are constant, then, we have

$$u(x,t) = U(\zeta), \frac{\partial^2 u}{\partial t^2} = d^2 \frac{d^2 U}{d\zeta^2}, \frac{\partial^2 u}{\partial x^2} = e^2 \frac{d^2 U}{d\zeta^2} \tag{4.3}$$

Substituting (4.3) in (4.1), we get:

$$\frac{1}{c^2} d^2 \frac{d^2 U}{d\zeta^2} - e^2 \frac{d^2 U}{d\zeta^2} + \mu^2 U(\zeta) = 0$$

$$\left(\frac{d^2}{c^2} - e^2\right) U(\zeta) + \mu^2 U(\zeta) = 0$$

$$a_o = \mu^2, a_1 = 0, a_2 = \frac{d^2}{c^2} - e^2$$

$$L[U(\zeta)] = \frac{P(s)}{\sum_{i=0}^{2} a_i s^i} = \frac{P(s)}{a_o + a_2 s^2} = \frac{P(s)}{\mu^2 \left(\frac{d^2}{c^2} - e^2\right) s^2}, \text{ then } s_1 = \frac{\mu c}{\sqrt{d^2 - e^2 c^2}} i, s_2 = \frac{-\mu c}{\sqrt{d^2 - e^2 c^2}} i$$

$$\operatorname{Re} f(f, s_1) = B_1 e^{\frac{\mu c \zeta i}{\sqrt{d^2 - e^2 c^2}}} \text{ and } \operatorname{Re} f(f, s_2) = B_2 e^{\frac{-\mu c \zeta i}{\sqrt{d^2 - e^2 c^2}}}$$

$U(\zeta) = B_1 e^{\frac{\mu c \eta i}{\sqrt{d^2 - e^2 c^2}}} + B_2 e^{\frac{-\mu c \eta i}{\sqrt{d^2 - e^2 c^2}}} = d_1 \cos \frac{\mu c \eta}{\sqrt{d^2 - e^2 c^2}} + d_2 \sin \frac{\mu c \eta}{\sqrt{d^2 - e^2 c^2}}$, where $d_1 = B_1 + B_2$, and $d_2 = B_1 i - B_2 i$
Then, the general solution is

$$u(x,t) = d_1 \cos \frac{\mu c}{\sqrt{d^2 - e^2 c^2}} (\mathrm{ex} + \mathrm{dt}) + d_2 \sin \frac{\mu c}{\sqrt{d^2 - e^2 c^2}} (\mathrm{ex} + \mathrm{dt})$$

Example2: To find the general solution of the equation

$$u_{tt} = -\alpha^2 u_{xxxx}, \tag{4.4}$$

where $\alpha^2 = \frac{k}{p}$
Sol: let $\zeta = x + \alpha t$, where α is constant

$$u_t = \frac{\partial U}{\partial t}\frac{\partial \zeta}{\partial \zeta} = \frac{\partial U}{\partial \zeta}\frac{\partial \zeta}{\partial \mathrm{t}} = \alpha\frac{\mathrm{dU}}{\mathrm{d}\zeta}, u_{tt} = -\alpha^2\frac{d^2U}{d\zeta^2} \text{ and } u_{xxxx} = \frac{d^4U}{d\zeta^4}$$

Now, we substitute are into the equation (4.4), we get

$\alpha^2 U^{(2)} = -\alpha^2 U^{(4)}, U^{(2)} + U^{(4)} = 0,$ where $a_o = 0, a_1 = 0, a_2 = 1, a_3 = 0, a_4 = 1$

$L[U(\zeta)] = \dfrac{P(s)}{h(s)\sum_{i=0}^{4} a_i s^i} = \dfrac{P(s)}{a_o + a_1 s^1 + a_2 s^2 + a_3 s^3 + a_4 s^4} = \dfrac{P(s)}{s^2 + s^4}$, then $s_1 = 0 = s_2, s_3 = i, s_4 = -i$

$\operatorname{Ref}(f, s_1) = A_1, \operatorname{Ref}(f, s_2) = A_2, \operatorname{Ref}(f, s_3) = A_3 e^{i\zeta}$ and $\operatorname{Re} f(f, s_4) = A_4 e^{-i\zeta}$

$U(\zeta) = A_1 + A_2\zeta + A_3 e^{i\zeta} + A_4 e^{-i\zeta} = c_1 + c_2\zeta + c_3\cos\zeta + c_4\sin\zeta,$

where $c_1 = A_1, c_2 = A_2, c_3 = A_3 + A_4$ and $c_4 = A_3 i - A_4 i$

Then, the general solution is $u(x,t) = c_1 + c_2(\mathrm{x} + \alpha \mathrm{t}) + c_3\cos(\mathrm{x} + \alpha\mathrm{t}) +$ $c_4\sin(x + \alpha t)$
Second we explain the ST with the same example to find the general solution
Sol: let $\vartheta = x + \omega t$, where α is constant

$$u_t = \frac{\partial U}{\partial t}\frac{\partial \vartheta}{\partial \vartheta} = \frac{\partial U}{\partial \vartheta}\frac{\partial \vartheta}{\partial \mathrm{t}} = \alpha\frac{\mathrm{dU}}{\mathrm{d}\vartheta}, u_{tt} = -\omega^2\frac{d^2U}{d\vartheta^2} \text{ and } u_{xxxx} = \frac{d^4U}{d\zeta^4}$$

Now, we substitute are into the equation (4.4), we get

$\omega^2 U^{(2)} = -\omega^2 U^{(4)}, U^{(2)} + U^{(4)} = 0,$ where $a_o = 0, a_1 = 0, a_2 = 1, a_3 = 0, a_4 = 1$

$L[U(\vartheta)] = \dfrac{P(s)}{h(s)\sum_{i=0}^{4} a_i s^i} = \dfrac{P(s)}{a_o + a_1 s^1 + a_2 s^2 + a_3 s^3 + a_4 s^4} = \dfrac{P(s)}{s^2 + s^4}$, then $s_1 = 0 = s_2, s_3 = i, s_4 = -i$

$\operatorname{Re} f(f, s_1) = B_1, \operatorname{Re} f(f, s_2) = B_2, \operatorname{Ref}(f, s_3) = B_3 e^{i\vartheta}$ and $\operatorname{Re} f(f, s_4) = B_4 e^{-i\vartheta}$

$U(\zeta) = B_1 + B_2\vartheta + B_3 e^{i\vartheta} + B_4 e^{-i\vartheta} = c_1 + c_2\vartheta + c_3\cos\vartheta + c_4\sin\vartheta,$

where $c_1 = B_1, c_2 = B_2, c_3 = B_3 + B_4$ and $c_4 = B_3 i - B_4 i$

Then, the general solution is $u(x,t) = c_1 + c_2(\mathrm{x} + \omega \mathrm{t}) + c_3\cos(\mathrm{x} + \omega\mathrm{t}) +$ $c_4\sin(\mathrm{x} + \omega\mathrm{t})$
Example 3: Consider the linear telegraph equation

$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + (\alpha + \beta)\frac{\partial u}{\partial t} + (\alpha\beta)u, \tag{4.5}$$

of the second order, where $\alpha = \frac{G}{c}, \beta = \frac{R}{L}$ and $c^2 = \frac{1}{LG}$, to find the general solution of the above equation.

Sol: let $\zeta = \text{kx} + \text{wt}$, where k, w are constant, then, we have

$$c^2k^2\frac{d^2U}{d\zeta^2} = w^2\frac{d^2U}{d\zeta^2} + w(\alpha+\beta)\frac{dU}{d\zeta} + (\alpha\beta)U(\zeta)$$

$$\left(c^2k^2 - w^2\right)U(\zeta) - w(\alpha+\beta)\dot{U}(\zeta) - (\alpha\beta)U(\zeta) = 0,$$

$$a_o = -(\alpha\beta), a_1 = -w(\alpha+\beta), a_2 = \left(c^2k^2 - w^2\right)$$

$$L[U(\zeta)] = \frac{P(s)}{\sum_{i=0}^{2} a_i s^i} = \frac{P(s)}{-(\alpha\beta) - w(\alpha+\beta)s + \left(c^2k^2 - w^2\right)s^2}$$

$$= s_i\frac{w(\alpha+\beta) \pm \sqrt{w^2(\alpha+\beta)^2 + 4\alpha\beta\left(c^2k^2 - w^2\right)}}{2\left(c^2k^2 - w^2\right)}, \text{ then } s_1 = \gamma_1, s_2 = \gamma_2, \text{ where}$$

$\text{Ref}\,(f, s_1) = A_1e^{\zeta\gamma_1}$ and $\text{Re}\,f\,(f, s_2) = A_2e^{\zeta\gamma_2}$

$$U(\zeta) = A_1e^{\zeta\gamma_1} + A_2e^{\zeta\gamma_2} = A_1e^{\frac{\left[w(\alpha+\beta) + \sqrt{w^2(\alpha+\beta)^2 + 4\alpha\beta\left(c^2k^2 - w^2\right)}\right](\text{kx}+\text{wt})}{2\left(c^2k^2 - w^2\right)}} + A_2e^{\frac{\left[w(\alpha+\beta) + \sqrt{w^2(\alpha+\beta)^2 + 4\alpha\beta\left(c^2k^2 - w^2\right)}\right](\text{kx}+\text{wt})}{2\left(c^2k^2 - w^2\right)}}$$

Then, the general solution is $u(x,t) = A_1e^{\frac{\left[w(\alpha+\beta) + \sqrt{w^2(\alpha+\beta)^2 + 4\alpha\beta\left(c^2k^2 - w^2\right)}\right](\text{kx}+\text{wt})}{2\left(c^2k^2 - w^2\right)}} + A_2e^{\frac{\left[w(\alpha+\beta) + \sqrt{w^2(\alpha+\beta)^2 + 4\alpha\beta\left(c^2k^2 - w^2\right)}\right](\text{kx}+\text{wt})}{2\left(c^2k^2 - w^2\right)}}$

5. Open Problems

The given method can be applied to solve:

1- Linear Integro-differential equations.

2- Linear Delay differential equation

3- Linear fractional differential equation.

6. Conclusion

In this paper, an effective method was implemented to obtain the general solution of linear partial differential equations using two integral transforms (LT and ST) with the canonical coordinates and the residues. The attractive feature of this method is that it is easy to use manually and gives the solution in a few steps.

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