Int. J. Nonlinear Anal. Appl. 13 (2022) 1, 2219-2225
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.5917

Forms of ϖ-continuous functions between bitopological spaces

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(Communicated by Madjid Eshaghi Gordji)

Abstract

TThe paper introduces the concepts of ϖ-strongly (resp., ϖ-closure, ϖ-weakly) form of continuous functions on bitopological spaces, furthermore, we introduce theorems, characterizing on the class of functions, show how it can be studied from a different point of view.

Keywords: ϖ-strongly continuous, ϖ-closure continuous, ϖ-weakly continuous, bitopological spaces.
2010 MSC: 54C05, 54D10, 05C10

1. Introduction and notations

Let $\mathscr{X}$ be anon empty set and $\mathscr{T}_1$, $\mathscr{T}_2$ are two topologies on $\mathscr{X}$, then the triple $(\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2)$ is called bitopological spaces [1]. For other notions or notations not defined here we follow closely S. Willard [2].

Definition 1.1. *[3] A point x of a space $\mathscr{X}$ is called a condensation point of the sub set $\mathscr{W} \subseteq \mathscr{X}$ if every neighbourhood of x contains an uncountable subset of this set.*

Definition 1.2. *[3] A subset $\mathscr{W}$ of a space $\mathscr{X}$ is called ϖ-closed if all its condensation points are contains it. Also the ϖ-closure of a set $\mathscr{W}$ is the intersection of all ϖ-closed sets that contains $\mathscr{W}$, and denoted by $Cl^{\varpi}\mathscr{W}$, then $\mathscr{W}$ is ϖ-closed if and only if $\mathscr{W} = Cl^{\varpi}\mathscr{W}$. The complete of a ϖ-closed is called ϖ-open. Similarly, the ϖ-interior of a set $\mathscr{W}$ in a space $\mathscr{X}$, denoted by Int^{ϖ}, consists points x of $\mathscr{W}$ such that for some open set $\mathscr{U}$ containing x such that $Cl^{\varpi}\mathscr{U} \subseteq \mathscr{W}$, then $\mathscr{W}$ is ϖ-open if and only if $\mathscr{W} = Int^{\varpi}\mathscr{W}$, or we can write it as $\mathscr{X} - \mathscr{W}$ is ϖ-closed. Form above, we have every closed set is ϖ-closed and every open set is ϖ-open.*

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Received: November 2021 *Accepted:* December 2021

Definition 1.3. *[1] A bitopological space ($\mathscr{X}$, $\mathscr{T}_1$, $\mathscr{T}_2$) is called pairwise Hausdorff space if for each pair of difference points x_1 and x_2 in ($\mathscr{X}$, $\mathscr{T}_1$, $\mathscr{T}_2$), then there is $\mathscr{T}_1$-open set $\mathscr{A}$ and $\mathscr{T}_2$-open set $\mathscr{N}$ such that $x_1 \in \mathscr{A}$ and $x_2 \in \mathscr{N}$,where $\mathscr{A}$ and $\mathscr{N}$ are disjoint.*

Definition 1.4. *[1] A function $f : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$ is call pairwise continuous if $f : (\mathscr{X}, \mathscr{T}_1) \to (\mathscr{Y}, \mathscr{F}_1)$ and $f : (\mathscr{X}, \mathscr{T}_2) \to (\mathscr{Y}, \mathscr{F}_2))$ are continueus.*

Let $f : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$ be a function, we will use the following symbol in this work as follow:
$\mathscr{T}_1 Cl^{\varpi}(\mathscr{A})$ denoted the $\mathscr{T}_1 - \varpi$-closed of a set $\mathscr{A} \subseteq (\mathscr{X}, \mathscr{T}_1)$
$\mathscr{F}_1 Cl^{\varpi}(\mathscr{N})$ denoted the $\mathscr{F}_1 - \varpi$-closed of a set $\mathscr{N} \subseteq (\mathscr{Y}, \mathscr{F}_1)$
$\mathscr{T}_1 Int^{\varpi}(\mathscr{A})$ denoted the $\mathscr{T}_1 - \varpi$-interior of a set $\mathscr{A} \subseteq (\mathscr{X}, \mathscr{T}_1)$
$\mathscr{F}_1 Int^{\varpi}(\mathscr{N})$ denoted the $\mathscr{F}_1 - \varpi$-interior of a set $\mathscr{N} \subseteq (\mathscr{Y}, \mathscr{F}_1)$
same as for $\mathscr{T}_2$ and $\mathscr{F}_2$ with respect to $(\mathscr{X}, \mathscr{T}_2)$ and $(\mathscr{Y}, \mathscr{F}_2)$ respectively.
A set $\mathscr{A}$ is called $\mathscr{T}_1 - \varpi$-closed if and only if $\mathscr{T}_1 Cl^{\varpi}(\mathscr{A}) = \mathscr{A}$,
A set $\mathscr{N}$ is called $\mathscr{F}_1 - \varpi$-closed if and only if $\mathscr{F}_1 Cl^{\varpi}(\mathscr{A}) = \mathscr{A}$,
A set $\mathscr{A}$ is called $\mathscr{T}_1 - \varpi$-open if and only if $\mathscr{T}_1 Int^{\varpi}(\mathscr{A}) = \mathscr{A}$,
A set $\mathscr{N}$ is called $\mathscr{F}_1 - \varpi$-open if and only if $\mathscr{F}_1 Int^{\varpi}(\mathscr{N}) = \mathscr{N}$,
same as for $\mathscr{T}_2$ and $\mathscr{F}_2$ with respect to $(\mathscr{X}, \mathscr{T}_2)$ and $(\mathscr{Y}, \mathscr{F}_2)$ respectively.

2. Main Result

The author in [6, 7, 8, 9] define ϖ-strongly (resp., ϖ-closure, ϖ-weakly) continuous functions as follows: A function $f : \mathscr{X} \to \mathscr{Y}$ is called ϖ-strongly (resp., ϖ-closure, ϖ-weakly) continuous, if for each point $x \in \mathscr{X}$ and every open set $\mathscr{N}$ of $f(x)$ in $\mathscr{Y}$, there exists an open set $\mathscr{A}$ containing x in $\mathscr{X}$ such that $f(Cl^{\varpi}(\mathscr{A}) \subseteq \mathscr{N}$ (resp., $f(Cl^{\varpi}(\mathscr{A}) \subseteq Cl^{\varpi}(\mathscr{N})$, $f(\mathscr{A}) \subseteq Cl^{\varpi}(\mathscr{N})$).

Now, we present the main definition in this work.

Definition 2.1. *A function $f : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$ is call pairwise ϖ-strongly (resp., ϖ-closure, ϖ-weakly) continuous, if either $f : (\mathscr{X}, \mathscr{T}_1) \to (\mathscr{Y}, \mathscr{F}_1)$ is ϖ-strongly (resp., ϖ-closure, ϖ-weakly) continuous or $f : (\mathscr{X}, \mathscr{T}_2) \to (\mathscr{Y}, \mathscr{F}_2))$ is ϖ-strongly (resp., ϖ-closure, ϖ-weakly) continuous (i.e., for each point $x \in (\mathscr{X}, \mathscr{T}_1)$ and every $\mathscr{F}_1$-opening set $\mathscr{N}_1$ of $f(x)$ in $\mathscr{Y}$, there exists an $\mathscr{T}_1$-opening set $\mathscr{A}_1$ contain x in $\mathscr{X}$ such that $f(\mathscr{T}_1 Cl^{\varpi}(\mathscr{A}_1) \subseteq \mathscr{N}_1$ (resp., $f(\mathscr{T}_1 Cl^{\varpi}(\mathscr{A}_1) \subseteq Cl^{\varpi}(\mathscr{N}_1)$, $f(\mathscr{A}_1) \subseteq \mathscr{F}_1 Cl^{\varpi}(\mathscr{N}_1)$ or for each point $x \in (\mathscr{X}, \mathscr{T}_2)$ and every $\mathscr{F}_2$-opening set $\mathscr{N}_2$ of $f(x)$ in $\mathscr{Y}$, there exist an $\mathscr{T}_2$-opening set $\mathscr{A}_2$ contain x in $\mathscr{X}$ such that $f(\mathscr{T}_2 Cl^{\varpi}(\mathscr{A}_2) \subseteq \mathscr{N}_2$ (resp., $f(\mathscr{T}_2 Cl^{\varpi}(\mathscr{A}_2) \subseteq Cl^{\varpi}(\mathscr{N}_2)$, $f(\mathscr{A}_2) \subseteq \mathscr{F}_2 Cl^{\varpi}(\mathscr{N}_2)$*

Definition 2.2. *If (x_α) is a net in a space $\mathscr{X}$, then (x_α) is called ϖ-convergence to $x \in \mathscr{X}$ denoted by ($x_\alpha \xrightarrow{\varpi} x$), if for each neighbourhood $\mathscr{A}$ of x, there is some $\alpha_0 \in \Lambda$ such that $\alpha \leq \alpha_0$ implies $x_\alpha \in Cl^{\varpi}(\mathscr{A})$. Thus $x_\alpha \xrightarrow{\varpi} x$ if and only if each ϖ-closure nbd of x contains a tail of (x_α), this is sometime said; (x_α) ϖ-converges to x if it is eventually in every ϖ-closure nbd of x.*

Theorem 2.3. *For any $f : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2))$ the follow are equivalent:*
(a) f is pairwise ϖ-strongly continuous,
(b) The inverses images of every $\mathscr{F}_1$-closed sets is $\mathscr{T}_1$-ϖ-closed and the inverses images of every $\mathscr{F}_2$-closed sets is $\mathscr{T}_2$-ϖ-closed,
(c) The inverses images of every $\mathscr{F}_1$-opening set is $\mathscr{T}_1$-ϖ-opening and the inverses images of every

$\mathscr{F}_2$-opening sets is $\mathscr{T}_2$-ϖ-opening,
(d) For each $x \in (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2)$ and each net $x_\alpha \xrightarrow{\varpi} x$, we have $f(x_\alpha) \to f(x)$.

Proof . *(a) ⇒ (b) Let* $\mathscr{N}_1$ *be* $\mathscr{F}_1$*-closed sets in* $(\mathscr{Y}, \mathscr{F}_1)$ *and* $\mathscr{N}_2$ *be* $\mathscr{F}_2$*-closed set in* $(\mathscr{Y}, \mathscr{F}_2)$*. Suppose that* $f^{-1}(\mathscr{N}_1)$ *is not* $\mathscr{F}_1$*-closed in* $(\mathscr{X}, \mathscr{T}_1)$ *and* $f^{-1}(\mathscr{N}_2)$ *is not* $\mathscr{F}_2$*-closed in* $(\mathscr{X}, \mathscr{T}_2)$*. Then there is a point* $x \notin f^{-1}(\mathscr{N}_1) \cup f^{-1}(\mathscr{N}_2)$ *such that for every* $\mathscr{T}_1$*-open set* $\mathscr{A}_1$ *and every* $\mathscr{T}_2$*-open set* $\mathscr{A}_2$ *both containing* x *we have* $\mathscr{T}_1Cl^{\varpi}(\mathscr{A}_1) \cap f^{-1}(\mathscr{N}_1) \neq \emptyset$ *and* $\mathscr{T}_2Cl^{\varpi}(\mathscr{A}_2) \cap f^{-1}(\mathscr{N}_2) \neq \emptyset$*. Since* $f(x) \notin \mathscr{N}_1 \cup \mathscr{N}_2$, $\mathscr{Y} \setminus \mathscr{N}_1$ *is* $\mathscr{F}_1$*-open and* $\mathscr{Y} \setminus \mathscr{N}_2$ *is* $\mathscr{F}_2$*-open, both containing* $f(x)$*, having the property that no* ϖ*-closed neighbourhood of* x *will map into* $\mathscr{Y} \setminus \mathscr{N}_1$ *and* $\mathscr{Y} \setminus \mathscr{N}_2$ *under* f*. Consequently,* f *is not pairwise* ϖ*-strongly continuous [10] at* x*. This contradiction implies that* $f^{-1}(\mathscr{N}_1)$ *is* $\mathscr{T}_1$*-*ϖ*-closed in* $(\mathscr{X}, \mathscr{T}_1)$ *and* $f^{-1}(\mathscr{N}_2)$ *is* $\mathscr{T}_2$*-*ϖ*-closed in* $(\mathscr{X}, \mathscr{T}_2)$*.*

(b) ⇒ (c) Let $\mathscr{N}_1$ *be* $\mathscr{F}_1$*-opening sets in* $(\mathscr{Y}, \mathscr{F}_1)$ *and* $\mathscr{N}_2$ *be* $\mathscr{F}_2$*-open set in* $(\mathscr{Y}, \mathscr{F}_2)$*. Then* $\mathscr{Y} \setminus \mathscr{N}_1$ *is* $\mathscr{F}_1$*-closed and* $\mathscr{Y} \setminus \mathscr{N}_2$ *is* $\mathscr{F}_2$*-closed. By (b)* $f^{-1}(\mathscr{Y} \setminus \mathscr{N}_1)$ *is* $\mathscr{T}_1$*-*ϖ*-closed in* $(\mathscr{X}, \mathscr{T}_1)$ *and* $f^{-1}(\mathscr{Y} \setminus \mathscr{N}_2)$ *is* $\mathscr{T}_2$*-*ϖ*-closed in* $(\mathscr{X}, \mathscr{T}_2)$*. But* $\mathscr{X} \setminus f^{-1}(\mathscr{Y} \setminus \mathscr{N}_1) = f^{-1}(\mathscr{N}_1)$ *is* $\mathscr{T}_1$*-*ϖ*-opening in* $(\mathscr{X}, \mathscr{T}_1)$ *and* $\mathscr{X} \setminus f^{-1}(\mathscr{Y} \setminus \mathscr{N}_2) = f^{-1}(\mathscr{N}_2)$ *is* $\mathscr{T}_2$*-*ϖ*-open in* $(\mathscr{X}, \mathscr{T}_2)$*.*

(c) ⇒ (d) Let $x \in (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2)$ *and let a net* $x_\alpha \xrightarrow{\varpi} x$*. Let* $\mathscr{N}_1$ *be* $\mathscr{F}_1$*-open set in* $(\mathscr{Y}, \mathscr{F}_1)$ *and* $\mathscr{N}_2$ *be* $\mathscr{F}_2$*-open set in* $(\mathscr{Y}, \mathscr{F}_2)$*.), both contain* $f(x)$*. Thus by (c),* $f^{-1}(\mathscr{N}_1)$ *is* $\mathscr{T}_1$*-*ϖ*-open in* $(\mathscr{X}, \mathscr{T}_1)$ *and* $f^{-1}(\mathscr{N}_2)$ *is* $\mathscr{T}_2$*-*ϖ*-open in* $(\mathscr{X}, \mathscr{T}_2)$ *both containing* x*. Thus there exists an* $\mathscr{T}_1$*-open* $\mathscr{A}_1$ *and* $\mathscr{T}_2$*-open* $\mathscr{A}_2$ *such that* $x \in \mathscr{A}_1 \subseteq \mathscr{T}_1Cl^{\varpi}(\mathscr{A}_1) \subseteq f^{-1}(\mathscr{N}_1)$ *and* $x \in \mathscr{A}_2 \subseteq \mathscr{T}_2Cl^{\varpi}(\mathscr{A}_2) \subseteq f^{-1}(\mathscr{N}_2)$*. The* $\mathscr{T}_1$*-*ϖ*-convergence and* $\mathscr{T}_2$*-*ϖ*–convergence of* x_α *is eventually in* $\mathscr{T}_1Cl^{\varpi}(\mathscr{A}_1)$ *and* $\mathscr{T}_2Cl^{\varpi}(\mathscr{A}_2)$ *respectively. So that* $f(x_\alpha)$ *is eventually in* $\mathscr{N}_1$ *and* $\mathscr{N}_2$*. This shows that*$f(x_\alpha) \to f(x)$*.*

(d) ⇒ (a) *Suppose that* f *is not pairwise* ϖ*-strongly continuous for some* $x \in (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2)$*. Thus there is an* $\mathscr{F}_1$*-open set* $\mathscr{N}_1$ *in* $(\mathscr{Y}, \mathscr{F}_1)$ *and* $\mathscr{F}_2$*-opening set* $\mathscr{N}_2$ *in* $(\mathscr{Y}, \mathscr{F}_2)$*, both contain* $f(x)$ *such that for every* $\mathscr{T}_1$*-open set* $\mathscr{A}_1$ *in* $(\mathscr{X}, \mathscr{T}_1)$ *and* $\mathscr{T}_2$*-opening sets* $\mathscr{A}_2$ *in* $(\mathscr{X}, \mathscr{T}_2)$*, both contain* x*, such that* $\mathscr{T}_1Cl^{\varpi}(\mathscr{A}_1) \not\subset \mathscr{N}_1$ *and* $\mathscr{T}_2Cl^{\varpi}(\mathscr{A}_2) \not\subset \mathscr{N}_2$*. Now consider the directed sets* $\mathscr{D}_1 = \{\ x_\alpha : \mathscr{T}_1Cl^{\varpi}(\mathscr{A}_1)\}$ *and* $\mathscr{D}_2 = \{\ x_\alpha : \mathscr{T}_2Cl^{\varpi}(\mathscr{A}_2)\}$ *using by reverse inclusion where* $\mathscr{A}_{1\varsigma}$ *and* $\mathscr{A}_{2\varsigma}$ *both contains* x *and* $x \in \mathscr{T}_1Cl^{\varpi}(\mathscr{A}_{1\varpi}) \cup \mathscr{T}_2Cl^{\varpi}(\mathscr{A}_{2\varpi})$ *such that* $f(x_\alpha) \not\subset \mathscr{N}_1 \cup \mathscr{N}_2$*. Then the net* $g_1 : \mathscr{D}_1 \to (\mathscr{X}, \mathscr{T}_1)$ *and* $g_2 : \mathscr{D}_2 \to (\mathscr{X}, \mathscr{T}_2)$ *defined by* $g_1(x_\alpha, \mathscr{A}_1) = x_\alpha$ *,* $\mathscr{T}_1$*-*ϖ*-converges to* x *and* $g_2(x_\alpha, \mathscr{A}_2) = x_\alpha$ *,* $\mathscr{T}_2$*-*ϖ*-converges to* x*, but the net* fog *does not converge to* $f(x)$ *. The contradiction we obtained implies that* f *is pairwise* ϖ*-strongly continuous function.* □

Similarly, we proving the follow theorems:

Theorem 2.4. *For any* $f : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2))$ *the follow are equivalent:*
(a) f *is pairwise* ϖ*-closure continuous,*
(b) The inverses images of every $\mathscr{F}_1$*-*ϖ*-closed sets is* $\mathscr{T}_1$*-*ϖ*-closed and the inverses images of every* $\mathscr{F}_2$*-*ϖ*-closed sets is* $\mathscr{T}_2$*-*ϖ*-closed,*
(c) The inverses images of every $\mathscr{F}_1$*-*ϖ*-opening sets is* $\mathscr{T}_1$*-*ϖ*-opening and the inverses images of every* $\mathscr{F}_2$*-*ϖ*-opening sets is* $\mathscr{T}_2$*-*ϖ*-open,*
(d) For each $x \in (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2)$ *and each net* $x_\alpha \xrightarrow{\varpi} x$*, we have* $f(x_\alpha) \xrightarrow{\varpi} f(x)$*.*

Theorem 2.5. *For any* $f : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2))$ *the follow are equivalent:*
(a) f *is pairwise* ϖ*-weakly continuous,*
(b) The inverses images of every $\mathscr{F}_1$*-*ϖ*-closed sets is* $\mathscr{T}_1$*-closed and the inverses images of every* $\mathscr{F}_2$*-*ϖ*-closed set is* $\mathscr{T}_2$*-closed,*
(c) The inverses images of every $\mathscr{F}_1$*-*ϖ*-opening sets is* $\mathscr{T}_1$*-open and the inverses images of every* $\mathscr{F}_2$*-*ϖ*-opening sets is* $\mathscr{T}_2$*-opening,*
(d) For each $x \in (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2)$ *and each net* $x_\alpha \to x$*, we have* $f(x_\alpha) \xrightarrow{\varpi} f(x)$*.*

Definition 2.6. *A bitopological space $(\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2)$ is called pairwise ϖ-Urysohn if for each pairs of different point x_1 and x_2 in $(\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2)$ then there is a $\mathscr{T}_1$-opening sets $\mathscr{A}$ and $\mathscr{T}_2$-opening sets $\mathscr{N}$ such that $x_1 \in \mathscr{A}$ and $x_2 \in \mathscr{N}$, $\mathscr{T}_1 Cl^{\varpi}(\mathscr{A}) \cap \mathscr{T}_2 Cl^{\varpi}(\mathscr{N}) = \phi$.*

Theorem 2.7. *If $f : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$ be a pairwise ϖ-strongly continuous invective function and $(\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$ be pairwise Hausdorff. Then $(\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise ϖ-Urysohn.*
Proof . *Let $x_1, x_2 \in \mathscr{X}$ such that $x_1 \neq x_2$. Then $f(x_1), \neq f(x_2)$. By hypothesis $(\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$ is pairwise Hausdorff, then there exist disjointing sets $\mathscr{F}_1$-opening $\mathscr{N}_1$ and $\mathscr{F}_2$-opening $\mathscr{N}_2$ contain $f(x_1)$ and $f(x_2)$ respective. Since f is pairwise ϖ-strongly continuous, there exist $\mathscr{T}_1$-opening sets $\mathscr{A}$ and $\mathscr{T}_2$-opening sets $\mathscr{A}_2$ containing x_1 and x_2 respectively, such that $f(\mathscr{T}_1 Cl^{\varpi}(\mathscr{A}_1) \subseteq \mathscr{N}_1$ and $f(\mathscr{T}_2 Cl^{\varpi}(\mathscr{A}_2) \subseteq \mathscr{N}_2$. It follows that $f^{-1}(f(\mathscr{T}_1 Cl^{\varpi}(\mathscr{A}_1)) \subseteq f^{-1}(\mathscr{N}_1)$ and $f^{-1}(f(\mathscr{T}_2 Cl^{\varpi}(\mathscr{A}_2)) \subseteq f^{-1}(\mathscr{N}_2)$, therefore $\mathscr{T}_1 Cl^{\varpi}(\mathscr{A}_1) \subseteq f^{-1}(\mathscr{N}_1)$ and $\mathscr{T}_2 Cl^{\varpi}(\mathscr{A}_2) \subseteq f^{-1}(\mathscr{N}_2)$. Then $\mathscr{T}_1 Cl^{\varpi}(\mathscr{A}_1) \cap \mathscr{T}_2 Cl^{\varpi}(\mathscr{A}_2) = \phi$, So $(\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise ϖ-Urysohn* □

Similarly, we can proving the follow theorems:

Theorem 2.8. *If $f : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$ be a pairwise ϖ-closure continuous invectively function and let $(\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$ be pairwise ϖ-Urysohn. Then $(\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise ϖ-Urysohn.*

Theorem 2.9. *If $f : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$ be a pairwise ϖ-closure continuous invectively function and let $(\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$ be pairwise ϖ-Urysohn. Then $(\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise Hausdorf.*

Now, we are study the composition of difference form of pairwise ω-continuous functions.

Theorem 2.10. *If $f : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$ be a pairwise ϖ-strongly continuous and $g : (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2) \to (\mathscr{P}, \mathscr{K}_1, \mathscr{K}_2)$ be pairwise ϖ-strongly continuous. Then $gof : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{P}, \mathscr{K}_1, \mathscr{K}_2)$ is pairwise ϖ-strongly continuous.*
Proof . *take $x \in (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2)$. Let $\mathscr{W}_1$ be $\mathscr{K}_1$-open set in $(\mathscr{P}, \mathscr{K}_1)$ and $\mathscr{W}_2$ be $\mathscr{K}_2$-open set in $(\mathscr{P}, \mathscr{K}_2)$ both containing $(gof)(x)$ in $\mathscr{K}$, since g is pairwise ϖ-strongly continuous, there is $\mathscr{F}_1$-open set $\mathscr{N}_1$ in $(\mathscr{Y}, \mathscr{F}_1)$ and $\mathscr{F}_2$-opening set $\mathscr{N}_2$ in $(\mathscr{Y}, \mathscr{F}_2)$ both contain $f(x)$ in $\mathscr{Y}$ such that $g(\mathscr{F}_1 Cl^{\varpi}(\mathscr{N}_1) \subseteq \mathscr{W}_1$ and $g(\mathscr{F}_2 Cl^{\varpi}(\mathscr{N}_2) \subseteq \mathscr{W}_1$. Since f is pairwise ϖ-strongly continuous, there is $\mathscr{T}_1$-opening sets $\mathscr{A}_1$ in $(\mathscr{X}, \mathscr{T}_1)$ and $\mathscr{T}_2$-opening sets $\mathscr{A}_2$ in $(\mathscr{X}, \mathscr{T}_2)$ both contain x in $\mathscr{X}$ such that $f(\mathscr{T}_1 Cl^{\varpi}(\mathscr{A}_1) \subseteq \mathscr{N}_1$ and $f(\mathscr{T}_2 Cl^{\varpi}(\mathscr{A}_2) \subseteq \mathscr{N}_2$, since $\mathscr{N}_1) \subseteq \mathscr{F}_1 Cl^{\varpi}(\mathscr{N}_1)$ and $\mathscr{N}_2) \subseteq \mathscr{F}_1 Cl^{\varpi}(\mathscr{N}_2)$, then $f(\mathscr{T}_1 Cl^{\varpi}(\mathscr{A}_1) \subseteq \mathscr{T}_1 Cl^{\varpi}(\mathscr{N}_1)$ and $f(\mathscr{T}_C l^{\varpi}(\mathscr{A}_2) \subseteq \mathscr{T}_2 Cl^{\varpi}(\mathscr{N}_2)$, so $g(f(\mathscr{T}_1 Cl^{\varpi}(\mathscr{A}_1)) \subseteq g(\mathscr{T}_1 Cl^{\varpi}(\mathscr{N}_1))$ and $g(f(\mathscr{T}_2 Cl^{\varpi}(\mathscr{A}_2)) \subseteq g(\mathscr{T}_2 Cl^{\varpi}(\mathscr{N}_2))$, also $gof(\mathscr{T}_1 Cl^{\varpi}(\mathscr{A}_1) \subseteq g(\mathscr{T}_1 Cl^{\varpi}(\mathscr{N}_1))$ and $gof(\mathscr{T}_C l^{\varpi}(\mathscr{A}_2) \subseteq g(\mathscr{T}_2 Cl^{\varpi}(\mathscr{N}_2))$. Therefore, found is $\mathscr{T}_1$-opening sets $\mathscr{A}_1$ in $(\mathscr{X}, \mathscr{T}_1)$ and $\mathscr{T}_2$-opening sets $\mathscr{A}_2$ in $(\mathscr{X}, \mathscr{T}_2)$ both contain x in $\mathscr{X}$ such that $(gof)(\mathscr{T}_1 Cl^{\varpi}(\mathscr{A}_1) \subseteq \mathscr{W}_1$ and $(gof)(\mathscr{T}_2 Cl^{\varpi}(\mathscr{A}_2) \subseteq \mathscr{W}_2$ and gof is pairwise ϖ-strongly continuous.* □

Theorem 2.11. *If $f : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$ be a pairwise ϖ-strongly continuous and $g : (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2) \to (\mathscr{P}, \mathscr{K}_1, \mathscr{K}_2)$ be pairwise continuous. Then $gof : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{P}, \mathscr{K}_1, \mathscr{K}_2)$ is pairwise ϖ-strongly continuous.*
Proof . *Let $\mathscr{W}_1$ be $\mathscr{K}_1$-open set in $(\mathscr{P}, \mathscr{K}_1)$ and $\mathscr{W}_2$ be $\mathscr{K}_2$-open set in $(\mathscr{P}, \mathscr{K}_2)$ Since g is pairwise continuous, we have $g^{-1}(\mathscr{W}_1$ is $\mathscr{F}_1$-opening sets in $(\mathscr{Y}, \mathscr{F}_1)$ and $g^{-1}(\mathscr{W}_2$ is $\mathscr{F}_2$-opening set in $(\mathscr{Y}, \mathscr{F}_2)$. By Theorem 2.3 (c) we have $f^{-1}(g^{-1}(\mathscr{W}_1) = (gof)-1(\mathscr{W}_1$ is $\mathscr{T}_1$-ϖ-opening sets in $(\mathscr{X}, \mathscr{T}_1)$ and $f^{-1}(g^{-1}(\mathscr{W}_2) = (gof)-1(\mathscr{W}_2$ is $\mathscr{T}_2$-ϖ-open set in $(\mathscr{X}, \mathscr{T}_2)$. Therefore, gof is pairwise ω-strongly continuous. Both contain $(gof)(x)$ in $\mathscr{K}$, since g is pairwise ϖ-strongly continuous, there is $\mathscr{F}_1$-opening sets $\mathscr{N}_1$ in $(\mathscr{Y}, \mathscr{F}_1)$ and $\mathscr{F}_2$-opening sets $\mathscr{N}_2$ in $(\mathscr{Y}, \mathscr{F}_2)$ both contain $f(x)$ in $\mathscr{Y}$ such that*

$g(\mathscr{F}_1Cl^{\varpi}(\mathscr{N}_1) \subseteq \mathscr{W}_1$ and $g(\mathscr{F}_2Cl^{\varpi}(\mathscr{N}_2) \subseteq \mathscr{W}_1$. Since f is pairwise ϖ-strongly continuous, there is $\mathscr{T}_1$-opening sets $\mathscr{A}_1$ in $(\mathscr{X}, \mathscr{T}_1)$ and $\mathscr{T}_2$-opening sets $\mathscr{A}_2$ in $(\mathscr{X}, \mathscr{T}_2)$ both contain x in $\mathscr{X}$ such that $f(\mathscr{T}_1Cl^{\varpi}(\mathscr{A}_1) \subseteq \mathscr{N}_1$ and $f(\mathscr{T}_2Cl^{\varpi}(\mathscr{A}_2) \subseteq \mathscr{N}_2$, since $\mathscr{N}_1) \subseteq \mathscr{F}_1Cl^{\varpi}(\mathscr{N}_1)$ and $\mathscr{N}_2) \subseteq \mathscr{F}_1Cl^{\varpi}(\mathscr{N}_2)$, then $f(\mathscr{T}_1Cl^{\varpi}(\mathscr{A}_1) \subseteq \mathscr{T}_1Cl^{\varpi}(\mathscr{N}_1)$ and $f(\mathscr{T}_Cl^{\varpi}(\mathscr{A}_2) \subseteq \mathscr{T}_2Cl^{\varpi}(\mathscr{N}_2)$, so $g(f(\mathscr{T}_1Cl^{\varpi}(\mathscr{A}_1)) \subseteq g(\mathscr{T}_1Cl^{\varpi}(\mathscr{N}_1))$ and $g(f(\mathscr{T}_2Cl^{\varpi}(\mathscr{A}_2)) \subseteq g(\mathscr{T}_2Cl^{\varpi}(\mathscr{N}_2))$, also $gof(\mathscr{T}_1Cl^{\varpi}(\mathscr{A}_1) \subseteq g(\mathscr{T}_1Cl^{\varpi}(\mathscr{N}_1))$ and $gof(\mathscr{T}_Cl^{\varpi}(\mathscr{A}_2) \subseteq g(\mathscr{T}_2Cl^{\varpi}(\mathscr{N}_2))$. Therefore, there is $\mathscr{T}_1$-opening sets $\mathscr{A}_1$ in $(\mathscr{X}, \mathscr{T}_1)$ and $\mathscr{T}_2$-opening sets $\mathscr{A}_2$ in $(\mathscr{X}, \mathscr{T}_2)$ both contain x in $\mathscr{X}$ such that $(gof)(\mathscr{T}_1Cl^{\varpi}(\mathscr{A}_1) \subseteq \mathscr{W}_1$ and $(gof)(\mathscr{T}_2Cl^{\varpi}(\mathscr{A}_2) \subseteq \mathscr{W}_2$ and gof is pairwise ϖ-strongly continuous. □

Similarly, we can proving the follow theorems:

Theorem 2.12. *If $f : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$ be a pairwise ϖ-weakly continuous and $g : (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2) \to (\mathscr{P}, \mathscr{K}_1, \mathscr{K}_2)$ be pairwise ϖ-strongly continuous. Then $gof : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{P}, \mathscr{K}_1, \mathscr{K}_2)$ is pairwise continuous.*

Theorem 2.13. *If $f : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$ be a pairwise continuous and $g : (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2) \to (\mathscr{P}, \mathscr{K}_1, \mathscr{K}_2)$ be pairwise ϖ-weakly continuous. Then $gof : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{P}, \mathscr{K}_1, \mathscr{K}_2)$ is pairwise ϖ-weakly continuous.*

Theorem 2.14. *If $f : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$ be a pairwise ϖ-closure continuous and $g : (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2) \to (\mathscr{P}, \mathscr{K}_1, \mathscr{K}_2)$ be pairwise ϖ-closure continuous. Then $gof : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{P}, \mathscr{K}_1, \mathscr{K}_2)$ is pairwise ϖ-closure continuous.*

Theorem 2.15. *If $f : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$ be a pairwise ϖ-weakly continuous and $g : (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2) \to (\mathscr{P}, \mathscr{K}_1, \mathscr{K}_2)$ be pairwise ϖ-closure continuous. Then $gof : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{P}, \mathscr{K}_1, \mathscr{K}_2)$ is pairwise ϖ-weakly continuous.*

Lemma 2.16. *If $f : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$ be a pairwise ϖ-strongly continuous if and only if for each pairwise sub basis $\mathscr{F}_1$-open subset $\mathscr{S}$ and $\mathscr{F}_2$-open subset $\mathscr{T}$ of $(\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$, then $f^{-1}(\mathscr{S})$ and $f^{-1}(\mathscr{T})$ are $\mathscr{T}_1$-ϖ-open in $(\mathscr{X}, \mathscr{T}_1)$ and $\mathscr{T}_2$-ϖ-open in $(\mathscr{X}, \mathscr{T}_2)$.*

Proof . *(⇒) Follows from Theorem 2.4.*
(⇐) Let $\{ \mathscr{S}_\alpha, \mathscr{T}_\alpha ; \alpha \in \Lambda \}$ be a pairwise sub basis for $(\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$ and suppose that $f^{-1}(\mathscr{S}_\alpha)$ and $f^{-1}(\mathscr{T}_\alpha)$ are $\mathscr{T}_1$-ϖ-opening sets in $(\mathscr{X}, \mathscr{T}_1)$ and $\mathscr{T}_2$-ϖ-opening sets in $(\mathscr{X}, \mathscr{T}_2)$ for each $\alpha \in \Lambda$. Every $\mathscr{F}_1$-open subset $\mathscr{S}$ and $\mathscr{F}_2$-open subset $\mathscr{T}$ of $(\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$ can be written as
$\mathscr{S} = \cup\{ \mathscr{S}_{\alpha 1} \cap \mathscr{S}_{\alpha 2} \cap ....\mathscr{S}_{\alpha n}; \{ \alpha 1, \alpha 2, ..., \alpha n \subseteq \Lambda \} \}$
and $\mathscr{T} = \cup\{ \mathscr{T}_{\alpha 1} \cap \mathscr{T}_{\alpha 2} \cap ....\mathscr{T}_{\alpha n}; \{ \alpha 1, \alpha 2, ..., \alpha n \subseteq \Lambda \} \}$
then $f^{-1}(\mathscr{S}) = \cup\{ f^{-1}(\mathscr{S}_{\alpha 1}) \cap f^{-1}(\mathscr{S}_{\alpha 2}) \cap ....f^{-1}(\mathscr{S}_{\alpha n}) \}$
and $f^{-1}(\mathscr{T}) = \cup\{ f^{-1}(\mathscr{T}_{\alpha 1}) \cap f^{-1}(\mathscr{T}_{\alpha 2}) \cap ....f^{-1}(\mathscr{T}_{\alpha n}) \}$.
The finite intersect of $\mathscr{T}_1$-ϖ-opening sets is $\mathscr{T}_1$-ϖ-opening and the finite intersect of $\mathscr{T}_2$-ϖ-opening sets is $\mathscr{T}_2$-ϖ-opening and the union of $\mathscr{T}_1$-ϖ-open sets is $\mathscr{T}_1$-ϖ-opening and the union of $\mathscr{T}_2$-ϖ-open sets is $\mathscr{T}_1$-ϖ-opening. Therefore $f^{-1}(\mathscr{S})$ is $\mathscr{T}_1$-ϖ-open and $f^{-1}(\mathscr{T})$ is $\mathscr{T}_2$-ϖ-open and hence by Theorem 2.3, f is pairwise ϖ-strongly continuous. □

Similarly, we can prove the following lemmas:

Lemma 2.17. *If $f : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$ be a pairwise ϖ-closed contineous if and only if for each pairwise sub basis $\mathscr{F}_1$-ϖ-open subset $\mathscr{S}$ and $\mathscr{F}_2$-ϖ-open subset $\mathscr{T}$ of $(\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$, then $f^{-1}(\mathscr{S})$ and $f^{-1}(\mathscr{T})$ are $\mathscr{T}_1$-ϖ-open in $(\mathscr{X}, \mathscr{T}_1)$ and $\mathscr{T}_2$-ϖ-open in $(\mathscr{X}, \mathscr{T}_2)$.*

Lemma 2.18. *If $f : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$ be a pairwise ϖ-weakly continuous if and only if for each pairwise sub basis $\mathscr{F}_1$-ϖ-open subset $\mathscr{S}$ and $\mathscr{F}_2$-ϖ-opening subset $\mathscr{T}$ of $(\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$, then $f^{-1}(\mathscr{S})$ and $f^{-1}(\mathscr{T})$ are $\mathscr{T}_1$-open in $(\mathscr{X}, \mathscr{T}_1)$ and $\mathscr{T}_2$-open in $(\mathscr{X}, \mathscr{T}_2)$.*

Theorem 2.19. *the function $f : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\prod \mathscr{X}_\alpha, \mathscr{F}_1, \mathscr{F}_2)$ is a pairwise ϖ-strongly continuous if and only if the composting with each pairwise continuous project function $\prod_\alpha$ is pairwise ϖ-strongly continuous.*

Proof . *($\Rightarrow$) Follows from Theorem 2.11*
($\Leftarrow$) Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be a pairwise sub basis $\mathscr{F}_1$-open set in $(\prod \mathscr{X}_\alpha, \mathscr{F}_1)$ and $\mathscr{F}_2$-open set in $(\prod \mathscr{X}_\alpha, \mathscr{F}_2)$ for each $\alpha \in \Lambda$. Then $\mathscr{S}_1 = \prod_\alpha^{-1}(\mathscr{T}_1)$ for some $\mathscr{F}_1$-open set $\mathscr{T}_1$ in $(\mathscr{X}_\alpha, \mathscr{F}_1)$ and $\mathscr{S}_2 = \prod_\alpha^{-1}(\mathscr{T}_2)$ for some $\mathscr{F}_2$-open set $\mathscr{T}_2$ in $(\mathscr{X}_\alpha, \mathscr{F}_2)$. Thus $f^{-1}(\mathscr{S}_1) = f^{-1}(\prod_\alpha^{-1}(\mathscr{T}_1)) = (\prod_\alpha of)^{-1}(\mathscr{T}_1)$ is $\mathscr{T}_1$-ϖ-open and $f^{-1}(\mathscr{S}_2) = f^{-1}(\prod_\alpha^{-1}(\mathscr{T}_2)) = (\prod_\alpha of)^{-1}(\mathscr{T}_2)$ is $\mathscr{T}_2$-ϖ-open. By Lemma 2.16, f is pairwise ϖ-strongly continuous. □

Similarly, we can proving the follow theorems:

Theorem 2.20. *the function $f : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\prod \mathscr{X}_\alpha, \mathscr{F}_1, \mathscr{F}_2)$ is a pairwise ϖ-closure continuous if and only if the compost with each pairwise continuous project function $\prod_\alpha$ is pairwise ϖ-closure continuous.*

Theorem 2.21. *the function $f : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\prod \mathscr{X}_\alpha, \mathscr{F}_1, \mathscr{F}_2)$ is a pairwise ϖ-weakly continuous if and only if the compost with each pairwise continuous project function $\prod_\alpha$ is pairwise ϖ-weakly continuous.*

The following propositions is follow from Theorem 2.19, Theorem 2.20 and Theorem 2.21.

Proposition 2.22. *If $f : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$ be a function and let $g : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{X} \times \mathscr{Y}, \mathscr{T}_1 \times \mathscr{F}_1, \mathscr{T}_2 \times \mathscr{F}_2)$ be the pairwise graphic function of f given by $g(x) = (x, f(x))$ for every point $x \in \mathscr{X}$. Then f is pairwise ϖ-strongly continuous if and only if g is pairwise ϖ-strongly continuous.*

Proposition 2.23. *If $f : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$ be a function and let $g : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{X} \times \mathscr{Y}, \mathscr{T}_1 \times \mathscr{F}_1, \mathscr{T}_2 \times \mathscr{F}_2)$ be the pairwise graphic function of f given by $g(x) = (x, f(x))$ for every point $x \in \mathscr{X}$. Then f is pairwise ϖ-closure continuous if and only if g is pairwise ϖ-closure continuous.*

Proposition 2.24. *If $f : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{Y}, \mathscr{F}_1, \mathscr{F}_2)$ be a function and let $g : (\mathscr{X}, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{X} \times \mathscr{Y}, \mathscr{T}_1 \times \mathscr{F}_1, \mathscr{T}_2 \times \mathscr{F}_2)$ be the pairwise graphic function of f given by $g(x) = (x, f(x))$ for every point $x \in \mathscr{X}$. Then f is pairwise ϖ-weakly continuous if and only if g is pairwise ϖ-weakly continuous.*

Lemma 2.25. *Let $(\mathscr{X}_{\alpha i}, \mathscr{T}_1, \mathscr{T}_2)$ be a bitopological spaces and let $\mathscr{W}_{\alpha i}$ and $\mathscr{A}_{\alpha i}$ be subsets of $(\mathscr{X}_{\alpha i}, \mathscr{T}_1)$ and $(\mathscr{X}_{\alpha i}, \mathscr{T}_2)$ respectively, for each $i = 1, 2, ..., n$. Then $\mathscr{W}_{\alpha 1} \times \mathscr{W}_{\alpha 2} \times .... \times \mathscr{W}_{\alpha n} \times \prod_{\alpha \neq \alpha 0}(\mathscr{X}_\alpha, \mathscr{T}_1) \subseteq \prod_{\alpha \in \Lambda}(\mathscr{X}_\alpha, \mathscr{T}_1)$. and $\mathscr{A}_{\alpha 1} \times \mathscr{A}_{\alpha 2} \times .... \times \mathscr{A}_{\alpha n} \times \prod_{\alpha \neq \alpha 0}(\mathscr{X}_\alpha, \mathscr{T}_2) \subseteq \prod_{\alpha \in \Lambda}(\mathscr{X}_\alpha, \mathscr{T}_2)$ are $\mathscr{T}_1$-ϖ-open and $\mathscr{T}_2$-ϖ-open respectively if and only if $\mathscr{W}_i$ is $\mathscr{T}_1$-ϖ-open in $(\mathscr{X}_{\alpha i}, \mathscr{T}_1)$ and $\mathscr{A}_i$ is $\mathscr{T}_2$-ϖ-open in $(\mathscr{X}_{\alpha i}, \mathscr{T}_1)$ for each $i = 1, 2, ..., n$.*

Proof . *($\Leftarrow$) Suppose that $\mathscr{W}_i$ is $\mathscr{T}_1$-ϖ-open in $(\mathscr{X}_{\alpha i}, \mathscr{T}_1)$ and $\mathscr{A}_i$ is $\mathscr{T}_2$-ϖ-open in $(\mathscr{X}_{\alpha i}, \mathscr{T}_2)$ for each $i = 1, 2, ..., n$.*
Then for each i and each $x_i \in \mathscr{S}_{\alpha i} \subset \mathscr{T}_1 Cl^\varpi(\mathscr{S}_{\alpha i}) \subset \mathscr{W}_{\alpha i}$, $x_i \in \mathscr{T}_{\alpha i} \subset \mathscr{T}_1 Cl^\varpi(\mathscr{E}_{\alpha i}) \subset \mathscr{A}_{\alpha i}$

Thus, for each $\{x_\alpha\} \in \mathscr{W}_{\alpha 1} \times \mathscr{W}_{\alpha 2} \times .... \times \mathscr{W}_{\alpha n} \times \prod_{\alpha\neq\alpha 0}(\mathscr{X}_\alpha, \mathscr{T}_1) \subseteq \prod_{\alpha\in\Lambda}(\mathscr{X}_\alpha, \mathscr{T}_1)$,
$\{x_\alpha\} \in \mathscr{A}_{\alpha 1} \times \mathscr{A}_{\alpha 2} \times .... \times \mathscr{A}_{\alpha n} \times \prod_{\alpha\neq\alpha 0}(\mathscr{X}_\alpha, \mathscr{T}_2) \subseteq \prod_{\alpha\in\Lambda}(\mathscr{X}_\alpha, \mathscr{T}_2) \subset \mathscr{T}_1 Cl^{\varpi}(\mathscr{S}_{\alpha 1}) \times \mathscr{T}_1 Cl^{\varpi}(\mathscr{S}_{\alpha 2}) \times .... \times \mathscr{T}_1 Cl^{\varpi}(\mathscr{S}_{\alpha n}) \times \prod_{\alpha\neq\alpha 0}(\mathscr{X}_\alpha, \mathscr{T}_1) \subset \mathscr{W}_{\alpha 1} \times \mathscr{W}_{\alpha 2} \times .... \times \mathscr{W}_{\alpha n} \times \prod_{\alpha\neq\alpha 0}(\mathscr{X}_\alpha, \mathscr{T}_1) \subseteq \prod_{\alpha\in\Lambda}(\mathscr{X}_\alpha, \mathscr{T}_1)$. *This show that* $\mathscr{W}_{\alpha 1} \times \mathscr{W}_{\alpha 2} \times .... \times \mathscr{W}_{\alpha n} \times \prod_{\alpha\neq\alpha 0}(\mathscr{X}_\alpha, \mathscr{T}_1) \subseteq \prod_{\alpha\in\Lambda}(\mathscr{X}_\alpha, \mathscr{T}_1)$ *is* $\mathscr{T}_1$-ϖ-*open.*

By a similar way, we get $\mathscr{A}_{\alpha 1} \times \mathscr{A}_{\alpha 2} \times .... \times \mathscr{A}_{\alpha n} \times \prod_{\alpha\neq\alpha 0}(\mathscr{X}_\alpha, \mathscr{T}_2) \subseteq \prod_{\alpha\in\Lambda}(\mathscr{X}_\alpha, \mathscr{T}_2)$ *is* $\mathscr{T}_2$-ϖ-*open*
(⇒) Straightforward. □

Theorem 2.26. *The function* $\prod_\alpha f_\alpha : \prod_\alpha(\mathscr{X}_\alpha, \mathscr{T}_1, \mathscr{T}_2) \to \prod_\alpha(\mathscr{Y}_\alpha, \mathscr{F}_1, \mathscr{F}_2)$ *define by* $\{\mathscr{X}_\alpha\} \to \{f_\alpha(\mathscr{X}_\alpha)\}$ *is a pairwise* ϖ-*strongly continuous if and only if each* $f_\alpha : (\mathscr{X}_\alpha, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{Y}_\alpha, \mathscr{F}_1, \mathscr{F}_2)$ *is pairwise* ϖ-*strongly continuous.*

Proof . *(⇒) Suppose that* $\prod_\alpha f_\alpha$ *is pairwise* ϖ-*strongly continuous. Let* $\mathscr{W}_{\alpha i}$ *be* $\mathscr{F}_1$-*open in* $(\mathscr{Y}_{\alpha i}, \mathscr{F}_1)$ *and* $\mathscr{A}_{\alpha i}$ *be* $\mathscr{F}_2$-*open in* $(\mathscr{Y}_{\alpha i}, \mathscr{F}_2)$. *Then* $\mathscr{W} = \mathscr{W}_{\alpha i} \times \prod_{\alpha\neq\alpha 0}(\mathscr{Y}_\alpha, \mathscr{F}_1)$ *and* $\mathscr{A} = \mathscr{A}_{\alpha i} \times \prod_{\alpha\neq\alpha 0}(\mathscr{Y}_\alpha, \mathscr{F}_2)$ *are pairwise sub basic* $\mathscr{F}_1$-*open in* $\prod_\alpha(\mathscr{Y}_\alpha, \mathscr{F}_1)$ *and* $\mathscr{F}_2$-*open in* $\prod_\alpha(\mathscr{Y}_\alpha, \mathscr{F}_2)$, *respectively. And*

$$(\prod_\alpha f_\alpha)^{-1}(\mathscr{W}) = f_{\alpha 0}^{-1}(\mathscr{W}_{\alpha i}) \times \prod_{\alpha\neq\alpha 0}(\mathscr{X}_\alpha, \mathscr{T}_1)$$

is $\mathscr{T}_1$-ϖ-*open and* $(\prod_\alpha f_\alpha)^{-1}(\mathscr{A}) = f_{\alpha 0}^{-1}(\mathscr{A}_{\alpha i}) \times \prod_{\alpha\neq\alpha 0}(\mathscr{X}_\alpha, \mathscr{T}_)$ *is* $\mathscr{T}_2$-ϖ-*open. Thus* $f^{-1}(\mathscr{W}_{\alpha i}$ *is* $\mathscr{T}_1$-ϖ-*open in* $(\mathscr{X}_{\alpha i}, \mathscr{T}_1)$ *and* $f^{-1}(\mathscr{A}_{\alpha i}$ *is* $\mathscr{T}_2$-ϖ-*open in* $(\mathscr{X}_{\alpha i}, \mathscr{T}_2)$ *by Theorem 2.3 implies that* $f_{\alpha i}$ *is pairwise* ϖ-*strongly continuous.*

(⇐) $\mathscr{W} = \mathscr{W}_{\alpha 1} \times \mathscr{W}_{\alpha 2} \times ..... \times \mathscr{W}_{\alpha n} \times \prod_{\alpha\neq\alpha 0}(\mathscr{Y}_\alpha, \mathscr{F}_1)$ *be a base* $\mathscr{F}_1$-*open in* $\prod_\alpha(\mathscr{Y}_\alpha, \mathscr{F}_1)$ *and* $\mathscr{A} = \mathscr{A}_{\alpha 1} \times \mathscr{A}_{\alpha 2} \times ..... \times \mathscr{A}_{\alpha n} \times \prod_{\alpha\neq\alpha 0}(\mathscr{Y}_\alpha, \mathscr{F}_2)$ *be a base* $\mathscr{F}_2$-*open in* $\prod_\alpha(\mathscr{Y}_\alpha, \mathscr{F}_2)$. *Then* $f_{\alpha 0}^{-1}(\mathscr{W}_{\alpha i})$ *is* $\mathscr{T}_1$-ϖ-*open in* $(\mathscr{X}_{\alpha i}, \mathscr{T}_1)$ *and* $f_{\alpha 0}^{-1}(\mathscr{A}_{\alpha i})$ *is* $\mathscr{T}_2$-ϖ-*open in* $(\mathscr{X}_{\alpha i}, \mathscr{T}_2)$ *for each* αi, *where* $i = 1, 2, ..., n$. *Then by Lemma 2.25 we have* $(\prod_\alpha f_\alpha)^{-1}(\mathscr{W}) = f_{\alpha 0}^{-1}(\mathscr{W}_{\alpha i}) \times \prod_{\alpha\neq\alpha 0}(\mathscr{X}_\alpha, \mathscr{T}_1)$ *is* $\mathscr{T}_1$-ϖ-*open in* $\prod_\alpha(\mathscr{X}_\alpha, \mathscr{T}_1)$ *and* $(\prod_\alpha f_\alpha)^{-1}(\mathscr{A}) = f_{\alpha 0}^{-1}(\mathscr{A}_{\alpha i}) \times \prod_{\alpha\neq\alpha 0}(\mathscr{X}_\alpha, \mathscr{T}_1)$ *is* $\mathscr{T}_2$-ϖ-*open in* $(\prod_\alpha \mathscr{X}_\alpha, \mathscr{T}_2)$. *This shows that* $\prod_\alpha f_\alpha$ *is pairwise* ϖ-*strongly continuous.* □

Similarly, we can prove the following theorems:

Theorem 2.27. *The function* $\prod_\alpha f_\alpha : \prod_\alpha(\mathscr{X}_\alpha, \mathscr{T}_1, \mathscr{T}_2) \to \prod_\alpha(\mathscr{Y}_\alpha, \mathscr{F}_1, \mathscr{F}_2)$ *define by* $\{\mathscr{X}_\alpha\} \to \{f_\alpha(\mathscr{X}_\alpha)\}$ *is a pairwise* ϖ-*closure continuous if and only if each* $f_\alpha : (\mathscr{X}_\alpha, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{Y}_\alpha, \mathscr{F}_1, \mathscr{F}_2)$ *is pairwise* ϖ-*closure continuous.*

Theorem 2.28. *The function* $\prod_\alpha f_\alpha : \prod_\alpha(\mathscr{X}_\alpha, \mathscr{T}_1, \mathscr{T}_2) \to \prod_\alpha(\mathscr{Y}_\alpha, \mathscr{F}_1, \mathscr{F}_2)$ *define by* $\{\mathscr{X}_\alpha\} \to \{f_\alpha(\mathscr{X}_\alpha)\}$ *is a pairwise* ϖ-*weakly continuous if and only if each* $f_\alpha : (\mathscr{X}_\alpha, \mathscr{T}_1, \mathscr{T}_2) \to (\mathscr{Y}_\alpha, \mathscr{F}_1, \mathscr{F}_2)$ *is pairwise* ϖ-*weakly continuous.*

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