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Discussion on delay Bessel's problems

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Abstract

This paper presents the delay Bessel's problem, and therefore the basic definitions, theorems, applications, and corollaries will be reviewed during this paper.

Keywords: Eventually positive, eventually negative, oscillatory, Bessel's problem.

1. Introduction

Delay differential (DDEs) are an oversized and vital category of projectile system, they typically arise in either natural or technological management problems, there are completely different sorts of delay differential equations, several authors are curious about the study of delay differential equation like O. Acino, M. L. Hbid and E.Ait Dads [2], Tony Humphries [10], George E.chatzarakis [6], William R.Derrick, Stanley.Grossman [11].
In this paper we study the delay Bessel problems which they're denoted by (DBP), we have an interest in learning definitions, examples and theorems for the delay Bessel problems.
As well as, the Bessel equation studied by Haniye Dehestani, Yadolloh Ordokhani [7], Dragana Jankov maširević and Tibork. Pogàny [4], Ciemens markett [3], Fethi Bin Muhammad Belgacem [5], martin Kerh [8], Ahmed Fitauhi and M.moncef Hamza [1], Orin J.Farrell [9].

2. Discussion on Delay Bessel's Problems

During this section, the DBP are studied and given with necessary and spare conditions, in this work a lot of elaborated definitions, examples and therefore theorems, preposition the delay Bessel's problems.

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Definition 2.1. *The DBP are variety of differential equations, in math's from second order, it's the 2 inequalities and the following equation.*

$$x^2y''(t) + a(t)xy'(t) + p(t)\left(x^2 - n^2\right)y(t-\tau) = 0 \tag{2.1}$$
$$x^2y''(t) + a(t)xy'(t) + p(t)\left(x^2 - n^2\right)y(t-\tau) \leq 0 \tag{2.2}$$
$$x^2y''(t) + a(t)xy'(t) + p(t)\left(x^2 - n^2\right)y(t-\tau) \geq 0 \tag{2.3}$$

Where $a(t) \geq 0,\ p(t) > 0$ *are continuous function on some interval* $a \leq t \leq b$ *and* $n \geq 0$, τ *is a positive constant, we give the following sufficient condition:*

- *Equation* (2.1) *has oscillatory solutions.*
- *Inequalities* (2.2) *has eventually negative solutions.*
- *Inequalities* (2.3) *has eventually positive solutions.*

Theorem 2.2. *Consider the delay Bessel inequality.*

$$x^2y''(t) + a(t)xy'(t) + p(t)\left(x^2 - n^2\right)y(t-\tau) \leq 0 \tag{2.4}$$

Where τ *is positive constant and* $a(t) \geq 0,\ p(t) > 0$ *are continuous functions for* $t \in R^+$ *and* $n \geq 0, x \neq 0$. *Also,*

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} \left(x^2 - n^2\right)p(s)\,ds > \liminf_{t\to\infty} \left[-x^2\tau - \int_{t-\tau}^{t} xa(s)\,ds\right] \tag{2.5}$$
$$\liminf_{t\to\infty} \int_{t-\tau}^{t} (x^2 - n^2)p(s)ds > 0 \tag{2.6}$$

Then (2.2) *has eventually negative solutions.*

Proof . *Let* $y(t)$ *be a solution to* (2.2), *to show that* $y(t)$ *is eventually positive, which leads to a contradiction*

$$y(t) > 0, \qquad t > t_0$$
$$y(t-\tau) > 0, \qquad t > t_0 + \tau$$

From (2.2), $y''(t) < 0, \quad t > t_0 + \tau,\ \ y(t) < y(t-\tau), t > t_0 + 2\tau$.
Dividing (2.2) *by* $y(t)$ *and integrating both sides from* $t-\tau$ *to* t *for* $t > t_0 + 3\tau$ *and by supposing that* $y(t) = e^t$, *then are obtaining the following:*

$$\frac{y(t-\tau)}{y(t)} \int_{t-\tau}^{t} (x^2 - n^2)p(s)ds \leq -x^2\tau - \int_{t-\tau}^{t} xa(s)ds \tag{2.7}$$
$$\int_{t-\tau}^{t} (x^2 - n^2)p(s)ds \leq \frac{y(t)}{y(t-\tau)} \left[-x^2\tau - \int_{t-\tau}^{t} xa(s)ds\right] \tag{2.8}$$

Since $\frac{y(t-\tau)}{y(t)} < 1$, *then*

$$\int_{t-\tau}^{t} (x^2 - n^2)p(s)ds < -x^2\tau - \int_{t-\tau}^{t} xa(s)ds \tag{2.9}$$

Take the limit inferiors on both sides (2.9), *it leads to the following results.*

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} (x^2 - n^2)p(s)ds < \liminf_{t\to\infty} \left[-x^2\tau - \int_{t-\tau}^{t} xa(s)ds\right] \tag{2.10}$$

Which leads to contradiction (2.4) *and the proof is complete.* □

Theorem 2.3. *Consider the delay Bessel inequality.*

$$x^2y''(t) + a(t)y'(t) + p(t)(x^2 - n^2)y(t-\tau) \geq 0 \tag{2.11}$$

Subject to the hypotheses of Theorem 2.2, and $\liminf_{t\to\infty} \int_{t-\tau}^{t} (x^2 - n^2)p(s)ds > 0$, *then* (2.3) *has eventually positive solution only.*

Proof . *The same steps in Theorem* (2.1) *and by supposing* $y(t)$ *is a solution to* (2.3), *to prove that* $-y(t)$ *is eventually negative which leads to a contradiction.* $\square$

Theorem 2.4. *Suppose that the delay Bessel's problems* (2.1) *exists and,* $\liminf_{t\to\infty} \int_{t-\tau}^{t} (x^2-n^2)p(s)ds > 0$. *Then* (2.1) *has oscillatory solutions only.*

Proof . *when applying these same proof steps for the previous Theorems 2.2 and 2.3, we assume the opposite to get the contradiction.* $\square$

Implementation

In this part of the work are some illustrative examples of delay Bessel's problem and are a clear application of the theories mentioned earlier.

Example 2.5.

$$x^2y''(t) + 0.27xy(t) + \left(x^2 - 1\right)y(t-1) \leq 0$$

Has eventually negative solution, $y(t) = -e^t, \;\; x = 1$

Example 2.6. *Consider the delay Bessel*

$$x^2y''(t) + (x^2 - 3)y(t-\tau) = 0$$

has oscillatory solution, $y(t) = \sin(t)$, *when* $t = \pm n\pi$

Example 2.7. *Consider the delay Bessel inequality*

$$x^2y''(t) - 4xy'(t) + (x^2 - 4)y(t - 2\pi) \geq 0$$

Than it has a positive solution, $y(t) = t^2, x = 2$, *when* $t \leq 1/2$

Example 2.8. *Let the delay Bessel's equation*

$$x^2y''(t) + xy'(t) + (x^2 - \frac{1}{2})y(t - 3\pi) = 0$$

has oscillatory solution, $y(t) = t^{-1}$, *when* $t = \sqrt{2}/2$.

3. Conclusion

The most objective of this work is to review the delay Bessel's issues and this comprise finding out the condition that understand eventually (positive, negative), oscillation wherever examples, theorems and corollaries are given to clarify every case.

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