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Some properties of fuzzy soft $\mathfrak{n}-\widetilde{\mathcal{N}}$ quasi normal operators

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Abstract

In this work, we invested a kind of fuzzy soft quasi normal operator namely fuzzy soft $(\mathfrak{n}-\widetilde{\mathcal{N}})$-quasi normal operator this modification of fuzzy soft bounded linear quasi normal operator appear in recently many papers. Some properties and operation about this operator have been given, also more conditions given to get some theorems in this study.

Keywords: fuzzy soft bounded linear operator, Fuzzy soft Quasi normal operator, fuzzy soft $(\mathfrak{n}-\widetilde{\mathcal{N}})$ quasi normal operator, fuzzy soft Hilbert space

1. Introduction

Functional analysis is a branch of pure mathematics. It was first developed in about century ago. It aims to solve many problems in pure mathematics. Therefore it provides an indispensable tool for solving those problems. It also provides us with techniques for estimating error in the solutions of infinite and finite dimensional problems. In our everyday life, we often faced with uncertainty that arises from the ambiguity of the phenomenon under study. This type of problems arises in areas like economics, medical science, business and engineering. Our classical mathematical methods often fail to tackle such problems. Thus, In 1965, a generalization of set theory was presented, by L. Zadeh [13]. The resulting theory was called fuzzy set theory. Fuzzy set theory soon became an excellent tool to deal with problems that associate with uncertainty. In classical set theory, a set $\mathcal{X}$ is define with its characteristic function from $\mathcal{X}$ to set $\{0,1\}$. On other hand, in fuzzy set theory, a set is define with its membership function form $\mathcal{X}$ to the closed interval $[0,1]$. Also In 1999, a yet another

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generalization was introduced by Molodsov [6] to deals with uncertainty. The resulting theory was called soft set theory. Since then, it was used to solve complicated problems in subject like computer science, medicine, engineering, etc. A soft set: a parametrized collection of a universal set. The concept soft set was then applied on various mathematical concepts in functional analysis resulting in concepts like soft point [1], Soft normed Space [13], Soft inner Product Space [3], and Soft Hilbert spaces [2]. In 2001, Maji et.al [4], was first to introduce the concept of a fuzzy soft set. The concept resulted from combining the concept of a fuzzy and a soft set. The necessity of combining the two concepts was to provide more accurate and general results. The other concepts consequently followed this structure resulting in the introduction fuzzy soft point [5], and fuzzy soft normed space [6]. In 2020, fuzzy soft Hilbert spaces [7] were presented by Faried et al. Furthermore the Fuzzy soft linear operators [8]. In this work, Submitting a new kind of fuzzy soft normal operators is said to be fuzzy soft $(n-\widetilde{\mathcal{N}})$-quasi normal, and give some theories related to this operator and its characteristics.

2. Basic Concepts

Some basic concepts of fuzzy set theory and soft set theory are introduced. Dealing with the $\mathcal{FS}$-set and $\mathcal{FS}$- inner product space and giving some theorem related to $\mathcal{FS}$-inner product space. And the $\mathcal{FS}$-convergent sequence is studied and some of its properties are given.

Definition 2.1. *[14] If $\widehat{A}$ is fuzzy set over $\mathcal{X}$ is set characterized by a membership function where $\mu_{\widehat{A}}$: $\mathcal{X} \longrightarrow \mathfrak{T}$, such as $\mathfrak{T} = [0,1]$ and $\widehat{A} = \left\{ \frac{\mu_{\widehat{A}}(x)}{x} \ : x \in \mathcal{X} \right\}$. And $\mathfrak{T}^{\mathcal{X}} = \left\{ \widehat{A} \ : \widehat{A}\right.$ is a function from X into $\mathfrak{T}\}$*

Definition 2.2. *[9] suppose that $\mathcal{P}(\mathcal{X})$ the power set of $\mathcal{X}$ and E be set of parameters and $\subseteq E$. The mapping $\mathcal{G} : A \to \mathcal{P}(\mathcal{X})$, when $(\mathcal{G}, A) = \{\mathcal{G}(a) \in \mathcal{P}(\mathcal{X}) : a \in A\}$. such that $(\mathcal{G}, A)$ we say that soft set.*

Definition 2.3. *[4] The soft set $\mathcal{G}_A$ we say that fuzzy soft set ($\mathcal{FS}-$set) over $\mathcal{X}$, when $\mathcal{G} : A \to \mathfrak{T}^{\mathcal{X}}$, and $\{\mathcal{G}(a) \in \mathfrak{T}^{\mathcal{X}} : a \in A\}$. The collection of all $\mathcal{FS}-$sets, denoted by $\mathcal{FSS}(\widetilde{\mathcal{X}})$.*

Definition 2.4. *[5] If $(\mathcal{G},A) \in \mathcal{FSS}(\widetilde{\mathcal{X}})$ we say that $\mathcal{FS}-$point over $\mathcal{X}$, denoted by $\widetilde{a}_{\mu_{\mathcal{G}(e)}}$, if $e \in A$, $x \in \mathcal{X}$,*

$$\mu_{\mathcal{G}(e)}(x) = \begin{cases} \gamma, & if \ a = a_o \in \mathcal{X} \ and \quad \varepsilon = \varepsilon_0 \in A \\ 0, & if \ a \in \mathcal{X} - \{a_0\} \quad or \quad \varepsilon \in A - \{\varepsilon_0\} \end{cases}, such\ that \gamma \in (0,1]$$

Remark 2.5. *[5] The collection of all $\mathcal{FS}-$Complex number denoted by $\widetilde{\mathcal{C}}(A)$ and the collection of all $\mathcal{FS}-$Real number denoted by $\widetilde{\mathcal{R}}(A)$.*

Definition 2.6. *[6] A mapping $\widetilde{\|.\|}$: $\widetilde{\mathcal{X}} \to \widetilde{\mathcal{R}}^{+}(A)$ where $\widetilde{\mathcal{X}}$ be $\mathcal{FS}-$vector space is said to be $\mathcal{FS}-$norm on $\widetilde{\mathcal{X}}$ if $\widetilde{\|.\|}$ satisfies :*

(1) $\widetilde{\|\widetilde{a}_{\mu_{\mathcal{G}(e)}}\|} \geq \widetilde{0}$, $\forall\ \widetilde{a}_{\mu_{\mathcal{G}(1)}} \in \widetilde{\mathcal{X}}$,and $\widetilde{\| \widetilde{a}_{\mu_{\mathcal{G}(e)}} \|} = \widetilde{0}$ if and only if $\widetilde{a}_{\mu_{\mathcal{G}(e)}} = \widetilde{\theta}$

(2) $\widetilde{\|\widetilde{r}\widetilde{a}_{\mu_{\mathcal{G}(e)}}\|} = \|\widetilde{r}\|\widetilde{\|\widetilde{a}_{\mu_{\mathcal{G}(e)}}\|}$, for all $\widetilde{a}_{\mu_{\mathcal{G}(e)}} \in \widetilde{\mathcal{X}}$, $\widetilde{r} \in \mathcal{C}(A)$.

(3) $\widetilde{\|\widetilde{a}_{\mu_{\mathcal{G}(e)}} + \widetilde{b}_{\mu_{2\mathcal{G}(e_2)}}\|} \leq \widetilde{\| \widetilde{a}_{\mu_{\mathcal{G}(e)}}\|} + \widetilde{\|\widetilde{b}_{\mu_{2\mathcal{G}(e_2)}}\|}$, $\forall\ \widetilde{a}_{\mu_{\mathcal{G}(e)}}, \widetilde{b}_{\mu_{2\mathcal{G}(e_2)}} \in \widetilde{\mathcal{X}}$.

Then $\left(\widetilde{\mathcal{X}}, \widetilde{\|.\|}\right)$ is called $\mathcal{FS}-$normed vector space ($\mathcal{FSN}-$space)

Definition 2.7. *[8] A mapping* $\widetilde{\langle .,. \rangle} : \widetilde{\mathcal{X}} \times \widetilde{\mathcal{X}} \to (\mathcal{C}(A)$ *or* $\mathcal{R}(A))$ *where* $\widetilde{\mathcal{X}}$ *be FSV−space is called* $\mathcal{FS}-$*inner product on* $\widetilde{\mathcal{X}}$ $(FS\mathfrak{I})$ *if* $\widetilde{\langle .,. \rangle}$ *satisfies:*

(1) $\left\langle \widetilde{a}_{\mu_{\mathcal{G}(e)}}, \widetilde{a}_{\mu_{\mathcal{G}(e)}} \right\rangle \geq \widetilde{0}$, $\forall\, \widetilde{a}_{\mu_{\mathcal{G}(e)}} \in \widetilde{\mathcal{X}}$ *and* $\left\langle \widetilde{a}_{\mu_{\mathcal{G}(e)}}, \widetilde{a}_{\mu_{\mathcal{G}(e)}} \right\rangle = \widetilde{0}$ *if and only if* $\widetilde{a}_{\mu_{\mathcal{G}(e)}} = \widetilde{\theta}$

(2) $\left\langle \widetilde{a}_{\mu_{1\mathcal{G}(e_1)}}, \widetilde{b}_{\mu_{2\mathcal{G}(e_2)}} \right\rangle = \left\langle \widetilde{b}_{\mu_{2\mathcal{G}(e_2)}}, \widetilde{a}_{\mu_{1\mathcal{G}(e_1)}} \right\rangle$ *, for all* $\widetilde{a}_{\mu_{1\mathcal{G}(e_1)}}, \widetilde{b}_{\mu_{2\mathcal{G}(e_2)}} \in \widetilde{\mathcal{X}}$,

(3) $\left\langle \widetilde{\alpha}\widetilde{a}_{\mu_{1\mathcal{G}(e_1)}}, \widetilde{b}_{\mu_{2\mathcal{G}(e_2)}} \right\rangle = \widetilde{\alpha} \left\langle \widetilde{a}_{\mu_{1\mathcal{G}(e_1)}}, \widetilde{b}_{\mu_{2\mathcal{G}(e_2)}} \right\rangle$ *, for all* $\widetilde{a}_{\mu_{1\mathcal{G}(e_1)}}, \widetilde{b}_{\mu_{2\mathcal{G}(e_2)}} \in \widetilde{\mathcal{X}}$ *, for all* $\widetilde{\alpha} \in \mathcal{C}(A)$.

(4) $\left\langle \widetilde{a}_{\mu_{1\mathcal{G}(e_1)}} + \widetilde{b}_{\mu_{2\mathcal{G}(e_2)}}, \widetilde{c}_{\mu_{3\mathcal{G}(e_3)}} \right\rangle = \left\langle \widetilde{a}_{\mu_{1\mathcal{G}(e_1)}}, \widetilde{c}_{\mu_{3\mathcal{G}(e_3)}} \right\rangle + \left\langle \widetilde{b}_{\mu_{2\mathcal{G}(e_2)}}, \widetilde{c}_{\mu_{3\mathcal{G}(e_3)}} \right\rangle$

For all $\widetilde{a}_{\mu_{1\mathcal{G}(e_1)}}, \widetilde{b}_{\mu_{2\mathcal{G}(e_2)}}, \widetilde{c}_{\mu_{3\mathcal{G}(e_3)}} \in \widetilde{\mathcal{X}}$

Then $\left(\widetilde{\mathcal{X}}, \widetilde{\langle .,. \rangle}\right)$ *is called FS−inner product space* $(FS\mathfrak{T}-$*space*$)$

Definition 2.8. *[12]* $\left(\widetilde{\mathcal{X}}, \widetilde{\| \bullet \|}\right)$ *is* $\mathcal{FSN}-$*space we say that* $\mathcal{FS}-$*complete if all* $\mathcal{FS}-$ *Cauchy is* $\mathcal{FS}-$ *convergence.*

Definition 2.9. *[8] The* $\mathcal{FS}-$ *complete inner product space* $\left(\widetilde{\mathcal{X}}, \widetilde{\langle .,. \rangle}\right)$ *is called* $\mathcal{FS}-$*Hilbert space* $(\mathcal{FSH}-$*space*$)$ *, and symbolized by* $\left(\widetilde{\mathcal{H}}, \widetilde{\langle\ ,\ \rangle}\right)$.

Definition 2.10. *[11] If* $\widetilde{\mathcal{H}}$ *be* $\mathcal{FSH}-$*space and* $\widetilde{\mathfrak{T}} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ *be* $\mathcal{FS}-$*operator .Then* $\widetilde{\mathfrak{T}}$ *is called* $\mathcal{FS}-$*linear operator* $(FSL-$ *operator*$)$ *if:*

$$\widetilde{\mathfrak{T}}\left(\widetilde{\alpha}\widetilde{a}_{\mu_{1\mathcal{G}(e_1)}} + \widetilde{\beta}\widetilde{b}_{\mu_{2\mathcal{G}(e_2)}}\right) = \widetilde{\alpha}\widetilde{\mathfrak{T}}\left(\widetilde{a}_{\mu_{1\mathcal{G}(e_1)}}\right) + \widetilde{\beta}\widetilde{\mathfrak{T}}\left(\widetilde{b}_{\mu_{2\mathcal{G}(e_2)}}\right),\ \forall\, \widetilde{a}_{\mu_{1\mathcal{G}(e_1)}}, \widetilde{b}_{\mu_{2\mathcal{G}(e_2)}} \in \widetilde{\mathcal{H}} \text{ and } \widetilde{\alpha}, \widetilde{\beta} \in \widetilde{\mathcal{C}}(A)$$

Definition 2.11. *[11] If* $\widetilde{\mathcal{H}}$*be* $\mathcal{FSH}-$ *space and* $\widetilde{\mathfrak{T}} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ *be* $\mathcal{FS}-$*operator is called* $\mathcal{FS}-$*bounded operator* $(\mathcal{FSB}-$*operator*$)$ *, if* $\exists\ \widetilde{m} \in \mathcal{R}(A)$ *such that* $\|\widetilde{\mathfrak{T}}\left(\widetilde{a}_{\mu_{1\mathcal{G}(e_1)}}\right)\| \leq \widetilde{m}\|\widetilde{a}_{\mu_{1\mathcal{G}(e_1)}}\|$, *for all* $\widetilde{a}_{\mu_{1\mathcal{G}(e_1)}} \in \widetilde{\mathcal{H}}$

Now, the collection of all $\mathcal{FS}-$ bounded linear operator is symbolized by $\widetilde{B}(\widetilde{\mathcal{H}})$.

Example 2.12. *[11] The* $\mathcal{FS}-$*operator* $\widetilde{I} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ *defined by* $\widetilde{I}\left(\widetilde{a}_{\mu_{1\mathcal{G}(e_1)}}\right) = \widetilde{a}_{\mu_{1\mathcal{G}(e_1)}}$ *,* $\forall\, \widetilde{a}_{\mu_{1\mathcal{G}(e_1)}} \in \widetilde{\mathcal{H}}$ *, it is called* $\mathcal{FS}-$*identity operator*

Definition 2.13. *[11] If* $\widetilde{\mathcal{H}}$ *be* $\mathcal{FSH}-$*space and* $\widetilde{\mathfrak{T}} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ *be* $\mathcal{FSB}-$*operator, then The* $\mathcal{FS}-$*adjoint operator* $\widetilde{\mathfrak{T}}^*$ *is defined by*

$$\left\langle \widetilde{\mathfrak{T}}\widetilde{a}_{\mu_{1\mathcal{G}(e_1)}}, \widetilde{b}_{\mu_{2\mathcal{G}(e_2)}} \right\rangle = \left\langle \widetilde{a}_{\mu_{1\mathcal{G}(e_1)}}, \widetilde{\mathfrak{T}}^*\, \widetilde{b}_{\mu_{2\mathcal{G}(e_2)}} \right\rangle \text{ , for all } \widetilde{a}_{\mu_{1\mathcal{G}(e_1)}}, \widetilde{b}_{\mu_{2\mathcal{G}(e_2)}} \in \widetilde{\mathcal{H}}$$

Theorem 2.14. *[11] If* $\widetilde{\mathfrak{T}}, \widetilde{\varphi} \in \widetilde{B}(\widetilde{\mathcal{H}})$ *,where* $\widetilde{\mathcal{H}}$ *is* $\mathcal{FSH}-$*space and* $\widetilde{\beta} \in \mathcal{C}(A)$ *, then* $\widetilde{\widetilde{T}^*}^* = \widetilde{\mathfrak{T}}$ *,* $\left(\widetilde{\beta T}\right)^* = \widetilde{\overline{\beta}}\widetilde{T}^*$ *,* $\left(\widetilde{\mathfrak{T}} + \widetilde{\varphi}\right)^* = \widetilde{\mathfrak{T}}^* + \widetilde{R}^*$ *and* $\left(\widetilde{\mathfrak{T}}\widetilde{\varphi}\right)^* = \widetilde{\varphi}^*\widetilde{\mathfrak{T}}^*$.

Theorem 2.15. *[11] if* $\widetilde{\mathfrak{T}} \in \widetilde{B}(\widetilde{\mathcal{H}})$ *,where* $\widetilde{\mathcal{H}}$ *is* $\mathcal{FSH}-$*space, then* $\widetilde{\|\widetilde{\mathfrak{T}}^*\|} = \widetilde{\|\ \widetilde{\mathfrak{T}}\|}$ *and* $\widetilde{\|\widetilde{\mathfrak{T}}^*\widetilde{\mathfrak{T}}\|} = \widetilde{\|\ \widetilde{\mathfrak{T}}\|}^2$.

Definition 2.16. *[7] The* $\mathcal{FS}-$*operator* $\widetilde{\mathfrak{T}}$ *of* $\mathcal{FSH}-$*space* $\widetilde{\mathcal{H}}$ *is called* $\mathcal{FS}-$*self adjoint operator if* $\widetilde{\mathfrak{T}} = \widetilde{\mathfrak{T}}^*$.

3. Main results

Submitting a new kind of fuzzy soft normal operators is said to be fuzzy soft $(\mathfrak{n}-\widetilde{\mathcal{N}})$-quasi normal, and give some theories related to this operator and its characteristics. In addition to the generalizations of many definitions that were relied upon in the work

Definition 3.1. *If $\widetilde{\mathfrak{T}}$ be $\mathcal{FS}-$operator of $\mathcal{FSH}-$space $\widetilde{\mathcal{H}}$ is called $\mathcal{FS}-$normal operator if $\widetilde{\mathfrak{T}}\widetilde{\mathfrak{T}}^* = \widetilde{\mathfrak{T}}^*\widetilde{\mathfrak{T}}$.*

Definition 3.2. *Let $\widetilde{\mathfrak{T}}$ be $\mathcal{FS}-$operator of $\mathcal{FSH}-$space $\widetilde{\mathcal{H}}$ is called $\mathcal{FS}-n-$normal operator if $\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^* = \widetilde{\mathfrak{T}}^*\widetilde{\mathfrak{T}}^n$*

Definition 3.3. *Let $\widetilde{\mathfrak{T}} \in \widetilde{B}(\widetilde{\mathcal{H}})$ is called $\mathcal{FS}-$quasi normal operator if $\widetilde{\mathfrak{T}}\left(\widetilde{\mathfrak{T}}^*\widetilde{\mathfrak{T}}\right) = \left(\widetilde{\mathfrak{T}}^*\widetilde{\mathfrak{T}}\right)\widetilde{\mathfrak{T}}$*

Definition 3.4. *Let $\widetilde{\mathfrak{T}} \in \widetilde{B}(\widetilde{\mathcal{H}})$ is called FS$-\widetilde{\mathcal{N}}-$quasi normal operator if $\widetilde{\mathfrak{T}}\left(\widetilde{\mathfrak{T}}^*\widetilde{\mathfrak{T}}\right) = \widetilde{\mathcal{N}}\left(\widetilde{\mathfrak{T}}^*\widetilde{\mathfrak{T}}\right)\widetilde{\mathfrak{T}}$*

Definition 3.5. *A fuzzy soft bounded liner operator $\widetilde{\mathfrak{T}} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ is said to be fuzzy soft $\left(\mathfrak{n}-\widetilde{\mathcal{N}}\right)-$ quasi normaloperator is defined by $\left(\mathcal{FS}-\left(\mathfrak{n}-\widetilde{\mathcal{N}}\right)-\text{quasi normal operator}\right)$, if $\left(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\right)\left(\widetilde{\mathfrak{T}}+\widetilde{\mathfrak{T}}^*\right) = \widetilde{\mathcal{N}}\left(\widetilde{\mathfrak{T}}+\widetilde{\mathfrak{T}}^*\right)\left(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\right)$*

Theorem 3.6. *Let $\widetilde{\mathfrak{T}} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ be $\mathcal{FS}-\left(\mathfrak{n}-\widetilde{\mathcal{N}}\right)-$ quasi normal operator and $\widetilde{S} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ be $\mathcal{FS}-$self adjoint operator such that $\widetilde{\mathfrak{T}}\widetilde{S} = \widetilde{S}\widetilde{\mathfrak{T}}$, $\widetilde{\mathcal{N}}\widetilde{S} = \widetilde{S}\widetilde{\mathcal{N}}$, then $\widetilde{\mathfrak{T}}\ \widetilde{S}$ is $\mathcal{FS}-\left(\mathfrak{n}-\widetilde{\mathcal{N}}\right)-$ quasi normal operator.*

Proof .

$$\left(\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)^n\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)^*\right)\left(\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)+\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)^*\right) = \left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)^n\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)^*\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)+\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)^n\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)^*\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)^*$$
$$=\left(\widetilde{\mathfrak{T}}^n\ \widetilde{S}^n\right)\left(\widetilde{S}^*\ \widetilde{\mathfrak{T}}^*\right)\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)+\left(\widetilde{\mathfrak{T}}^n\ \widetilde{S}^n\right)\left(\widetilde{S}^*\ \widetilde{\mathfrak{T}}^*\right)\left(\widetilde{S}^*\ \widetilde{\mathfrak{T}}^*\right)$$
$$=\left(\widetilde{S}^n\ \widetilde{\mathfrak{T}}^n\right)\left(\widetilde{\mathfrak{T}}^*\ \widetilde{S}^*\right)\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)+\left(\widetilde{S}^n\ \widetilde{\mathfrak{T}}^n\right)\left(\widetilde{\mathfrak{T}}^*\ \widetilde{S}^*\right)\left(\widetilde{\mathfrak{T}}^*\ \widetilde{S}^*\right)$$
$$=\left(\widetilde{S}^n\ \widetilde{\mathfrak{T}}^n\right)\ \widetilde{\mathfrak{T}}^*\left(\widetilde{S}^*\ \widetilde{\mathfrak{T}}\right)\ \widetilde{S}+\left(\widetilde{S}^n\ \widetilde{\mathfrak{T}}^n\right)\ \widetilde{\mathfrak{T}}^*\left(\widetilde{S}^*\ \widetilde{\mathfrak{T}}^*\right)\ \widetilde{\mathcal{S}}^*$$
$$=\left(\widetilde{S}^n\ \widetilde{\mathfrak{T}}^n\right)\ \widetilde{\mathfrak{T}}^*\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)\ \widetilde{S}+\left(\widetilde{S}^n\ \widetilde{\mathfrak{T}}^n\right)\ \widetilde{\mathfrak{T}}^*\left(\widetilde{\mathfrak{T}}^*\ \widetilde{S}^*\right)\ \widetilde{S}^*$$
$$=\widetilde{S}^n\left(\widetilde{\mathfrak{T}}^n\ \widetilde{\mathfrak{T}}^*\ \widetilde{\mathfrak{T}}\right)\ \widetilde{S}^2+\widetilde{S}^n\left(\widetilde{\mathfrak{T}}^n\ \widetilde{\mathfrak{T}}^*\ \widetilde{\mathfrak{T}}^*\right)\ \widetilde{S}^2$$
$$=\widetilde{S}^n\left(\widetilde{\mathfrak{T}}^n\ \widetilde{\mathfrak{T}}^*\ \left(\widetilde{\mathfrak{T}}+\widetilde{\mathfrak{T}}^*\right)\right)\ \widetilde{S}^2$$
$$=\widetilde{S}^n\left(\widetilde{\mathcal{N}}\ \left(\widetilde{\mathfrak{T}}+\widetilde{\mathfrak{T}}^*\right)\ \widetilde{\mathfrak{T}}^n\ \widetilde{\mathfrak{T}}^*\right)\ \widetilde{S}^2$$
$$=\widetilde{\mathcal{N}}\widetilde{S}^n\left(\widetilde{\mathfrak{T}}\ \widetilde{\mathfrak{T}}^n\ \widetilde{\mathfrak{T}}^*\right)\ \widetilde{S}^2+\widetilde{\mathcal{N}}\ \widetilde{S}^n\left(\widetilde{\mathfrak{T}}^*\ \widetilde{\mathfrak{T}}^n\ \widetilde{\mathfrak{T}}^*\right)\ \widetilde{S}^2$$
$$=\widetilde{\mathcal{N}}[\widetilde{S}\ \left(\widetilde{S}^{n-1}\ \widetilde{\mathfrak{T}}\right)\ \widetilde{\mathfrak{T}}^n\left(\widetilde{\mathfrak{T}}^*\ \widetilde{S}^*\right)\widetilde{S}^*+\widetilde{S}\left(\widetilde{S}^{n-1}\ \widetilde{\mathfrak{T}}^*\right)\ \widetilde{\mathfrak{T}}^n\left(\widetilde{\mathfrak{T}}^*\ \widetilde{S}^*\right)\ \widetilde{S}^*]$$
$$=\widetilde{\mathcal{N}}[\widetilde{S}\left(\widetilde{\mathfrak{T}}\ \widetilde{S}^{n-1}\right)\ \widetilde{\mathfrak{T}}^n\left(\widetilde{S}^*\ \widetilde{\mathfrak{T}}^*\right)\widetilde{S}^*+\widetilde{S}\left(\widetilde{\mathfrak{T}}^*\ \widetilde{S}^{n-1}\right)\ \widetilde{\mathfrak{T}}^n\left(\widetilde{S}^*\ \widetilde{\mathfrak{T}}^*\right)\widetilde{S}^*$$
$$=\widetilde{\mathcal{N}}[\left(\widetilde{S}\ \widetilde{\mathfrak{T}}\right)\left(\widetilde{S}^{n-1}\ \widetilde{\mathfrak{T}}^n\right)\widetilde{S}\left(\widetilde{\mathfrak{T}}^*\ \widetilde{S}^*\right)+\left(\widetilde{S}^*\ \widetilde{\mathfrak{T}}^*\right)\left(\widetilde{S}^{n-1}\ \widetilde{\mathfrak{T}}^n\right)\widetilde{S}\left(\widetilde{\mathfrak{T}}^*\ \widetilde{S}^*\right)]$$

$$= \widetilde{\mathcal{N}}[\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)\left(\widetilde{\mathfrak{T}}^n\ \widetilde{S}^n\right)\left(\widetilde{S}\ \widetilde{\mathfrak{T}}\right)^* + \left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)^*\left(\widetilde{\mathfrak{T}}^n\ \widetilde{S}^n\right)\left(\widetilde{S}\ \widetilde{\mathfrak{T}}\right)^*$$
$$= \widetilde{\mathcal{N}}[\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)^n\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)^* + \left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)^*\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)^n\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)^*]$$
$$= \widetilde{\mathcal{N}}\left(\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right) + \left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)^*\right)\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)^n\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)^*$$

Therefore $\left(\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)^n\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)^*\right)\left(\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right) + \left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)^*\right) = \widetilde{\mathcal{N}}\left(\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right) + \left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)^*\right)\left(\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)^n\left(\widetilde{\mathfrak{T}}\ \widetilde{S}\right)^*\right)$
And so, $\widetilde{\mathfrak{T}}\ \widetilde{S}$ *is* $\mathcal{FS}-\left(\mathfrak{n}-\widetilde{\mathcal{N}}\right)-$ *quasi normal operator.* □

Theorem 3.7. *Let* $\widetilde{\mathfrak{T}}:\widetilde{\mathcal{H}}\to\widetilde{\mathcal{H}}$ *be* $\mathcal{FS}-\left(\mathfrak{n}-\widetilde{\mathcal{N}}\right)-$*quasi normal operator. Then* $\widetilde{\lambda}\widetilde{\mathfrak{T}}$ *is* $\mathcal{FS}-\left(\mathfrak{n}-\widetilde{\mathcal{N}}\right)-$ *quasi normal operator, where* $\widetilde{\lambda}\in\mathcal{R}(A)$.

Proof . *We must proof the following*

$$\left(\left(\widetilde{\lambda}\widetilde{\mathfrak{T}}\right)^n\left(\widetilde{\lambda}\widetilde{\mathfrak{T}}\right)^*\right)\left(\left(\widetilde{\lambda}\widetilde{\mathfrak{T}}\right) + \left(\widetilde{\lambda}\widetilde{\mathfrak{T}}\right)^*\right) = \widetilde{\mathcal{N}}\left(\left(\widetilde{\lambda}\widetilde{\mathfrak{T}}\right) + \left(\widetilde{\lambda}\widetilde{\mathfrak{T}}\right)^*\right)\left(\left(\widetilde{\lambda}\widetilde{\mathfrak{T}}\right)^n\left(\widetilde{\lambda}\widetilde{\mathfrak{T}}\right)^*\right)$$

Now, we will take the first party to get the second party

$$\begin{aligned}\left(\left(\widetilde{\lambda}\widetilde{\mathfrak{T}}\right)^n\left(\widetilde{\lambda}\widetilde{\mathfrak{T}}\right)^*\right)\left(\widetilde{\lambda}\widetilde{\mathfrak{T}} + \left(\widetilde{\lambda}\widetilde{\mathfrak{T}}\right)^*\right) &= \left(\widetilde{\lambda}^n\widetilde{\mathfrak{T}}^n\widetilde{\lambda}\widetilde{\mathfrak{T}}^*\right)\left(\widetilde{\lambda}\widetilde{\mathfrak{T}} + \widetilde{\lambda}\widetilde{\mathfrak{T}}^*\right)\\ &= \widetilde{\lambda}\widetilde{\lambda}^n\widetilde{\lambda}\left(\widetilde{\mathfrak{T}}^*\widetilde{\mathfrak{T}}^n\right)\left(\widetilde{\mathfrak{T}}^* + \widetilde{\mathfrak{T}}\right)\\ &= \widetilde{\lambda}^n\widetilde{\lambda}\widetilde{\lambda}\widetilde{\mathcal{N}}\left(\widetilde{\mathfrak{T}} + \widetilde{\mathfrak{T}}^*\right)\left(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\right)\\ &= \widetilde{\mathcal{N}}\left(\widetilde{\lambda}\widetilde{\mathfrak{T}} + \widetilde{\lambda}\widetilde{\mathfrak{T}}^*\right)\left(\lambda^n\widetilde{\mathfrak{T}}^n\widetilde{\lambda}\widetilde{\mathfrak{T}}^*\right)\\ &= \widetilde{\mathcal{N}}\left(\widetilde{\lambda}\widetilde{\mathfrak{T}} + \left(\widetilde{\lambda}\widetilde{\mathfrak{T}}\right)^*\right)\left(\left(\widetilde{\lambda}\widetilde{\mathfrak{T}}\right)^n\left(\widetilde{\lambda}\widetilde{\mathfrak{T}}\right)^*\right)\end{aligned}$$
$$\Longrightarrow \left(\left(\widetilde{\lambda}\widetilde{\mathfrak{T}}\right)^n\left(\widetilde{\lambda}\widetilde{\mathfrak{T}}\right)^*\right)\left(\left(\widetilde{\lambda}\widetilde{\mathfrak{T}}\right) + \left(\widetilde{\lambda}\widetilde{\mathfrak{T}}\right)^*\right) = \widetilde{\mathcal{N}}\left(\left(\widetilde{\lambda}\widetilde{\mathfrak{T}}\right) + \left(\widetilde{\lambda}\widetilde{\mathfrak{T}}\right)^*\right)\left(\left(\widetilde{\lambda}\widetilde{\mathfrak{T}}\right)^n\left(\widetilde{\lambda}\widetilde{\mathfrak{T}}\right)^*\right)$$

Therefore $\widetilde{\lambda}\widetilde{\mathfrak{T}}$ *is* $\mathcal{FS}-\widetilde{\mathcal{N}}-$ *quasi* $n-$ *operator* □

Theorem 3.8. *Let* $\widetilde{\mathfrak{T}}:\widetilde{\mathcal{H}}\to\widetilde{\mathcal{H}}$ *and* $\widetilde{S}:\widetilde{\mathcal{H}}\to\widetilde{\mathcal{H}}$ *be two* $\mathcal{FS}-\left(\mathfrak{n}-\widetilde{\mathcal{N}}\right)-$ *quasi normal operators such that* $\widetilde{\mathfrak{T}}^*\widetilde{\mathcal{S}}^* = \widetilde{\mathfrak{T}}\widetilde{\mathcal{S}} = \widetilde{\mathfrak{T}}\widetilde{\mathcal{S}}^* = \widetilde{\mathfrak{T}}^*\widetilde{\mathcal{S}} = \widetilde{0}$ *then* $\widetilde{\mathcal{R}} + \widetilde{\mathcal{S}}$ *is* $\mathcal{FS}-\left(\mathfrak{n}-\widetilde{\mathcal{N}}\right)-$ *quasi normal operator.*

Proof . *We must proof the following*

$$\left(\left(\widetilde{\mathfrak{T}}+\widetilde{S}\right)^n\left(\widetilde{\mathfrak{T}}+\widetilde{S}\right)^*\right)\left(\left(\widetilde{\mathfrak{T}}+\widetilde{S}\right) + \left(\widetilde{\mathfrak{T}}+\widetilde{S}\right)^*\right) = \widetilde{\mathcal{N}}\left(\left(\widetilde{\mathfrak{T}}+\widetilde{S}\right) + \left(\widetilde{\mathfrak{T}}^*+\widetilde{S}^*\right)\right)\left((\widetilde{\mathfrak{T}}+\widetilde{S})^n\left(\widetilde{\mathfrak{T}}^*+\widetilde{S}^*\right)\right).$$

We take the first party

$$\begin{aligned}&\left(\left(\widetilde{\mathfrak{T}}+\widetilde{S}\right)^n\left(\widetilde{\mathfrak{T}}+\widetilde{S}\right)^*\right)\left(\left(\widetilde{\mathfrak{T}}+\widetilde{S}\right) + \left(\widetilde{\mathfrak{T}}+\widetilde{S}\right)^*\right)\\ &= \left(\left(\widetilde{\mathfrak{T}}^n + n\widetilde{\mathfrak{T}}^{n-1}\ \widetilde{S} + \ldots + \widetilde{S}^n\right)\left(\widetilde{\mathfrak{T}}^*+\widetilde{S}^*\right)\right)\left(\left(\widetilde{\mathfrak{T}}+\widetilde{S}\right) + \left(\widetilde{\mathfrak{T}}^*+\widetilde{S}^*\right)\right)\\ &= \left(\left(\widetilde{\mathfrak{T}}^n + \widetilde{S}^n\right)\left(\widetilde{\mathfrak{T}}^*+\widetilde{S}^*\right)\right)\left(\left(\widetilde{\mathfrak{T}}+\widetilde{S}\right) + \left(\widetilde{\mathfrak{T}}^*+\widetilde{S}^*\right)\right)\\ &= \left(\widetilde{\mathfrak{T}}^n\ \widetilde{\mathfrak{T}}^* + \widetilde{S}^n\ \widetilde{\mathfrak{T}}^* + \widetilde{\mathfrak{T}}^n\ \widetilde{S}^* + \widetilde{S}^n\ \widetilde{S}^*\right)\left(\left(\widetilde{\mathfrak{T}}+\widetilde{\mathfrak{T}}^*\right) + \left(\widetilde{S}+\widetilde{S}^*\right)\right)\end{aligned}$$

$$= \left(\widetilde{\mathfrak{T}}^n\ \widetilde{\mathfrak{T}}^* + \widetilde{S}^n\ \widetilde{S}^*\right)\left(\left(\widetilde{\mathfrak{T}} + \widetilde{\mathfrak{T}}^*\right) + \left(\widetilde{S} + \widetilde{S}^*\right)\right)$$
$$= \left(\widetilde{\mathfrak{T}}^n\ \widetilde{\mathfrak{T}}^*\left(\widetilde{\mathfrak{T}} + \widetilde{\mathfrak{T}}^*\right) + \widetilde{S}^n\ \widetilde{S}^*\ \widetilde{\mathfrak{T}} + \widetilde{S}^n\ \widetilde{S}^*\ \widetilde{\mathfrak{T}}^* + \widetilde{\mathfrak{T}}^n\ \widetilde{\mathfrak{T}}^*\ \widetilde{S} + \widetilde{\mathfrak{T}}^n\ \widetilde{\mathfrak{T}}^*\ \widetilde{S}^* + \widetilde{S}^n\widetilde{S}^*\left(\widetilde{S} + \widetilde{S}^*\right)\right)$$
$$= \left(\widetilde{\mathfrak{T}}^n\ \widetilde{\mathfrak{T}}^*\left(\widetilde{\mathfrak{T}} + \widetilde{\mathfrak{T}}^*\right) + \widetilde{S}^n\widetilde{S}^*\left(\widetilde{S} + \widetilde{S}^*\right)\right)$$
$$= \left(\widetilde{\mathcal{N}}\left(\widetilde{\mathfrak{T}} + \widetilde{\mathfrak{T}}^*\right)\widetilde{\mathfrak{T}}^n\ \widetilde{\mathfrak{T}}^* + \widetilde{\mathcal{N}}\left(\widetilde{S} + \widetilde{S}^*\right)\widetilde{S}^n\ \widetilde{S}^*\right) \tag{3.1}$$

Now, we will take the other side

$$\widetilde{\mathcal{N}}\left(\left(\widetilde{\mathfrak{T}} + \widetilde{S}\right) + \left(\widetilde{\mathfrak{T}}^* + \widetilde{S}^*\right)\right)\left((\widetilde{\mathfrak{T}} + \widetilde{S})^n\left(\widetilde{\mathfrak{T}}^* + \widetilde{S}^*\right)\right)$$
$$= \widetilde{\mathcal{N}}\left(\left(\left(\widetilde{\mathfrak{T}} + \widetilde{S}\right) + \left(\widetilde{\mathfrak{T}}^* + \widetilde{S}^*\right)\right)\left(\left(\widetilde{\mathfrak{T}}^n + n\widetilde{\mathfrak{T}}^{n-1}\ \widetilde{S} + \ldots + \widetilde{S}^n\right)\left(\widetilde{\mathfrak{T}}^* + \widetilde{S}^*\right)\right)\right)$$
$$= \widetilde{\mathcal{N}}\left(\left(\left(\widetilde{\mathfrak{T}} + \widetilde{S}\right) + \left(\widetilde{\mathfrak{T}}^* + \widetilde{S}^*\right)\right)\left(\left(\widetilde{\mathfrak{T}}^n + \widetilde{S}^n\right)\left(\widetilde{\mathfrak{T}}^* + \widetilde{S}^*\right)\right)\right)$$
$$= \widetilde{\mathcal{N}}\left(\left(\widetilde{\mathfrak{T}} + \widetilde{\mathfrak{T}}^*\right) + \left(\widetilde{S} + \widetilde{S}^*\right)\right)\left(\widetilde{\mathfrak{T}}^n\ \widetilde{\mathfrak{T}}^* + \widetilde{S}^n\ \widetilde{\mathfrak{T}}^* + \widetilde{\mathfrak{T}}^n\ \widetilde{S}^* + \widetilde{S}^n\ \widetilde{S}^*\right)$$
$$= \widetilde{\mathcal{N}}\left(\left(\widetilde{\mathfrak{T}} + \widetilde{\mathfrak{T}}^*\right) + \left(\widetilde{S} + \widetilde{S}^*\right)\right)\left(\widetilde{\mathfrak{T}}^n\ \widetilde{\mathfrak{T}}^* + \widetilde{S}^n\ \widetilde{S}^*\right)$$
$$= \widetilde{\mathcal{N}}\left(\left(\widetilde{\mathfrak{T}} + \widetilde{\mathfrak{T}}^*\right)\ \widetilde{\mathfrak{T}}^n\ \widetilde{\mathfrak{T}}^* + \widetilde{\mathfrak{T}}\ \widetilde{S}^n\ \widetilde{S}^* + \widetilde{\mathfrak{T}}^*\ \widetilde{S}^n\ \widetilde{S}^* + \widetilde{S}\ \widetilde{\mathfrak{T}}^n\ \widetilde{\mathfrak{T}}^* + \widetilde{S}^*\ \widetilde{\mathfrak{T}}^n\ \widetilde{\mathfrak{T}}^* + \left(\widetilde{S} + \widetilde{S}^*\right)\widetilde{S}^n\ \widetilde{S}^*\right)$$
$$= \widetilde{\mathcal{N}}\left(\left(\widetilde{\mathfrak{T}} + \widetilde{\mathfrak{T}}^*\right)\ \widetilde{\mathfrak{T}}^n\ \widetilde{\mathfrak{T}}^* + \left(\widetilde{S} + \widetilde{S}^*\right)\ \widetilde{S}^n\ \widetilde{S}^*\right)$$
$$= \left(\widetilde{\mathcal{N}}\left(\widetilde{\mathfrak{T}} + \widetilde{\mathfrak{T}}^*\right)\ \widetilde{\mathfrak{T}}^n\ \widetilde{\mathfrak{T}}^* + \widetilde{\mathcal{N}}\left(\widetilde{S} + \widetilde{S}^*\right)\widetilde{S}^n\ \widetilde{S}^*\right) \tag{3.2}$$

From (3.1) *and* (3.2) *we get*

$$\left(\left(\widetilde{\mathfrak{T}} + \widetilde{S}\right)^n\left(\widetilde{\mathfrak{T}} + \widetilde{S}\right)^*\right)\left(\left(\widetilde{\mathfrak{T}} + \widetilde{S}\right) + \left(\widetilde{\mathfrak{T}} + \widetilde{S}\right)^*\right) = \widetilde{\mathcal{N}}\left(\left(\widetilde{\mathfrak{T}} + \widetilde{S}\right) + \left(\widetilde{\mathfrak{T}}^* + \widetilde{S}^*\right)\right)\left((\widetilde{\mathfrak{T}} + \widetilde{S})^n\left(\widetilde{\mathfrak{T}}^* + \widetilde{S}^*\right)\right)$$

Hence $\widetilde{\mathfrak{T}} + \widetilde{S}$ *is* $\mathcal{FS}-\left(\mathfrak{n} - \widetilde{\mathcal{N}}\right) -$ *quasi normal operator.* □

Proposition 3.9. *Let* $\widetilde{\mathfrak{T}}\ :\widetilde{\mathcal{H}} \rightarrow \widetilde{\mathcal{H}}$ *is* $\mathcal{FS}-$ *self adjoint operator, then* $\widetilde{\mathfrak{T}}$ *is* $\mathcal{FS}-\left(\mathfrak{n} - \widetilde{\mathcal{N}}\right) -$ *quasi normal operatorif* $\widetilde{\mathcal{N}} = \widetilde{I}$ *.*

Proof . *We must proof the following* $\left(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\right)\left(\widetilde{\mathfrak{T}} + \widetilde{\mathfrak{T}}^*\right) = \left(\widetilde{\mathfrak{T}} + \widetilde{\mathfrak{T}}^*\right)\left(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\right)$

We take the first party

$$\begin{aligned}\left(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\right)\left(\widetilde{\mathfrak{T}} + \widetilde{\mathfrak{T}}^*\right) &= \left(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\widetilde{\mathfrak{T}}\right) + \left(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\widetilde{\mathfrak{T}}^*\right)\\ &= \left(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}\widetilde{\mathfrak{T}}\right) + \left(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}\widetilde{\mathfrak{T}}\right)\\ &= \widetilde{\mathfrak{T}}^{n+2} + \widetilde{\mathfrak{T}}^{n+2}\\ &= 2\widetilde{\mathfrak{T}}^{n+2}\end{aligned} \tag{3.3}$$

Now, we will take the other side

$$\begin{aligned}\left(\widetilde{\mathfrak{T}} + \widetilde{\mathfrak{T}}^*\right)\left(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\right) &= \widetilde{\mathfrak{T}}\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^* + \widetilde{\mathfrak{T}}^*\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\\ &= \widetilde{\mathfrak{T}}\ \widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}} + \widetilde{\mathfrak{T}}\ \widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}\\ &= \widetilde{\mathfrak{T}}^{n+2} + \widetilde{\mathfrak{T}}^{n+2}\\ &= 2\widetilde{\mathfrak{T}}^{n+2}\end{aligned} \tag{3.4}$$

From (3.3) *and* (3.4) *we get*

$$\left(\widetilde{\mathfrak{T}}^n\widetilde{\mathcal{R}}^*\right)\left(\widetilde{\mathfrak{T}}+\widetilde{\mathfrak{T}}^*\right)=\left(\widetilde{\mathfrak{T}}+\widetilde{\mathfrak{T}}^*\right)\left(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\right)$$

Hence $\widetilde{\mathfrak{T}}$ *is* $\mathcal{FS}-\left(\mathfrak{n}-\widetilde{\mathcal{N}}\right)-$ *quasi normal operator.* □

Corollary 3.10. *Let* $\widetilde{\mathfrak{T}}$ *be any* $\mathcal{FS}-$*operator on* $\mathcal{FSH}-$*space* $\widetilde{\mathcal{H}}$. *Then*

1) $\widetilde{\mathfrak{T}}+\widetilde{\mathfrak{T}}^*$ *is* $\mathcal{FS}-\left(\mathfrak{n}-\widetilde{\mathcal{N}}\right)-$ *quasi normal operator.*
2) $\widetilde{\mathfrak{T}}\widetilde{\mathfrak{T}}^*$ *is* $\mathcal{FS}-\left(\mathfrak{n}-\widetilde{\mathcal{N}}\right)-$ *quasi normal operator.*
3) $\widetilde{\mathfrak{T}}^*\widetilde{\mathfrak{T}}$ *is* $\mathcal{FS}-\left(\mathfrak{n}-\widetilde{\mathcal{N}}\right)-$ *quasi normal operator.*
4) $\widetilde{I}+\widetilde{\mathfrak{T}}\widetilde{\mathfrak{T}}^*$ *and* $\widetilde{I}+\widetilde{\mathfrak{T}}^*\widetilde{\mathfrak{T}}$ *are* $\mathcal{FS}-\left(\mathfrak{n}-\widetilde{\mathcal{N}}\right)-$ *quasi normal operator.*

Proposition 3.11. *Let* $\widetilde{\mathfrak{T}}:\widetilde{\mathcal{H}}\to\widetilde{\mathcal{H}}$ *be FSN* $-$ *operator then* $\widetilde{\mathfrak{T}}$ *is* $\mathcal{FS}-\left(\mathfrak{n}-\widetilde{\mathcal{N}}\right)-$ *quasi normal operator if* $\widetilde{\mathcal{N}}=\widetilde{I}$.
Proof . *We must proof the following*

$$\left(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\right)\left(\widetilde{\mathfrak{T}}+\widetilde{\mathfrak{T}}^*\right)=\left(\widetilde{\mathfrak{T}}+\widetilde{\mathfrak{T}}^*\right)\left(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\right)$$

Now, we will take the first party to get the second party

$$\begin{aligned}\left(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\right)\left(\widetilde{\mathfrak{T}}+\widetilde{\mathfrak{T}}^*\right)&=\left(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\widetilde{\mathfrak{T}}\right)+\left(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\widetilde{\mathfrak{T}}^*\right)\\&=\left(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}\ \widetilde{\mathfrak{T}}^*\right)+\left(\widetilde{\mathfrak{T}}^*\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\right)\\&=\left(\widetilde{\mathfrak{T}}\ \widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\right)+\left(\widetilde{\mathfrak{T}}^*\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\right)\\&=\left(\widetilde{\mathfrak{T}}+\widetilde{\mathfrak{T}}^*\right)\left(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\right)\end{aligned}$$

Therefor

$$\left(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\right)\left(\widetilde{\mathfrak{T}}+\widetilde{\mathfrak{T}}^*\right)=\left(\widetilde{\mathfrak{T}}+\widetilde{\mathfrak{T}}^*\right)\left(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\right)$$

Hence $\widetilde{\mathfrak{T}}$ *is* $\mathcal{FS}-\left(\mathfrak{n}-\widetilde{\mathcal{N}}\right)-$ *quasi normal operator.* □

Definition 3.12. *Let* $\widetilde{\mathfrak{T}}\in\widetilde{B}(\widetilde{\mathcal{H}})$ *is called a fuzzy soft skew adjoint operator* ($\mathcal{FS}-$*skew adjoint operator*) *if* $\widetilde{\mathfrak{T}}=-\widetilde{\mathfrak{T}}^*$.

Proposition 3.13. *Let* $\widetilde{\mathfrak{T}}:\widetilde{\mathcal{H}}\to\widetilde{\mathcal{H}}$ *is skew operator then* $\widetilde{\mathfrak{T}}$ *is* $\mathcal{FS}-\left(\mathfrak{n}-\widetilde{\mathcal{N}}\right)-$*quasi normal operator.*
Proof . *We must proof the following*

$$\left(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\right)\left(\widetilde{\mathfrak{T}}+\widetilde{\mathfrak{T}}^*\right)=\widetilde{\mathcal{N}}\left(\widetilde{\mathfrak{T}}+\widetilde{\mathfrak{T}}^*\right)\left(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\right)$$

We take the first party

$$\begin{aligned}\left(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\right)\left(\widetilde{\mathfrak{T}}+\widetilde{\mathfrak{T}}^*\right)&=\left(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*\right)\left(\widetilde{\mathfrak{T}}+(-\widetilde{\mathfrak{T}})\right)\\&=(\widetilde{\mathfrak{T}}^n\widetilde{\mathfrak{T}}^*)(\widetilde{0})=\widetilde{0}\end{aligned}\tag{3.5}$$

Now, we will take the other side

$$\begin{aligned}\widetilde{\mathcal{N}}\left(\widetilde{\mathfrak{T}}+\widetilde{\mathfrak{T}}^{*}\right)\left(\widetilde{\mathfrak{T}}^{n}\widetilde{\mathfrak{T}}^{*}\right) &= \widetilde{\mathcal{N}}\left(\widetilde{\mathfrak{T}}+(-\widetilde{\mathfrak{T}})\right)\left(\widetilde{\mathfrak{T}}^{n}\widetilde{\mathfrak{T}}^{*}\right)\\ &= \widetilde{\mathcal{N}}\left(\widetilde{0}\right)\left(\ \widetilde{\mathfrak{T}}^{n}\widetilde{\mathfrak{T}}^{*}\ \right)=\widetilde{0} \qquad (3.6)\end{aligned}$$

From (3.5) *and* (3.6) *we get*

$$\left(\widetilde{\mathfrak{T}}^{n}\widetilde{\mathfrak{T}}^{*}\right)\left(\widetilde{\mathfrak{T}}+\widetilde{\mathfrak{T}}^{*}\right)=\widetilde{\mathcal{N}}\left(\widetilde{\mathfrak{T}}+\widetilde{\mathfrak{T}}^{*}\right)\left(\widetilde{\mathfrak{T}}^{n}\widetilde{\mathfrak{T}}^{*}\right)$$

Hence $\widetilde{\mathfrak{T}}$ *is* $\mathcal{FS}-\left(\mathfrak{n}-\widetilde{\mathcal{N}}\right)-$ *quasi normal operator.* □

4. Conclusions

The necessity of combining the two concepts fuzzy and soft sets was to provide more accurate and general results. The other concepts consequently followed this structure resulting like $\mathcal{FS}$-normed spaces, $\mathcal{FS}$-inner product space, $\mathcal{FS}$-Hilbert space and $\mathcal{FS}$-bounded linear operators. we invested a kind of fuzzy soft quasi normal operator namely fuzzy soft $(\mathfrak{n}-\widetilde{\mathcal{N}})$ -quasi normal operator this modification of fuzzy soft bounded linear quasi normal operator appear in recently many papers. Some properties and operation about this operator have been given, also more conditions given to get some theorems in this study. In addition to the generalizations of many definitions that were relied upon in the work.

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