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Bayesian inference of fractional brownian motion of multivariate stochastic differential equations

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Abstract

There have been much interest in analysis of stochastic differential equation with long memory, represented by fractional diffusion process, this property have been proved itself in financial mathematic as intrinsic character of financial time series, so finding an appropriate method for estimate and analyze stochastic differential equations with long memory is a very important contemporary topic, in this paper we suggest a method for a system of stochastic differential equations with long memory, also we use the Bayesian methodology to incorporate the advanced knowledge , in addition we apply renormalized integral known in literature as Wick-Itô-Skorohod to solve problem of arbitrage in stochastic models (which yield inefficient mathematical stochastic models for financial market), some of conventional methods like quasi maximum likelihood , Separable Integral-Matching for Ordinary Differential Equations, and multivariate Brownian method are used to be compared with the suggested method. The suggested method has been proved to be very accurate. The estimated model used to calculate the portfolio of assets quantities allocation.

Keywords: Fractional Brownian motion, stochastic differential equations, maximum likelihood, prior distribution, Metropolis Hasting method, Hurst index, Langevin method

1. Introduction

Differential equations are equations which relates variables with their rates of change, they have a wide spread in most of modern sciences, because they describe the physical behavior in simple local way, fractional calculus is important tool to solve variational problems. An apparent relation between the calculus of variations and fractional calculus, calculus of variations interpret the dynamic

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of phenomenon, fractional calculus accompanied with fat tails distribution, widely used in financial risk management.
Financial phenomena as certified by almost all previous studies that it suffers from autocorrelation with long range dependence or long memory [6], also it proved to suffer from heavy tail distribution which make normal distribution is inappropriate for them, in addition financial phenomena in many times be skewed to the left with negative extreme values. The existence of long range dependence make ARIMA models are not a good choice for data because the estimate of autocorrelation is not consistence. In addition building the portfolio for risk management strategies requires a strong model to represent the phenomena under study.
Modeling dynamics of asset prices plays an important role in a lot of microeconomics problems. For example, by understanding the behavior of stock prices, one can take best decision for a portfolio (one of investment strategy and hedging of capital). Stochastic models are based primarily on continuous or discrete time random walks. Continuous-time random walk process is a suitable class of process for modeling the behavior of high frequency data.

1.1. Long range dependence

A fractional Brownian motion FBM is an irregular diffusion process with covariance as shown below [13]:

$$E\left(W_t^H W_s^H\right) = \frac{1}{2}\left\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right\} \tag{1.1}$$

Where $0 < H < 1$ is Hurst index
and the variance

$$E\left(W_t^H W_t^H\right) = E\left(W_t^H\right)^2 = \frac{1}{2}\left\{|t|^{2H} + |t|^{2H} - |t-t|^{2H}\right\} = |t|^{2H} \tag{1.2}$$

The covariance between two different Wiener process is:

$$E\left(W_{1,t}^{H_1} W_{2,s}^{H_2}\right) = \frac{1}{2}\left\{|t|^{2H_{12}} + |s|^{2H_{12}} - |t-s|^{2H_{12}}\right\} \tag{1.3}$$

The difference of FBM is called Fractional Gaussian Noise FGN and have variance covariance as:

$$E\left(dW_{i,k_1}^{H_i} dW_{j,k_2}^{H_j}\right) = \frac{1}{2}\left[(k_1+1)^{2H_{ij}} - (k_1+k_2)^{2H_{ij}} + (k_2-1)^{2H_{ij}}\right] \tag{1.4}$$

So, the likelihood of FGN becomes [7] [18] :

$$p(\underline{x}, H) = \frac{1}{(2\pi)^{\frac{N}{2}} \left|\Omega_H\right|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}\underline{x}' \Omega_H^{-1} \underline{x}\right\} \tag{1.5}$$

Where, Ω can be calculated by [15]:

$$\Omega = E\left(dW_{i,k_1}^{H_i} dW_{j,k_2}^{H_j}\right) = \frac{\sigma^2}{2}\left[(k_1+1)^{2H_{ij}} - (k_1+k_2)^{2H_{ij}} + (k_2-1)^{2H_{ij}}\right] \tag{1.6}$$

Explicit form for the estimate of Hurst index is impossible to be obtained, as a function of the data. However, the maximum of its object function could be found by numerical methods.

1.2. Maximum Likelihood Estimator MLE

The maximum likelihood method depends on the Gaussian likelihood function assumption. This mean it considers only the mean and variance of estimators. The maximum likelihood can be expressed by the joint normal distribution, which is equivalent to the multivariate normal with mean vector $\underline{\mu}$ and variance-covariance matrix Ω .
Multivariate diffusion process consists of p variables takes the form :

$$\begin{pmatrix} dx_{1,t} \\ \vdots \\ dx_{p,t} \end{pmatrix} = \begin{pmatrix} \mu_1\left(x_{1,t}, t\right) \\ \vdots \\ \mu_p\left(x_{p,t}, t\right) \end{pmatrix} dt + \Sigma^{\frac{1}{2}}\left(x_{i,t}, x_{j,t}, t\right) \begin{pmatrix} dW_{1,t}^{H_1} \\ \vdots \\ dW_{p,t}^{H_p} \end{pmatrix}$$

$$\begin{pmatrix} dW_{1,t}^{H_1} \\ \vdots \\ dW_{p,t}^{H_p} \end{pmatrix} = \Sigma^{-\frac{1}{2}}\left(x_{i,t}, x_{j,t}, t\right) \left(\begin{pmatrix} dx_{1,t} \\ \vdots \\ dx_{p,t} \end{pmatrix} - \begin{pmatrix} \mu_1\left(x_{1,t}, t\right) \\ \vdots \\ \mu_p\left(x_{p,t}, t\right) \end{pmatrix} dt \right)$$

and the joint distribution is:

$$p(\underline{x}; \underline{\theta}, \underline{H}) =$$

$$\frac{1}{(2\pi)^{\frac{np}{2}} \left|\Sigma\left(\underline{x}_{i,t}, \underline{x}_{j,t}, t\right) \otimes \Omega_H\right|^{\frac{1}{2}}} \exp \begin{Bmatrix} -\frac{1}{2} vec\left(d\underline{x}_1 - \mu_1(\underline{x}_1, t), ..., d\underline{x}_p - \mu_p(\underline{x}_p, t)\right)' \\ \left(\Sigma\left(\underline{x}_{i,t}, \underline{x}_{j,t}, t\right) \otimes \Omega_H\right)^{-1} vec\left(d\underline{x}_1 - \mu_1(\underline{x}_1, t), ..., d\underline{x}_p - \mu_p(\underline{x}_p, t)\right) \end{Bmatrix}$$

Where vec denotes vector with all observations of p variables with sample size $n$stacked vertically, $\otimes$ is matrix Kronecker product, $\Omega_H$is variance covariance block matrix of FGN, $\Sigma\left(\underline{x}_{i,t}, \underline{x}_{j,t}, t\right)$ is the diffusion matrix, and $\underline{\theta}$ is a vector contains all the parameters of drift and diffusion functions.
After we determine the drift and diffusion function form we make prior distribution of each parameter, according to the domain of parameter and the information we have in advance, we could suggest the priors of each parameter as follow :

1.3. Prior of Hurst exponent

Since the domain of Hurst exponents are in the interval $(0,1)$ [6], we can use a Matrix variate beta distribution that has the form [17]

$$p\left(\underline{H}; \alpha, \beta\right) = \frac{\Gamma_p\left(\alpha+\beta\right)}{\Gamma_p\left(\alpha\right)\Gamma_p\left(\beta\right)} \det(H)^{\alpha-(p+1)/2} \det\left(I_p - H\right)^{\beta-(p+1)/2} \tag{1.7}$$

Where H is $p \times p$ matrix, p number of system parameters, $\alpha, \beta > (p-1)/2$ and

$$\Gamma_p\left(\alpha\right) = \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma\left(\alpha - (i-1)/2\right)$$

1.4. Prior of Real values parameters

The domain of Real values parameters are in the interval $(-\infty, \infty)$ so we can use a multivariate normal distribution that has the form

$$\frac{1}{(2\pi)^{\frac{p}{2}} |\eta|^{\frac{1}{2}}} e^{-\frac{1}{2}(\mu-\underline{\theta})' \eta^{-1}(\mu-\underline{\theta})} \tag{1.8}$$

1.5. Prior of positive values parameters

The domain of positive value parameters are in the interval $(0,\infty)$ so we can use an Inverse matrix gamma distribution that has the form [12]

$$\frac{|\Psi|^{-\nu}}{\varphi^{p\nu}\Gamma_p(\nu)}|X|^{-\nu-(p+1)/2}e^{tr\left(-\frac{1}{\varphi}\Psi X^{-1}\right)} \quad ,\nu>2,\varphi>0 \tag{1.9}$$

Where X and Ψ are $p\times p$ matrices.

1.6. The posterior distribution

The posterior is the normalized product of the likelihood with prior distributions of parameters, let $\underline{\lambda}=(\underline{\theta},\underline{H})$ then

$$\begin{aligned} f(x|\underline{\lambda}) =& \frac{1}{(2\pi)^{\frac{np}{2}}\left|\Sigma\left(\underline{x}_{i,t},\underline{x}_{j,t},t\right)\otimes\Omega_H\right|^{\frac{1}{2}}} \\ &\times\exp\left\{\begin{array}{l} -\frac{1}{2}vec\left(d\underline{x}_1-\mu_1(\underline{x}_1,t),...,d\underline{x}_p-\mu_p(\underline{x}_p,t)\right)' \\ \left(\Sigma\left(\underline{x}_{i,t},\underline{x}_{j,t},t\right)\otimes\Omega_H\right)^{-1}vec\left(d\underline{x}_1-\mu_1(\underline{x}_1,t),...,d\underline{x}_p-\mu_p(\underline{x}_p,t)\right)\end{array}\right\} \\ p(\underline{\lambda}|x) =& \frac{1}{(2\pi)^{\frac{np}{2}}\left|\Sigma\left(\underline{x}_{i,t},\underline{x}_{j,t},t\right)\otimes\Omega_H\right|^{\frac{1}{2}}} \\ &\times\exp\left\{\begin{array}{l} -\frac{1}{2}vec\left(d\underline{x}_1-\mu_1(\underline{x}_1,t),...,d\underline{x}_p-\mu_p(\underline{x}_p,t)\right)' \\ \left(\Sigma\left(\underline{x}_{i,t},\underline{x}_{j,t},t\right)\otimes\Omega_H\right)^{-1}vec\left(d\underline{x}_1-\mu_1(\underline{x}_1,t),...,d\underline{x}_p-\mu_p(\underline{x}_p,t)\right)\end{array}\right\}p_p(\underline{H},\underline{\mu}) \end{aligned} \tag{1.10}$$

$$\begin{aligned} p(\underline{\lambda}|x) =& \frac{1}{(2\pi)^{\frac{np}{2}}\left|\Sigma\left(\underline{x}_{i,t},\underline{x}_{j,t},t\right)\otimes\Omega_H\right|^{\frac{1}{2}}} \\ &\times\exp\left\{\begin{array}{l} -\frac{1}{2}vec\left(d\underline{x}_1-\mu_1(\underline{x}_1,t),...,d\underline{x}_p-\mu_p(\underline{x}_p,t)\right)' \\ \left(\Sigma\left(\underline{x}_{i,t},\underline{x}_{j,t},t\right)\otimes\Omega_H\right)^{-1}vec\left(d\underline{x}_1-\mu_1(\underline{x}_1,t),...,d\underline{x}_p-\mu_p(\underline{x}_p,t)\right)\end{array}\right\}p(\underline{H})p(\underline{\mu}) \\ p(\underline{\lambda}|x) =& \frac{1}{(2\pi)^{\frac{np}{2}}\left|\Sigma\left(\underline{x}_{i,t},\underline{x}_{j,t},t\right)\otimes\Omega_H\right|^{\frac{1}{2}}} \\ &\times\exp\left\{\begin{array}{l} -\frac{1}{2}vec\left(d\underline{x}_1-\mu_1(\underline{x}_1,t),...,d\underline{x}_p-\mu_p(\underline{x}_p,t)\right)' \\ \left(\Sigma\left(\underline{x}_{i,t},\underline{x}_{j,t},t\right)\otimes\Omega_H\right)^{-1}vec\left(d\underline{x}_1-\mu_1(\underline{x}_1,t),...,d\underline{x}_p-\mu_p(\underline{x}_p,t)\right)\end{array}\right\} \\ &\times\frac{\Gamma_p(\alpha+\beta)}{\Gamma_p(\alpha)\Gamma_p(\beta)}\det(H)^{\alpha-(p+1)/2}\det(I_p-H)^{\beta-(p+1)/2}\frac{1}{(2\pi)^{\frac{p}{2}}|\eta|^{\frac{1}{2}}}e^{-\frac{1}{2}\left(\underline{\mu}-\underline{\theta}\right)'\eta^{-1}\left(\underline{\mu}-\underline{\theta}\right)} \end{aligned} \tag{1.11}$$

taking the log of the posterior distribution we have

$$\begin{aligned} \log p(\underline{\lambda}|x) \propto& -\left|\Sigma\left(\underline{x}_{i,t},\underline{x}_{j,t},t\right)\otimes\Omega_H\right|^{\frac{1}{2}}-\frac{1}{2}vec\left(d\underline{x}_1-\mu_1(\underline{x}_1,t),...,d\underline{x}_p-\mu_p(\underline{x}_p,t)\right)'* \\ &\left(\Sigma\left(\underline{x}_{i,t},\underline{x}_{j,t},t\right)\otimes\Omega_H\right)^{-1}*vec\left(d\underline{x}_1-\mu_1(\underline{x}_1,t),...,d\underline{x}_p-\mu_p(\underline{x}_p,t)\right) \\ &+\alpha-(p+1)/2\left(\log\det(H)\right)+\beta-(p+1)/2\log\det\left(I_p-H\right) \\ &-\frac{1}{2}\log|\eta|-\frac{1}{2}\left(\underline{\mu}-\underline{\theta}\right)'\eta^{-1}\left(\underline{\mu}-\underline{\theta}\right) \end{aligned} \tag{1.12}$$

Where we leave out all the constant that does not change with changing the parameters. The log posterior have no explicit form and cannot be solved analytically so we use adaptive Metropolis Hasting algorithm.

1.7. Adaptive Metropolis Hasting algorithm

Adaptive Metropolis Hasting is the idea of update the proposal distribution by using the knowledge we have so far acquired about the target distribution. Suppose, therefore, that at time $t-1$ we have sampled the states $X_0, X_1, ..., X_{t-1}$, where X_0 is the initial state. Then a candidate point Y is sampled from the (asymptotically symmetric) proposal distribution $q_t\left(.\,|X_0, ..., X_{t-1}\right)$, which now may rely on the whole history $(X_0, X_1, ..., X_{t-1})$. The candidate point $Y$is accepted with probability [9]

$$\alpha\left(X_{t-1}, Y\right) = \min\left(1, \frac{\pi\left(Y\right)}{\pi\left(X_{t-1}\right)}\right)$$

Where $\pi\left(x\right)$ is (unscaled) density
in which case we set $X_t = Y$, and otherwise $X_t = X_{t-1}$
The proposal distribution $q_t\left(.\,|X_0, ..., X_{t-1}\right)$ used in the AM algorithm is a Gaussian distribution with mean at the current point X_{t-1} and covariance $C_t = C_t\left(X_0, ..., X_{t-1}\right)$.
The decisive thing regarding the adaptation is how the covariance of the proposal distribution depends on the history of the chain. In the AM algorithm this is solved by setting $C_t = s_d cov\left(X_0, ..., X_{t-1}\right) + s_d \varepsilon I_d$ after an initial period , where s_d is a parameter that depends only on dimension d and $\varepsilon > 0$ is a constant that we may choose very small , and

$$C_t = \begin{cases} C_0 & t \le t_0 \\ s_d cov\left(X_0, ..., X_{t-1}\right) + s_d \varepsilon I_d & t > t_0 \end{cases}$$

the empirical covariance matrix determined by points $x_0, ..., x_k \in \mathrm{R}^d$

$$cov\left(x_0, ..., x_k\right) = \frac{1}{k}\left(\sum_{i=0}^{k} x_i x_i' - (k+1)\,\bar{x}_k \bar{x}_k'\right)$$

where $\bar{x}_k = \dfrac{1}{k+1}\displaystyle\sum_{i=0}^{k} x_i$
the covariance C_t satisfies the recursion formula

$$C_{t+1} = \frac{t-1}{t} C_t + \frac{s_d}{t}\left(t\bar{X}_{t-1}\bar{X}_{t-1}' - (t+1)\,\bar{X}_t\bar{X}_t' + X_t X_t' + \varepsilon I_d\right)$$

This allows one to calculate C_t without too much computational cost since the mean $\bar{X}_t$ also satisfies an obvious recursion formula, the parameter ε is just to avoid that C_t will become singular, a basic choice for the scaling parameter we have elected the value $s_d = (2.4)^d \big/ d$.
adaptive Metropolis-within Gibbs algorithm using the proposal distribution $N\left(0, e^{2l}\right)$ with l the logarithm of the standard deviation of the increment. This parameter is chosen so that the acceptance rate is approximately 0.44 which is proposed to be optimal in the Metropolis-within Gibbs sampler. It is proposed to add/subtract an adoption amount $\delta\left(n\right) = \min\left(0.1, n^{-1/2}\right)$ to/from t after every 50th iteration and adapt the proposal variance if the acceptance rate is smaller than 0.3 or larger than 0.6.

2. Quasi-maximum Likelihood Estimation

Consider a multidimensional diffusion process

$$dX_t = a\left(X_t, \theta_2\right) dt + b\left(X_t, \theta_1\right) dW_t, \qquad X_0 = x_0 \tag{2.1}$$

where W_t represents an r-dimensional classical Wiener process independent of the initial variable x_0. In addition, $\theta_1 \in \Theta_1 \subset \mathrm{R}^p$, $\theta_2 \in \Theta_2 \subset \mathrm{R}^q$, $a : \mathrm{R}^d \times \Theta_2 \to \mathrm{R}^d$ and $b : \mathrm{R}^d \times \Theta_1 \to \mathrm{R}^d \otimes \mathrm{R}^r$. The naming of θ_2 and θ_1 is theoretically natural because of the optimal convergence rates of the estimators for these parameters as we will see in the following. Given sampled data $X_n = (X_{t_i})_{i=0,\ldots,n}$, with $t_i = i\Delta_n$, $\Delta_n \to 0$ as $n \to \infty$, quasi-maximum likelihood estimator (QMLE) makes use of the following approximation of the true log-likelihood for multidimensional diffusions [11].

$$\ell_n (X_n, \theta) = -\frac{1}{2} \left\{ \log \det (\Sigma_{i-1} (\theta_1)) + \frac{1}{\Delta_n} \Sigma_{i-1}^{-1} (\theta_1) \left[(\Delta x_i - \Delta_n a_{i-1} (\theta_2))^{\otimes 2} \right] \right\} \tag{2.2}$$

Where $\theta = (\theta_1, \theta_2)$, $\Delta X_i = X_{t_i} - X_{t_{i-1}}$, $\Sigma_i (\theta_1) = \Sigma (\theta_1, X_{t_i})$, $a_i (\theta_2) = a (X_{t_i}, \theta_2)$, $\Sigma = b^{\otimes 2}$, $A^{\otimes 2} = AA^T$ and A^{-1} the inverse of A, $A [B] = tr (AB)$. Then the QMLE of θ is an estimator that satisfies

$$\hat{\theta} = \arg \max_{\theta} \ell_n (X_n, \theta)$$

exactly or approximately

3. Wick Product

Wick product is a renormalization operator. The Wick renormalization method reduces the problem to exponential integrability problem. Wick product is not a pointwise operation [5]

$$\left(\int_{\mathrm{R}} f dW^{(H)} \right) \diamondsuit \left(\int_{\mathrm{R}} g dW^{(H)} \right) = \left(\int_{\mathrm{R}} f dW^{(H)} \right) \cdot \left(\int_{\mathrm{R}} g dW^{(H)} \right) - \langle f, g \rangle_H$$

$$\langle f, g \rangle_H = \int_{\mathrm{R}} \int_{\mathrm{R}} f(s) g(t) \phi (s, t) \, ds dt \qquad f, g \in S(R) \tag{3.1}$$

Let $S(\mathrm{R})$ be the Schwartz space of rapidly decreasing smooth functions on R, and if $f \in S(\mathrm{R})$, denote

$$\|f\|_H^2 = \int_{\mathrm{R}} \int_{\mathrm{R}} f(t) f(s) \phi(s, t) \, ds dt < \infty \tag{3.2}$$

$S(R)$ is Schwartz space

$$\phi (s, t) = \phi_H (s, t) = H (2H - 1) |s - t|^{2H-2} \quad , \quad s, t \in \mathrm{R}$$

$$\int_0^t \int_0^s \phi (u, v) \, du dv = \frac{1}{2} \left(t^{2H} + s^{2H} - |s - t|^{2H} \right) = R_H (t, s)$$

and

$$\exp^{\diamondsuit} (\langle w, f \rangle) = \exp \left(\langle w, f \rangle - \frac{1}{2} \|f\|_H^2 \right) \tag{3.3}$$

Fractional Wick Itô Skorohod integral

$$\int_{\mathrm{R}} Y(t) \, dW_t^{(H)} = \int_{\mathrm{R}} Y(t) \diamondsuit W^{(H)} (t) \, dt \tag{3.4}$$

Where $W^{(H)}(t)$ is fractional white noise the integral on an interval can be defined as

$$\int_0^T Y(t)\, dW_t^{(H)} = \int_{\mathrm{R}} Y(t)\, I_{[0,T]} dW_t^{(H)}$$

Suppose

$$Y(t) = \sum_{i=1}^{n} F_i(w)\, I_{[t_i,t_{i+1}]}(t) \quad \text{where } \; F_i(w) \in (S)_H^*$$

$$\int_{\mathrm{R}} Y(t)\, dW_t^{(H)} = \sum_{i=1}^{n} F_i(w) \diamondsuit \left(W_{t_{i+1}}^{(H)} - W_{t_i}^{(H)}\right)$$

$\int_0^T f(t)\, d\diamondsuit W_t^H$ is normally distributed with zero mean and variance $\|f\|_\phi^2$. Therefore $\exp\left(\int_0^T f(t)\, d\diamondsuit W_t^H\right)$ is log-normally distributed with mean $\exp\left(\frac{1}{2}\|f\|_\phi^2\right)$.

Example 3.1. *Using stochastic Wick calculus and obtain the following integral*

$$\begin{aligned}
\int_0^t W_s^{(H)} dW_s^{(H)} &= \int_0^t W_s^{(H)} \diamondsuit W^{(H)}(s)\, ds \\
&= \int_0^t W_s^{(H)} \diamondsuit \frac{d}{ds} W_s^{(H)} ds = \frac{1}{2}\left(W_t^{(H)}\right)^{\diamondsuit 2} \\
&= \frac{1}{2}\left(W_t^{(H)} \diamondsuit W_t^{(H)}\right) \\
&= \frac{1}{2}\left(\int_0^t \chi_{[0,t]}(u)\, dW_u^{(H)} \diamondsuit \int_0^t \chi_{[0,t]}(u)\, dW_u^{(H)}\right) \\
&= \frac{1}{2}\left(\int_0^t \chi_{[0,t]}(u)\, dW_u^{(H)} \int_0^t \chi_{[0,t]}(u)\, dW_u^{(H)} - \int_0^t \int_0^t \phi(t,s)\, ds dt\right) \\
&= \frac{1}{2}\left(W_s^{(H)}\right)^2 - \frac{1}{2} t^{2H}
\end{aligned}$$

where

$$\chi_{[0,t]}(u) = \begin{cases} 1 & if \;\; u \in (0,t) \\ 0 & otherwise \end{cases}$$

If $f, g \in L_\phi^2([0,T])$ then $\int_0^T f d\diamondsuit W_t^H$, $\int_0^T g d\diamondsuit W_t^H$ are well defined Gaussian random variables. Moreover:

1. $E\left(\int_0^T f d\diamondsuit W_t^H\right) = 0$
2. $E\left(\int_0^T f d\diamondsuit W_t^H \int_0^T g d\diamondsuit W_t^H\right) = \langle f, g\rangle_\phi$
3. $E\left(\left(\int_0^T f d\diamondsuit W_t^H\right)^2\right) = \|f\|_\phi^2$ (*Wick-Ito isometry*)

$$\int_0^T f d\diamondsuit W_t^H \sim N\left(0, \|f\|_\phi^2\right)$$

Consider the fractional stochastic differential equation

$$dX\left(t\right) = \mu X\left(t\right)dt + \sigma X\left(t\right)dW_t^{(H)} \quad , \quad X\left(t\right) = x > 0$$

where x, μ and σ are constants, We rewrite this as the following equation in $(S)_H^*$:

$$\frac{dX\left(t\right)}{dt} = \mu X\left(t\right) + \sigma X\left(t\right)\diamondsuit \mathrm{W}^{(H)}\left(t\right)$$

or

$$\frac{dX\left(t\right)}{dt} = \left(\mu + \sigma \mathrm{W}^{(H)}\left(t\right)\right)\diamondsuit X\left(t\right)$$

Using Wick calculus, the solution of this equation becomes

$$\begin{aligned}
&\int_0^t \frac{dX\left(s\right)}{ds}ds = \int_0^t \left(\mu + \sigma \mathrm{W}^{(H)}\left(s\right)\right)\diamondsuit X\left(s\right)\mathrm{ds} \\
&\int_0^t \frac{dX\left(s\right)}{ds}ds\diamondsuit \frac{1}{X\left(s\right)} = \int_0^t \left(\mu + \sigma \mathrm{W}^{(H)}\left(s\right)\right)\mathrm{ds} \\
&\log \diamondsuit X\left(t\right) = \log x + \mu t + \sigma \int_0^t \mathrm{W}^{(H)}\left(s\right)ds \\
&X\left(t\right) = \exp \diamondsuit \left(\log x + \mu t + \sigma \int_0^t \mathrm{W}^{(H)}\left(s\right)ds\right) \\
&X\left(t\right) = x\exp \diamondsuit \left(\mu t + \sigma W_t^{(H)}\right)
\end{aligned}$$

Setting $f = \sigma \chi_{[0,t]}$ hence $\langle w, f\rangle = \sigma dW_t^H$ and using Wick exponential [1]

$$\begin{aligned}
X\left(t\right) &= x\exp \diamondsuit \left(\mu t + \sigma W_t^{(H)}\right) \\
&= x\exp \diamondsuit \left(\mu t\right)\exp \diamondsuit \left(\sigma W_t^{(H)}\right) \\
&= x\exp\left(\mu t\right)\exp\left(\sigma B_t^{(H)} - \frac{1}{2}\sigma^2 \int_0^t \int_0^t \phi\left(s,t\right)dsdt\right) \\
&= x\exp\left(\mu t\right)\exp\left(\sigma B_t^{(H)} - \frac{1}{2}\sigma^2 t^{2H}\right)
\end{aligned}$$

Note that

$$E\left(X\left(t\right)\right) = xe^{\mu t}$$

A semimartingale $(X_t, t \geq 0)$ concerning a Brownian motion can often be expressed as

$$X_t = X_0 + \int_0^t f_s dW_s + \int_0^t g_s ds$$

If $f \in L_H^2\left(\mathrm{R}_+\right)$, then

$$\exp \diamondsuit \left(\int_0^\infty f_s dW_s^{(H)}\right) = \exp\left(\int_0^\infty f_s dW_s^{(H)} - \frac{1}{2}\left\|f\right\|_H^2\right) \tag{3.5}$$

Proof . It follows that

$$\exp\diamondsuit\left(\int_0^\infty f_s dW_s^{(H)}\right)=\sum_{i=1}^{\infty}\frac{1}{n!}\left(\int_0^\infty f_s dW_s^{(H)}\right)^{\diamondsuit n} \tag{3.6}$$

$$=\sum_{i=1}^{\infty}\frac{1}{n!}\left\|f\right\|_H^n h_n\left(\frac{\int_0^\infty f_s dW_s^{(H)}}{\left\|f\right\|_H}\right) \tag{3.7}$$

$$=\exp\left(\left\|f\right\|_H\frac{\int_0^\infty f_s dW_s^{(H)}}{\left\|f\right\|_H}-\frac{1}{2}\left\|f\right\|_H^2\right) \tag{3.8}$$

$$=\exp\left(\int_0^\infty f_s dW_s^{(H)}-\frac{1}{2}\left\|f\right\|_H^2\right) \tag{3.9}$$

□

4. The Multivariate FBM

Let $X\left(t\right)$ of dimension p be defined as

$$X\left(t\right)=\int K_H\left(u,t\right)A_+ dW\left(u\right) \tag{4.1}$$

A_+ is a $p\times p$ matrix of reals. H is a diagonal matrix of parameters $H_j\in\left(0,1\right)$, $\forall j=1,...,p$, and $K_H\left(u,t\right)$ is a matrix of kernels that reads $\left(t-u\right)_+^{H-1/2}-\left(-u\right)_+^{H-1/2}$. In this notation, $\left(a\right)_+=\max\left(a,0\right)$ and t^H is understood as the exponential of a matrix $\exp\left(H\log\left(t\right)\right)$. As seen in the stochastic integral [8].

$X\left(t\right)$ represents a multivariate non-stationary Gaussian process that has stationary increments. In addition, the elements of $X\left(t\right)$ are correlated, and the structure of the correlation is brought from the existence of the mixing matrix A_+. And the correlation structure is sufficient to fully define the process since it is Gaussian and zero mean (as a linear transform of a zero mean Gaussian process).

$$A_{jj}=\frac{\sigma_j^2\sin(\pi H_j)}{B_{jj}}$$
$$B_{kj}=B\left(H_j+0.5,H_k+0.5\right)$$

where $B\left(x,y\right)$ is the beta function , and

$$A_{jk}=\begin{cases}\dfrac{\sigma_j\sigma_k\rho_{jk}\sin\left(\pi\left(H_j+H_k\right)\right)}{\left(\cos\left(\pi H_j\right)+\cos\left(\pi H_k\right)\right)B_{jk}} & \text{if}\quad H_j+H_k\neq 1\\[2ex] \dfrac{2\sigma_j\sigma_k\rho_{jk}}{\left(\sin\left(\pi H_j\right)+\sin\left(\pi H_k\right)\right)B_{jk}} & \text{if}\quad H_j+H_k=1\end{cases}$$

A $p-$variate stochastic process $X=\left\{X\left(t\right)=\left(X_1\left(t\right),...,X_p\left(t\right)\right),t\in \mathrm{R}\right\}$ is said operator self-similar (os-s) if there exists a $p\times p$ matrix H (called the exponent of X) such that for any $\lambda>0$ [4]

$$X\left(\lambda t\right)\overset{fidi}{=}\lambda^H X\left(t\right)$$

where $\overset{fidi}{=}$ means finite-dimensional distributions equality, and the $p\times p$ matrix λ^H can represented by the power series $\lambda^H=e^{H\log\lambda}=\sum_{i=1}^{\infty}H^i\left(\log\lambda\right)^i\Big/i!$ joint self-similarity put many constraints on the structure of correlation of the process

Thus the covariance function of the i^{th} elements is the usual function

$$E\left(x_i\left(s\right)x_i\left(t\right)\right)=\frac{\sigma_i^2}{2}\left\{\left|t\right|^{2H_i}+\left|s\right|^{2H_i}-\left|t-s\right|^{2H_i}\right\} \tag{4.2}$$

the general form of the (cross-)covariance function of vector self-similar process X with finite variance and exponent $H=diag\left(H_1,H_2,...,H_p\right)$, $0<H_i<1$. under some regularity condition, for any $i,j=1,2,...,p \quad i\neq j$ with $H_i+H_j\neq 1$, there exists $c_{ij},c_{ji}\in \mathrm{R}$ such that for any $s,t\in \mathrm{R}$

$$cov\left(x_i\left(s\right)x_j\left(t\right)\right)=\frac{\sigma_i\sigma_j}{2}\left\{c_{ij}\left(s\right)\left|s\right|^{H_i+H_j}+c_{ji}\left(t\right)\left|t\right|^{H_i+H_j}-c_{ji}\left(t-s\right)\left|t-s\right|^{H_i+H_j}\right\}, \tag{4.3}$$
$$\sigma_i^2=var\left(x_i\left(1\right)\right)$$

and

$$c_{ij}\left(t\right)=\begin{cases} c_{ij}, & t>0\\ c_{ji}, & t<0\end{cases}$$

A similar expression (involving additional logarithmic terms) for the covariance $cov\left(x_i\left(s\right)x_i\left(t\right)\right)$ is obtained in the case $H_i+H_j=1$

The double sided stochastic integral representation [14]

$$X\left(t\right)=\int_{\mathrm{R}}\left\{\left(\left(t-x\right)_+^{H-0.5}-\left(-x\right)_+^{H-0.5}\right)A_+ +\left(\left(t-x\right)_-^{H-0.5}-\left(-x\right)_-^{H-0.5}\right)A_-\right\}W\left(dx\right), \tag{4.4}$$

where $H-0.5=diag\left(H_1-0.5,...,H_p-0.5\right)$,$x_+=\max\left(x,0\right)$, $x_-=\max\left(-x,0\right)$, A_+,A_-, are real $p\times p$ matrices and $W\left(dx\right)=\left(W_1\left(dx\right),...,W_p\left(dx\right)\right)$ is a Gaussian white noise with zero mean, independent components and covariance $E\left(W_i\left(dx\right)W_j\left(dx\right)\right)=\delta_{ij}dx$, if $0<H_i<1$, $H_i+H_j\neq 1$, $i,j=1,2,...,p$ then the cross-covariance of X.

$$c_{ij}=2\tilde{c}_{ij}\phi_{ij}/\sigma_i\sigma_j$$
$$\phi_{ij}=B\left(H_i+0.5,H_j+0.5\right)/\sin\left(\left(H_i+H_j\right)\pi\right)$$

where the matrix $\tilde{C}=\left(\tilde{c}_{ij}\right)$ is given by

$$\begin{aligned}\tilde{C}=&\cos\left(H\pi\right)A_+A_+^*+A_-A_-^*\cos\left(H\pi\right)\\ &-\sin\left(H\pi\right)A_+A_-^*\cos\left(H\pi\right)-\cos\left(H\pi\right)A_+A_-^*\sin\left(H\pi\right)\end{aligned}$$

Here and below, A^* denotes the transposed matrix,

$$\sin\left(H\pi\right)=diag\left(\sin\left(H_1\pi\right),...,\sin\left(H_p\pi\right)\right),\cos\left(H\pi\right)=diag\left(\cos\left(H_1\pi\right),...,\cos\left(H_p\pi\right)\right)$$

If $i\neq j$ and $H_i+H_j=1$, then there exists $d_{ij},f_{ij}\in \mathrm{R}$ such that for any $s,t\in \mathrm{R}^2$, we have

$$E\left(X_i\left(s\right)X_j\left(t\right)\right)=\frac{\sigma_i\sigma_j}{2}\times\ \left\{d_{ij}\left(\left|s\right|+\left|t\right|-\left|t-s\right|\right)+f_{ij}\left(t\log\left|t\right|+s\log\left|s\right|-\left(t-s\right)\log\left|t-s\right|\right)\right\} \tag{4.5}$$

The matrix $R=\left(R_{ij}\right)_{i,j=1,...,p}$ is positive definite, where

$$R_{ij}=\begin{cases}1, & i=j & \\ c_{ij}+c_{ji} & i\neq j & H_i+H_j\neq 1\\ d_{ij} & i=j & H_i+H_j=1\end{cases}$$

The cross-covariances are given by [3].

$$r_{ij}\left(s,t\right) = E\left(x_i\left(s\right)x_j\left(t\right)\right) = \frac{\sigma_i\sigma_j}{2}\left\{w_{ij}\left(-s\right) + 2w_{ij}\left(t\right) - w_{ij}\left(t-s\right)\right\} \tag{4.6}$$

where the function $w_{ij}\left(h\right)$ is defined by

$$w_{ij}\left(h\right) = \begin{cases} \left(\rho_{ij} - \eta_{ij} sign\left(h\right)\right)\left|h\right|^{H_i+H_j} & \text{if} \quad H_i + H_j \neq 1 \\ \rho_{ij}\left|h\right| - \eta_{ij} h \log\left|h\right| & \text{if} \quad H_i + H_j = 1 \end{cases}$$

Parameters η_{ij} are connected with the time-reversibility of the process. They are characterized by the antisymmetry property. In special $\eta_{ij} = -\eta_{ji}$, if the process is time-reversible, they are all equal to zero, if the process permits a causal (or an anticausal) representation, they are function of ρ_{ij}, H_i, and H_j. In general otherwise, they are unconstrained
p-multivariate process fulfilling the three following conditions
Gaussianity,
Self-similarity with parameter $H \in (0,1)^p$,
Stationarity of the increments.

The mfBm has stationary increments. It is easy to derive the covariance structure of the increments process. Let $\Delta x\left(t\right) = x\left(t+1\right) - x\left(t\right)$ be stationary process (with increments of size 1) that refered to as the multivariate fractional Gaussian noise (mfGn). Then

$$\begin{aligned} \gamma_{ij}\left(h\right) &= E\left(\Delta x_i\left(t\right)\Delta x_j\left(t+h\right)\right) \\ &= \frac{\sigma_i\sigma_j}{2}\left(w_{ij}\left(h-1\right) - 2w_{ij}\left(h\right) + w_{ij}\left(h+1\right)\right) \end{aligned} \tag{4.7}$$

The asymptotic behavior as $|h| \to +\infty$

$$\gamma_{ij}\left(h\right) \sim \sigma_i\sigma_j\left|h\right|^{H_i+H_j-2}\kappa_{ij}\left(sign\left(h\right)\right)$$

Where

$$\kappa_{ij}\left(sign\left(h\right)\right) = \begin{cases} \left(\rho_{ij} - \eta_{ij} sign\left(h\right)\right)\left(H_i+H_j\right)\left(H_i+H_j-1\right) & \text{if} \;\; H_i + H_j \neq 1 \\ \eta_{ij} sign\left(h\right) & \text{if} \;\; H_i + H_j = 1 \end{cases}$$

$$\eta_{ij} = \frac{corr\left(X_i\left(1\right), X_j\left(-1\right)\right) - corr\left(X_i\left(-1\right), X_j\left(1\right)\right)}{2 - 2^{H_i+H_j}}$$

η_{ij} quantifies the dissymmetry between X_i and X_j.
The multivariate fractional Brownian motion is characterized by the Hurst indices of its components, by its covariance matrix at time 1, and also by an antisymmetric matrix η_{ij} which controls the time asymmetry of the multivariate process [8] .
The elements of the increments process could be long-range dependent individually, and could also present what is called long-range interdependence, this mean that their cross-correlation function may be not summable.
Modeling dynamics of asset prices plays important role in a lot of microeconomics problems. For example, by understanding the behavior of stock prices, one can take good decision for a portfolio. Continuous-time random walk process is a suitable class of process for modeling the behavior of high frequency data.

5. Markowitz portfolios

Markowitz (1952) proposed the portfolio selection approach. Markowitz path-breaking insight was that the risk/return profiles of single assets should not be thought separately but in their portfolio context. In this view, portfolios are considered to be efficient if they are either risk minimal for a given return level or have the maximum return for a given level of risk. Even though both thoughts of efficient portfolios are equivalent, the kind of portfolio optimization does differ for these two status. The former is a quadratic optimization with linear constraints, whereas in the latter the objective function is linear and the constraints are quadratic. In the following it is assumed that there are N assets and that these are infinitely divisible. The returns of these assets are jointly normally distributed. The portfolio return $\bar{r}$ is defined by the scalar product of the $(N*1)$ weight and return vectors $w$and μ. The portfolio risk is measured by the portfolio variance $\sigma_w^2 = w'\Sigma w$, where refers to the positive semi-definite variance–covariance matrix of the assets' returns. For the status of minimal variance portfolios for a given portfolio return, $\bar{r}$, the optimization problem can be stated as [16] :

$$\begin{aligned} &P = \arg\min_{w} \sigma_w^2 = w'\Sigma w \\ &w'\mu = \bar{r} \\ &w'\mathbf{1} = 1 \end{aligned} \tag{5.1}$$

where $\mathbf{1}$ is the $(N*1)$ vector of ones. In relation to this function, the weight vector for a minimal variance portfolio and a given target return is given by [19]

$$w^* = \bar{r}w_0^* + w_1^* \tag{5.2}$$

with

$$\begin{aligned} w_0^* &= \frac{1}{d}\left(c\Sigma^{-1}\mu + b\Sigma^{-1}\mathbf{1}\right) \\ w_1^* &= \frac{1}{d}\left(b\Sigma^{-1}\mu + a\Sigma^{-1}\mathbf{1}\right) \end{aligned}$$

The portfolio standard deviation is given by

$$\sigma = \sqrt{\frac{1}{d}\left(c\bar{r}^2 - 2b\bar{r} + a\right)}$$

with $a = \mu'\Sigma^{-1}\mu$, $b = \mu'\Sigma^{-1}\mathbf{1}$, $c = \mathbf{1}'\Sigma^{-1}\mathbf{1}$ and $d = ac - b^2$. Equation (5.2) results from a Lagrange optimization with the constraints for a given target return and weights summing to one. It can Volatility is a measure of risk.

A hyperbola for efficient mean–variance portfolios. The hyperbola is enclosed by the asymptotes $\bar{r} = b/c \pm \sqrt{d/c}\sigma$. The locus of the GMV portfolio is the apex of the hyperbola with weights given by $w^*_{GMV} = \Sigma^{-1}\mathbf{1}/\mathbf{1}'\Sigma^{-1}\mathbf{1}$.

6. Application

The Adaptive Metropolis Hasting algorithm method discussed earlier is used to estimate the posterior of a system of three fractional stochastic differential equations, with data represents daily three banking sector stock prices from 1 January 2017 to 11 March 2019 with sample size 509 for each variable, as shown in Figure 1. The first step is to find the form of drift and diffusion functions using the Langevin method [20], by calculating the conditional moments and determining the drift and

diffusion forms. The search takes a long time because of the high dimensionality of model parameters. The first and second conditional moments show a damped square variable amplitude sine function and a damped second order polynomial respectively as depicted in Figures 3-7. The method is programmed in R program. The suggested estimation method will be compared with the existing conventional methods like quasi-maximum likelihood in Yuima R package [11], and stochastic integral matching method in Simode R packages [21] . Although these methods are for standard Brownian motion we can use them as a benchmark for our suggested method.
The obtained result and estimation are used to build a portfolio for the three stock which is a very important quantity that required by investors to overcome their stock circulation risk and achieve a profit in return, and this will encourage the neutral risk investors to participate in speculation action and in return it will refresh the macroeconomic of country.

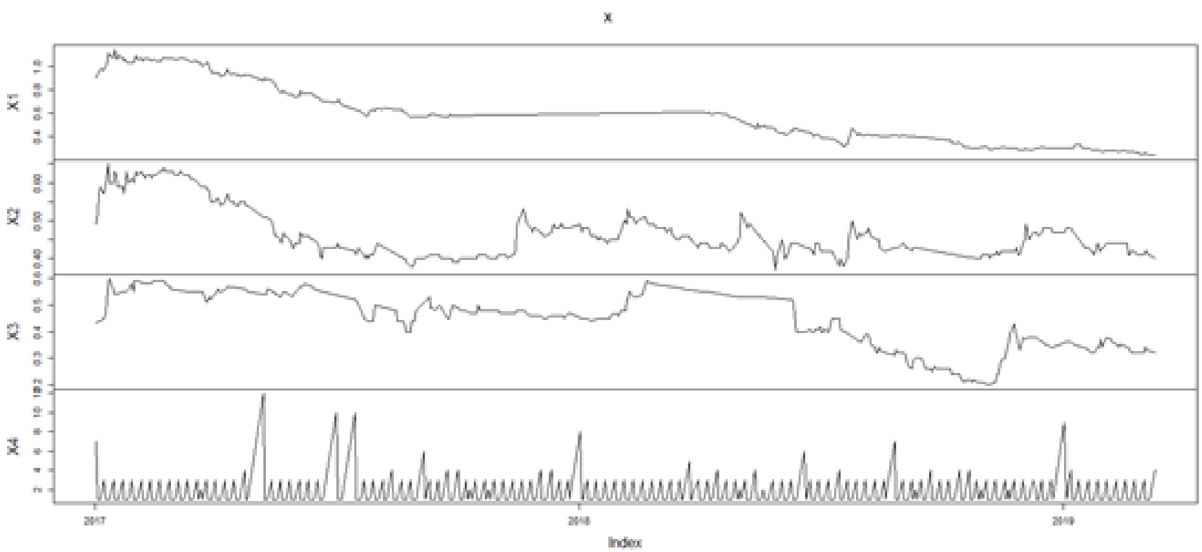


Figure 1: Plot of the stock prices time series and time between observations (fourth panel).

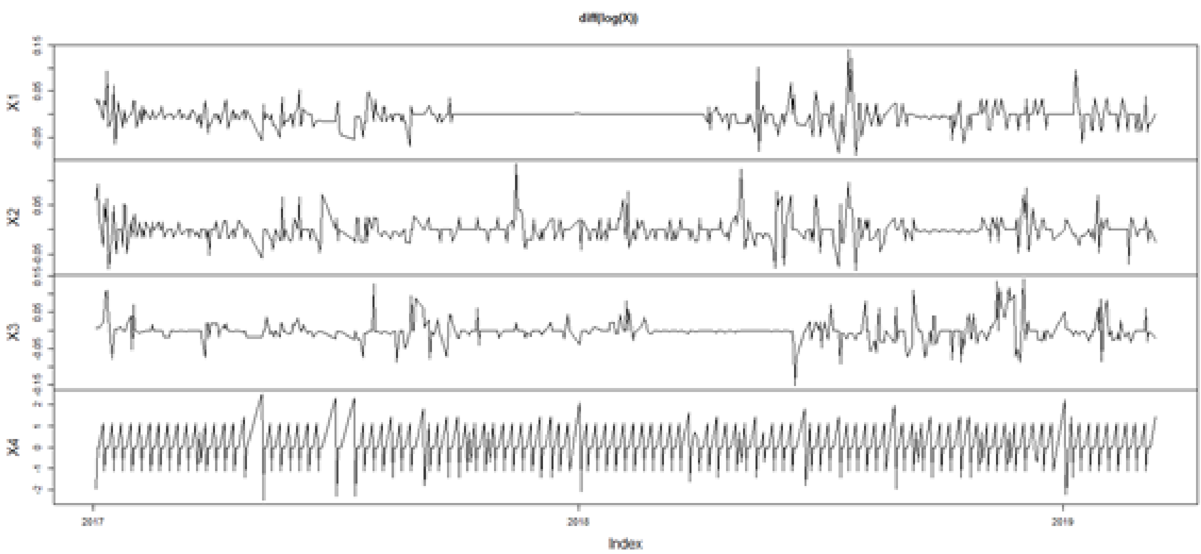


Figure 2: Plot of the returns of three series with time difference (fourth panel).

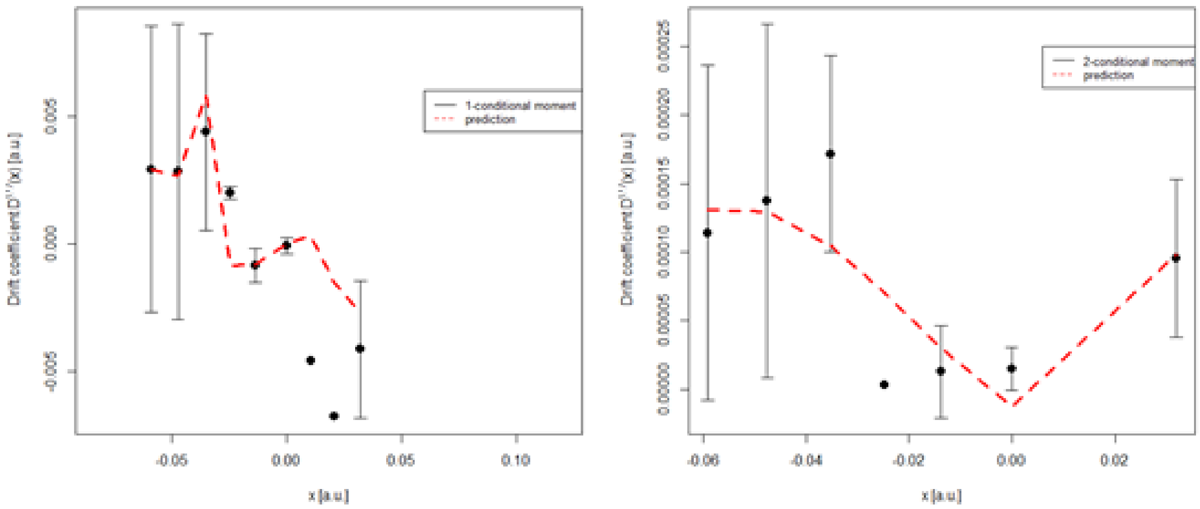


Figure 3: Drift and diffusion by Langevin method for series 1, conditional moments (black), error bar (bars), drift fitting (red)

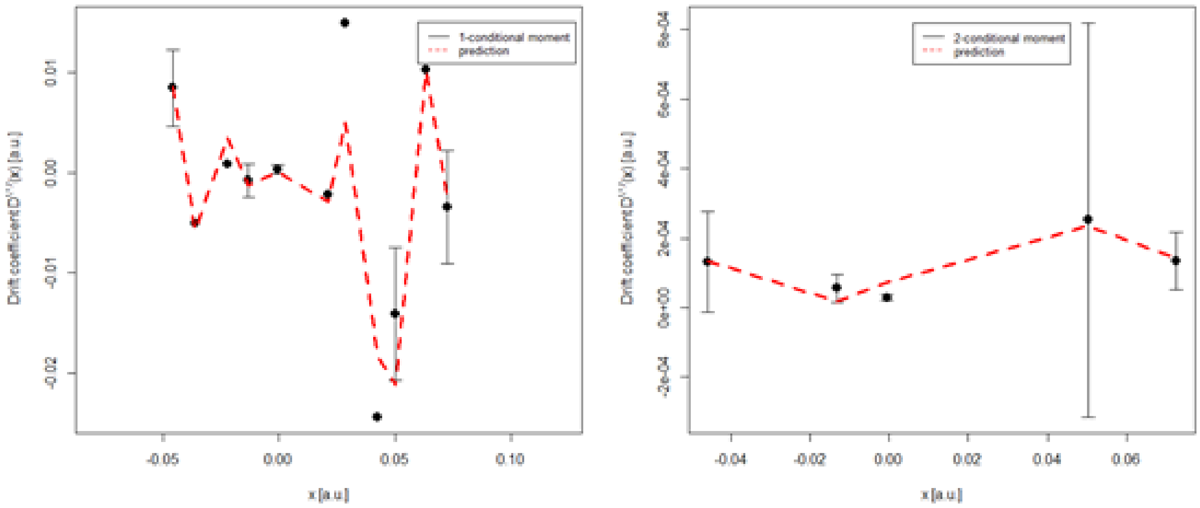


Figure 4: Drift and diffusion by Langevin method for series 2, conditional moments (black) ,error bar (bars), and drift fitting (red)

To decrease the search time, we use the nonlinear least squares in R program to determine the initial values for numerical search optimization method. We see from Figures 3-7, that the drifts and diffusions are not linear and they follow second order equation with a damping factor.

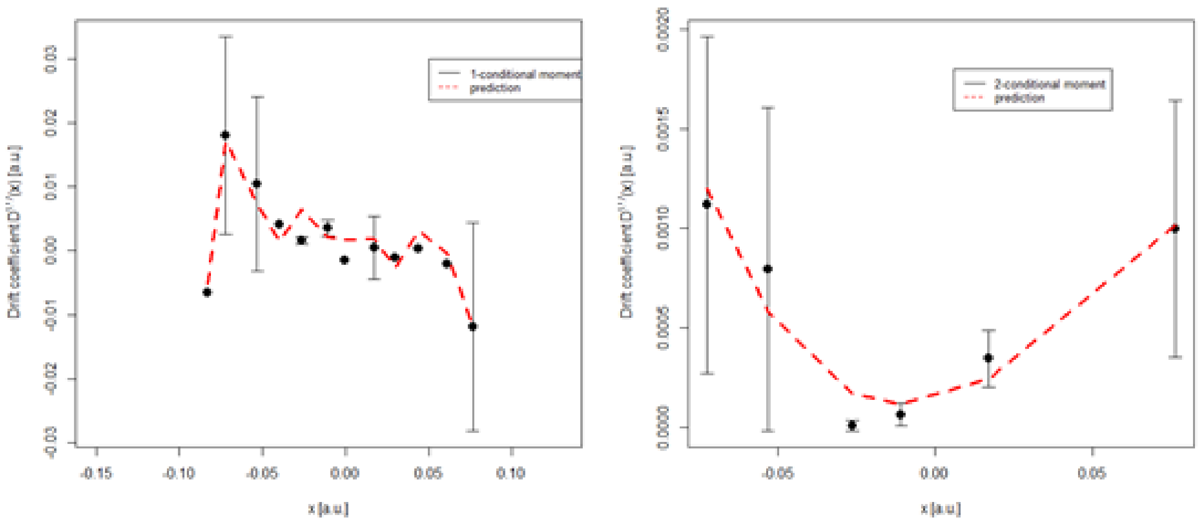


Figure 5: Drift and diffusion by Langevin method for series 3, conditional moments (black) ,error bar (bars), and drift fitting (red).

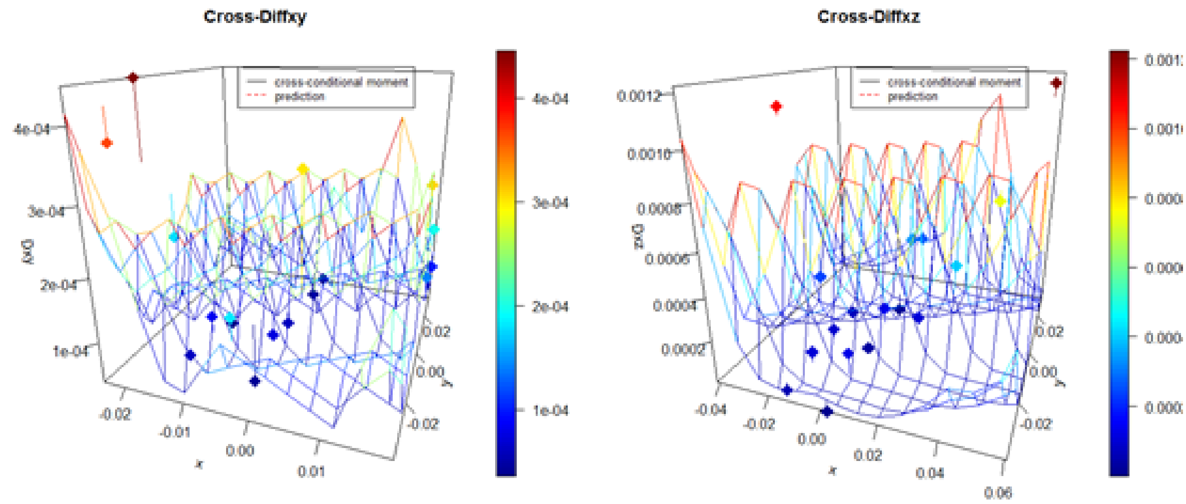


Figure 6: Cross-diffusion by Langevin method for series 1,2 and series 1,3, cross conditional moments (colored points), and diffusion fitting (colored surface).

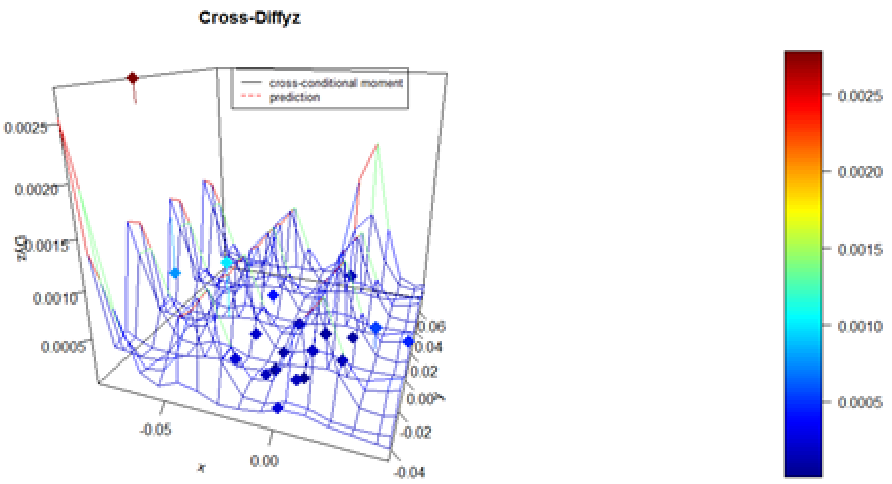


Figure 7: Cross-diffusion by Langevin method for series 2,3, cross conditional moments (colored points), and diffusion fitting (colored surface)

6.1. The models of FSDE

From the fitted conditional moments in Figures 3-7, we find the best functional forms of drift and diffusion for return rate $r_t = \log\left(x_t/x_{t-1}\right)$ [22], as follow:
Drift function:

$$\begin{aligned} dr_i =& f_i\left(\underline{r},\theta\right)dt + g_{ii}\left(\underline{r},\theta\right)dW_t^{H_i} \\ =& \left(\theta_{i0}r_i^2\sin\left(\theta_{i1}r_i+\theta_{i2}\right)+\theta_{i3}\exp\left(\theta_{i4}r_i\right)\right)dt + \left(\alpha_{110}\left(r_i+\alpha_{111}\right)^2+\alpha_{112}e^{\alpha_{113}r_i}\right)dW_t^{H_i} \qquad i=1,2,3 \end{aligned}$$

$$(r_{t+1}-r_t)=\left(\theta_{i0}r_{ti}^2\sin\left(\theta_{i1}r_{ti}+\theta_{i2}\right)+\theta_{i3}\exp\left(\theta_{i4}r_{ti}\right)\right)dt + \left(\alpha_{110}\left(r_{ti}+\alpha_{111}\right)^2+\alpha_{112}\exp(\alpha_{113}r_{ti})\right)dW_t^{H_i}$$

$$\begin{aligned} \left(\log\left(\frac{x_{i,t+1}}{x_{i,t}}\right)-\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)\right) =& \theta_{i0}\left(\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)\right)^2\sin\left(\theta_{i1}\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)+\theta_{i2}\right) \\ &+\theta_{i3}\exp\left(\theta_{i4}\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)\right)\Delta t + \left(\alpha_{110}\left(\log\left(\frac{x_{1,t}}{x_{1,t-1}}\right)+\alpha_{111}\right)^2+\alpha_{112}\left(\frac{x_{1,t}}{x_{1,t-1}}\right)^{\alpha_{113}}\right)dW_t^{H_i} \end{aligned}$$

$$\begin{aligned} \left(\log\left(\frac{x_{i,t+1}}{x_{i,t}}\right)-\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)\right) =& \theta_{i0}\left(\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)\right)^2\sin\left(\theta_{i1}\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)+\theta_{i2}\right) \\ &+\theta_{i3}\exp\left(\theta_{i4}\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)\right)\Delta t + \left(\alpha_{110}\left(\log\left(\frac{x_{1,t}}{x_{1,t-1}}\right)+\alpha_{111}\right)^2+\alpha_{112}\left(\frac{x_{1,t}}{x_{1,t-1}}\right)^{\alpha_{113}}\right)dW_t^{H_i} \end{aligned}$$

$$\begin{aligned} \log\left(\frac{x_{i,t+1}}{x_{i,t}}\Big/\frac{x_{i,t}}{x_{i,t-1}}\right) =& \theta_{i0}\left(\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)\right)^2\sin\left(\theta_{i1}\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)+\theta_{i2}\right) \\ &+\theta_{i3}\exp\left(\theta_{i4}\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)\right)\Delta t + \left(\alpha_{110}\left(\log\left(\frac{x_{1,t}}{x_{1,t-1}}\right)+\alpha_{111}\right)^2+\alpha_{112}\left(\frac{x_{1,t}}{x_{1,t-1}}\right)^{\alpha_{113}}\right)dW_t^{H_i} \end{aligned}$$

$$\log\left(\frac{x_{i,t+1}x_{i,t-1}}{x_{i,t}^2}\right)=\theta_{i0}\left(\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)\right)^2\sin\left(\theta_{i1}\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)+\theta_{i2}\right)+\theta_{i3}\exp\left(\theta_{i4}\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)\right)\Delta t$$
$$+\left(\alpha_{110}\left(\log\left(\frac{x_{1,t}}{x_{1,t-1}}\right)+\alpha_{111}\right)^2+\alpha_{112}\left(\frac{x_{1,t}}{x_{1,t-1}}\right)^{\alpha_{113}}\right)dW_t^{H_i}$$

where

$$E\left(e^{dW_t^{H_i}}\right)=e^{\frac{1}{2}\left(\Sigma\left(\underline{x}_{i,t},\underline{x}_{j,t},t\right)\otimes\Omega_H\right)}$$

$$E\left(x_{i,t+1}\right)=\frac{x_{i,t}^2}{x_{i,t-1}}e^{\theta_{i0}\left(\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)\right)^2\sin\left(\theta_{i1}\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)+\theta_{i2}\right)+\theta_{i3}\left(\frac{x_{i,t}}{x_{i,t-1}}\right)^{\theta_{i4}}\Delta t+\frac{1}{2}\left(\Sigma\left(\underline{x}_{i,t},\underline{x}_{j,t},t\right)\otimes\Omega_H\right)_{ii,t}}$$

We see that there is a spurious drift has been added to the original drift, this drift comes from fractional Brownian motion, so we use Wick-Itô-Skorohod calculus as discussed in section 3, To eliminate its effect from model drift.

$$\therefore\quad x_{i,t+1}=\frac{x_{i,t}^2}{x_{i,t-1}}e^{\left(\theta_{i0}r_{i,t}^2\sin(\theta_{i1}r_{i,t}+\theta_{i2})+\theta_{i3}\exp(\theta_{i4}r_{i,t})\right)\Delta t-\frac{1}{2}\left(\left(\Sigma\left(\underline{x}_{i,t},\underline{x}_{j,t},t\right)\otimes\Omega_H\right)_{ii,t}\right)+\Sigma^{\frac{1}{2}}\left(\underline{x}_{i,t},\underline{x}_{i,t},t\right)dW_t^{H_i}}$$

Where $\lim_{\Delta t\to 0}\Delta t=dt$

Diffusion function:

$$\operatorname{var}\left(x_{i,t+1}\right)=E(x_{i,t+1}^2)-\left(E(x_{i,t+1})\right)^2$$

$$E(x_{i,t+1}^2)=E\left(\left(\frac{x_{i,t}^2}{x_{i,t-1}}e^{\left(\theta_{i0}r_i^2\sin(\theta_{i1}r_i+\theta_{i2})+\theta_{i3}\exp(\theta_{i4}r_i)\right)\Delta t-\frac{1}{2}\left(\left(\Sigma\left(\underline{x}_{i,t},\underline{x}_{j,t},t\right)\otimes\Omega_H\right)_{ii}\right)+\Sigma^{\frac{1}{2}}\left(\underline{x}_{i,t},\underline{x}_{j,t},t\right)dW_t^{H_i}}\right)^2\right)$$
$$=\frac{x_{i,t}^4}{x_{i,t-1}^2}e^{2\left(\theta_{i0}r_i^2\sin(\theta_{i1}r_i+\theta_{i2})+\theta_{i3}\exp(\theta_{i4}r_i)\right)\Delta t-\left(\left(\Sigma\left(\underline{x}_{i,t},\underline{x}_{j,t},t\right)\otimes\Omega_H\right)_{ii}\right)+2\left(\left(\Sigma\left(\underline{x}_{i,t},\underline{x}_{j,t},t\right)\otimes\Omega_H\right)_{ii}\right)}$$
$$=\frac{x_{i,t}^4}{x_{i,t-1}^2}e^{2\left(\theta_{i0}r_i^2\sin(\theta_{i1}r_i+\theta_{i2})+\theta_{i3}\exp(\theta_{i4}r_i)\right)\Delta t+\left(\left(\Sigma\left(\underline{x}_{i,t},\underline{x}_{j,t},t\right)\otimes\Omega_H\right)_{ii}\right)}$$

$$\left(E(x_{i,t+1})\right)^2=\left(E\left(\frac{x_{i,t}^2}{x_{i,t-1}}e^{\left(\theta_{i0}r_{i,t}^2\sin(\theta_{i1}r_i+\theta_{i2})+\theta_{i3}\exp(\theta_{i4}r_{i,t})\right)\Delta t-\frac{1}{2}\left(\left(\Sigma\left(\underline{x}_{i,t},\underline{x}_{j,t},t\right)\otimes\Omega_H\right)_{ii,t}\right)+\Sigma^{\frac{1}{2}}\left(\underline{x}_{i,t},\underline{x}_{i,t},t\right)dW_t^{H_i}}\right)\right)^2$$
$$=\frac{x_{i,t}^4}{x_{i,t-1}^2}e^{2\left(\theta_{i0}r_{i,t}^2\sin(\theta_{i1}r_i+\theta_{i2})+\theta_{i3}\exp(\theta_{i4}r_{i,t})\right)\Delta t-\left(\left(\Sigma\left(\underline{x}_{i,t},\underline{x}_{j,t},t\right)\otimes\Omega_H\right)_{ii,t}\right)+\left(\left(\Sigma\left(\underline{x}_{i,t},\underline{x}_{i,t},t\right)\otimes\Omega_H\right)_{ii,t}\right)}$$
$$=\frac{x_{i,t}^4}{x_{i,t-1}^2}e^{2\left(\theta_{i0}r_{i,t}^2\sin(\theta_{i1}r_{i,t}+\theta_{i2})+\theta_{i3}\exp(\theta_{i4}r_{i,t})\right)\Delta t}$$

$\therefore$ V

7. Numerical calculation

We use the discrete form of the model to fit the data, and it could be used for forecasting or for portfolio building (investment strategy). Additionally, we can solve the model analytically and then estimate the parameter. However, this is not an easy task as it is frequently in non-closed form. The next step is to estimate the parameters of the model that have maximum log likelihood. We analyzed three stock price time series. The data is of size 509, taken from 1 January 2017 to 11 March 2019. Since the data is positive prices, we take logarithm of difference, $r(t)=\log\left(p\left(x_{t+1}\right)/p\left(x_t\right)\right)$ to transform it to return rate, which is very important quantity in financial investment, and to make series approximately normal distribution. The time difference is $\Delta t=1$, which reflects the difference in working days in the year. We fit the return with appropriate models for drift and diffusion to extract a primary models for fractional stochastic differential equation by plotting the scatter of the first difference of return with a lagged return and estimating the parameter of the model using the nonlinear least square. This will give us a glimpse about model function as shown below [22] :

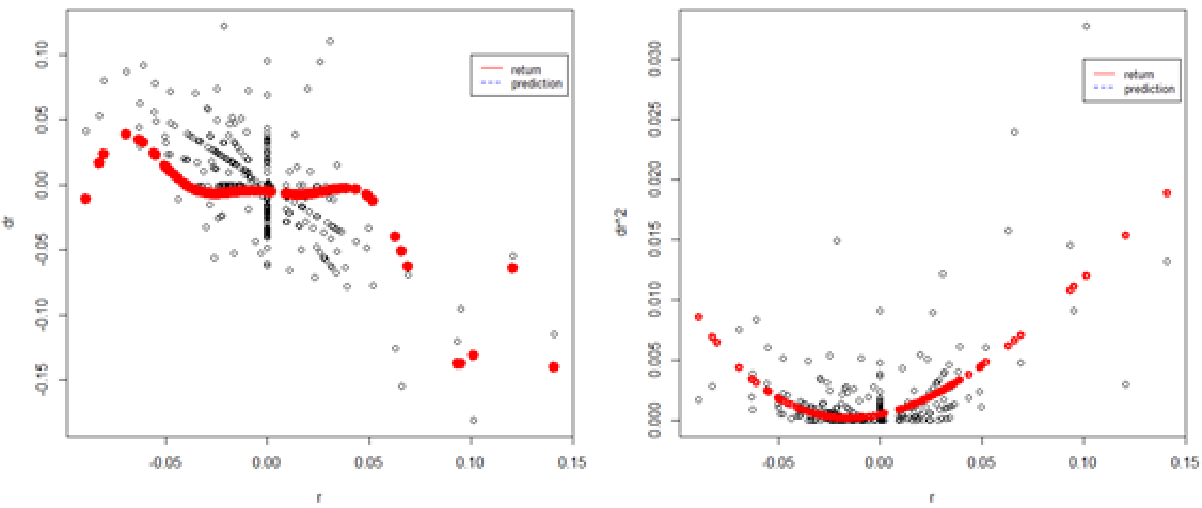


Figure 8: Return scatter plot with fitted drift and diffusion models by nonlinear least squares for series 1, return (black), and fitted (red)

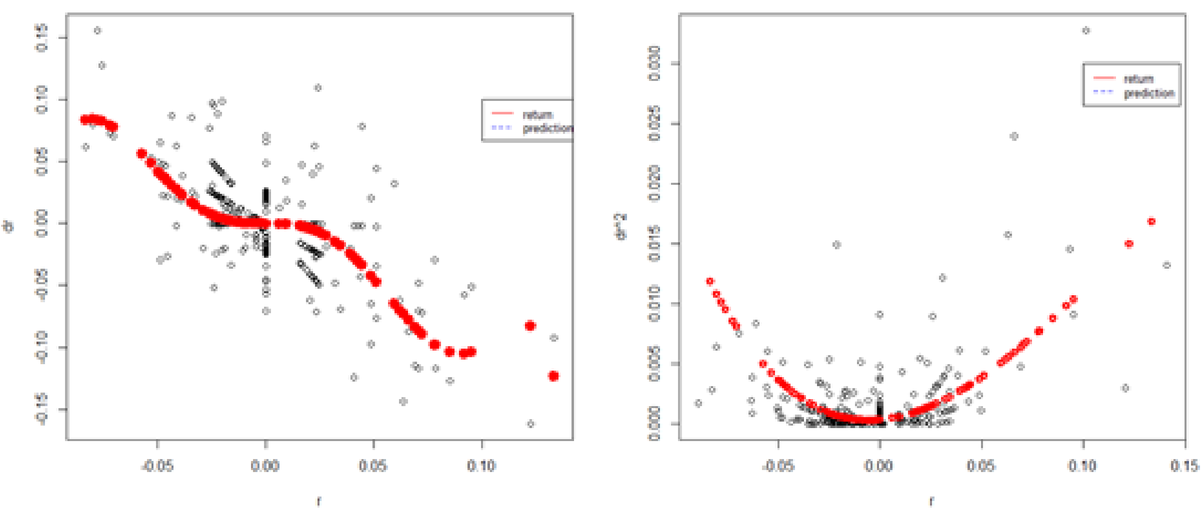


Figure 9: Return scatter plot with fitted drift and diffusion models by nonlinear least squares for series 2, return (black), and fitted (red)

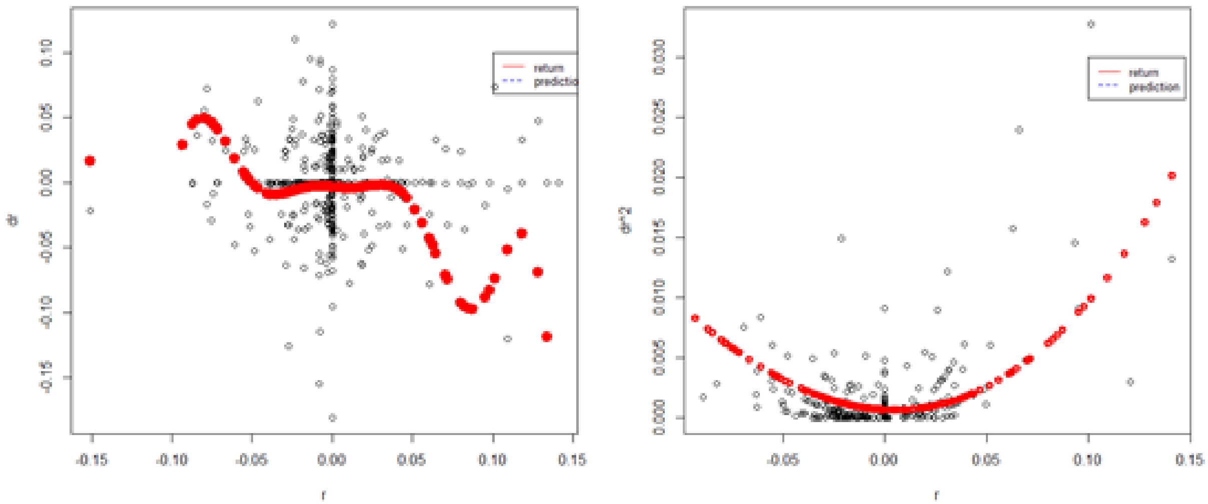


Figure 10: Return scatter plot with fitted drift and diffusion models by nonlinear least squares for series 3, return (black), and fitted (red)

We use Quasi-maximum Likelihood Estimation explained in section 2, to compare the conventional methods with the suggested method of maximum likelihood discussed in section 1, using yuima R package, and the estimated coefficients found to be as follow :

Table 1: Parameter estimation of quasi multivariate maximum likelihood

Equations	θ_{0i}	θ_{1i}	θ_{2i}	θ_{3i}	θ_{4i}	
Drift1	8.4506	193.2457	-17.3479	0.0020	-39.5415	
Drift2	14.1524	189.1097	-46.5658	-0.0007	43.6340	
Drift3	-7.1288	36.8217	28.7963	0.0001	-47.9658	
	α_{0i}	α_{1i}	α_{2i}	α_{3i}		
Dif1	20.7868	0.0065	0.0180	-7.8977		
Dif2	0.0622	0.0001	0.0025	0.0004		
Dif3	24.4776	39.5733	0.0654	-0.0042		
	α_{0ij}	α_{1ij}	α_{2ij}	α_{3ij}	α_{4ij}	α_{5ij}
Dif12	0.0008	-0.0189	-0.0341	0.0017	-0.6674	0.1678
Dif13	0.0501	9.1557	0.0742	0.0150	0.0684	0.0158
Dif23	-4.5272	-1.4994	1.2878	-0.1230	0.0760	-9.6380
	H_1	H_2	H_3	H_{12}	H_{13}	H_{23}
Hurst index	0.5	0.5	0.5	0.5	0.5	0.5
	Series1	Series2	Series3	mean		
MSE	0.0006541535	0.0005719739	0.003065913	0.00143068		

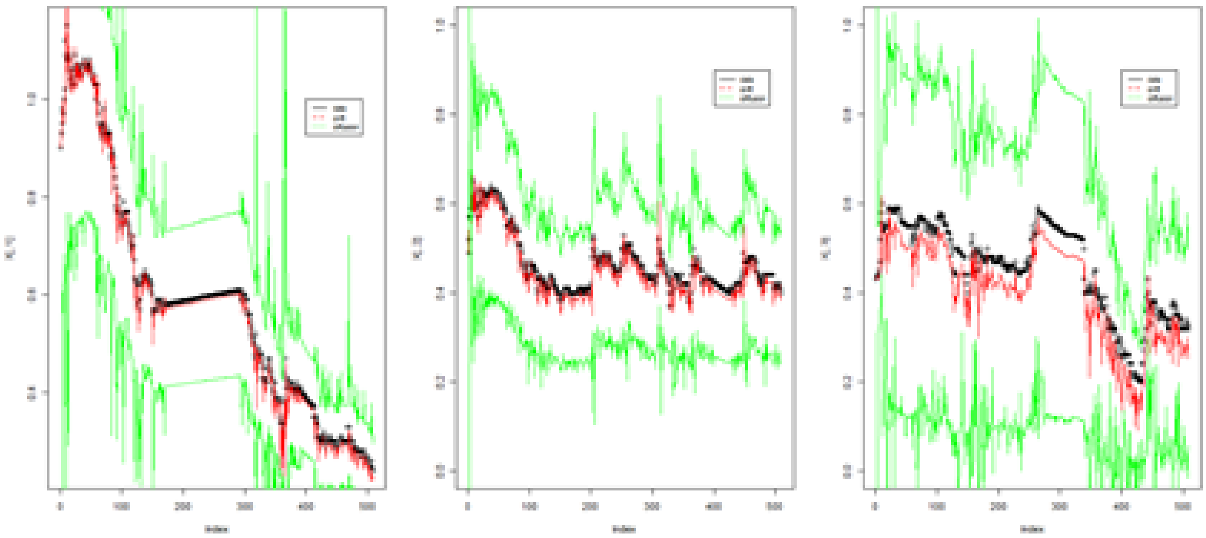


Figure 11: Plot of fitted models quasi multivariate maximum likelihood for three series, data (black points), fitted drift (red line), and fitted diffusion (green line).

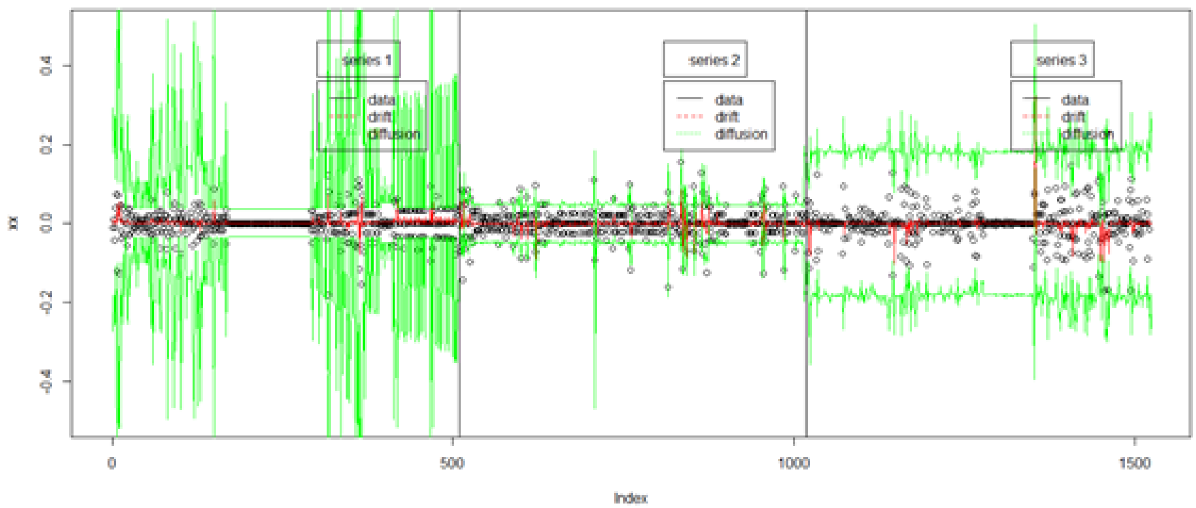


Figure 12: Plot of fitted models by quasi multivariate maximum likelihood for three returns, data (black points), fitted drift (red line), and fitted diffusion (green line).

From Figures 8-12, we perceive that the models represent the data in an accurate manner, and we can use them in the model. Secondly, we have estimated the parameters with Hurst indices simultaneously to obtain the final fitted model as follow:

$$\begin{pmatrix} X_{1,t+1} \\ X_{2,t+1} \\ X_{3,t+1} \end{pmatrix} = \begin{pmatrix} \frac{x_{1,t}^2}{x_{1,t-1}} \exp\Bigg(\theta_{10}\left(\log\left(\frac{x_{1,t}}{x_{1,t-1}}\right)\right)^2 \sin\left(\theta_{11}\log\left(\frac{x_{1,t}}{x_{1,t-1}}\right)+\theta_{12}\right)+\theta_{13}\left(\frac{x_{1,t}}{x_{1,t-1}}\right)^{\theta_{14}}\Delta t \\ -\frac{1}{2}\left(\left(\Sigma\left(\underline{x}_{1,t},\underline{x}_{1,t},t\right)\otimes\Omega_H\right)_{11,t}\right)+\Bigg(\alpha_{110}\left(\log\left(\frac{x_{1,t}}{x_{1,t-1}}\right)+\alpha_{111}\right)^2 \\ +\alpha_{112}\left(\frac{x_{1,t}}{x_{1,t-1}}\right)^{\alpha_{113}}\Bigg)\left(W_{t+1}^{H_1}-W_t^{H_1}\right)\Bigg) \\ \\ \frac{x_{2,t}^2}{x_{2,t-1}} \exp\Bigg(\theta_{20}\left(\log\left(\frac{x_{2,t}}{x_{2,t-1}}\right)\right)^2 \sin\left(\theta_{21}\log\left(\frac{x_{2,t}}{x_{2,t-1}}\right)+\theta_{22}\right)+\theta_{23}\left(\frac{x_{2,t}}{x_{2,t-1}}\right)^{\theta_{24}}\Delta t \\ -\frac{1}{2}\left(\left(\Sigma\left(\underline{x}_{2,t},\underline{x}_{2,t},t\right)\otimes\Omega_H\right)_{22,t}\right)+\Bigg(\alpha_{220}\left(\log\left(\frac{x_{2,t}}{x_{2,t-1}}\right)+\alpha_{221}\right)^2 \\ +\alpha_{222}\left(\frac{x_{2,t}}{x_{2,t-1}}\right)^{\alpha_{223}}\Bigg)\left(W_{t+1}^{H_2}-W_t^{H_2}\right)\Bigg) \\ \\ \frac{x_{3,t}^2}{x_{3,t-1}} \exp\Bigg(\theta_{30}\left(\log\left(\frac{x_{3,t}}{x_{3,t-1}}\right)\right)^2 \sin\left(\theta_{31}\log\left(\frac{x_{3,t}}{x_{3,t-1}}\right)+\theta_{32}\right)+\theta_{33}\left(\frac{x_{3,t}}{x_{3,t-1}}\right)^{\theta_{34}}\Delta t \\ -\frac{1}{2}\left(\left(\Sigma\left(\underline{x}_{3,t},\underline{x}_{3,t},t\right)\otimes\Omega_H\right)_{33,t}\right)+\Bigg(\alpha_{330}\left(\log\left(\frac{x_{3,t}}{x_{3,t-1}}\right)+\alpha_{331}\right)^2 \\ +\alpha_{332}\left(\frac{x_{3,t}}{x_{3,t-1}}\right)^{\alpha_{333}}\Bigg)\left(W_{t+1}^{H_3}-W_t^{H_3}\right)\Bigg) \end{pmatrix}$$

Where $\left(W_{t+1}^{H_i}-W_t^{H_i}\right)\sim N\left(0,\Omega_{H_i}\right)$.

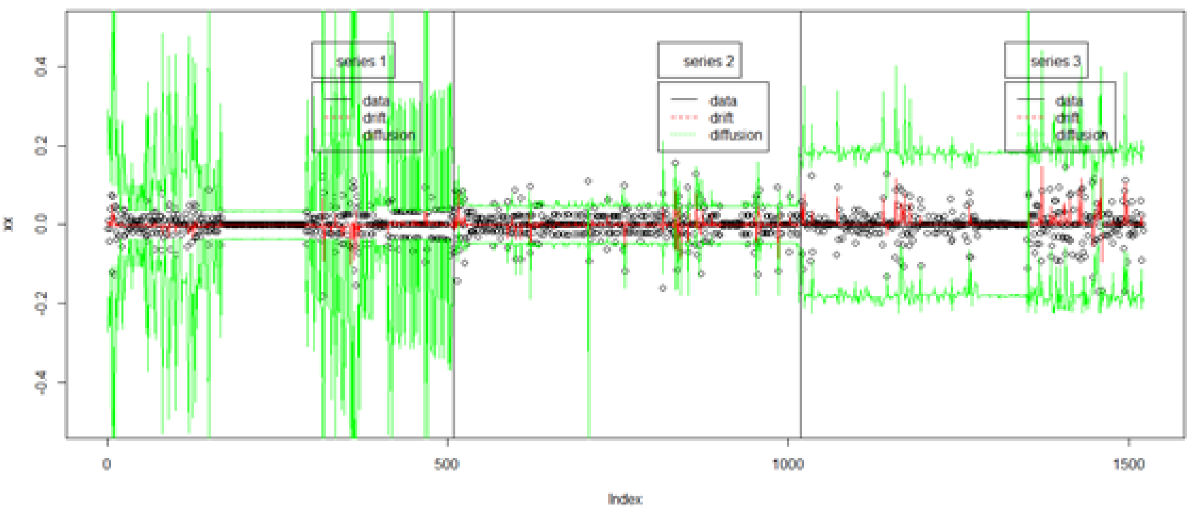


Figure 13: Plot of fitted models by multivariate maximum likelihood for three returns, data (black points), fitted drift (red line), and fitted diffusion (green line).

Table 2 shows the parameter estimation of drift and diffusion function with Hurst indices, and mean square error (MSE).

Table 2: Metropolis Hasting mean and incredible interval for parameters

Equations		θ_{0i}	θ_{1i}	θ_{2i}	θ_{3i}	θ_{4i}	
	mean	8.48465	193.245	-17.27933	-0.00172	-39.53321	
Drift1	0.025	8.48446	193.2445	-17.27943	-0.00187	-39.53338	
	0.975	8.48483	193.2455	-17.27924	-0.00152	-39.53304	
	mean	14.15058	189.1274	-46.63618	-0.00080	43.63983	
Drift2	0.025	14.15035	189.1272	-46.63655	-0.00116	43.63980	
	0.975	14.15077	189.1276	-46.63588	-0.00046	43.63987	
	mean	-7.29845	36.60179	26.96638	0.00097	-47.96723	
Drift3	0.025	-7.29853	36.60167	26.96630	0.00088	-47.96726	
	0.975	-7.29837	36.60193	26.96646	0.00105	-47.96719	
		α_{0i}	α_{1i}	α_{2i}	α_{3i}		
	mean	20.77102	-0.00140	0.00031	-7.89594		
Dift1	0.025	20.77049	-0.00158	0.00026	-7.89608		
	0.975	20.77156	-0.00121	0.00038	-7.89580		
	mean	0.00274	0.00819	0.00858	0		
Dift2	0.025	0.00264	0.00806	0.00852	0		
	0.975	0.00284	0.00831	0.00864	0.00001		
	mean	24.48003	39.57672	0.00041	0.00013		
Dift3	0.025	24.47989	39.57663	0.00024	-0.00005		
	0.975	24.48019	39.5768	0.00059	0.00029		
		α_{0ij}	α_{1ij}	α_{2ij}	α_{3ij}	α_{4ij}	α_{5ij}
	mean	-0.00152	0.0005	-0.03128	-0.00355	-0.27221	0.2051
Dift12	0.025	-0.00183	0.00038	-0.03151	-0.0038	-0.27257	0.20495
	0.975	-0.00113	0.00063	-0.03097	-0.00332	-0.27174	0.20527
	mean	0.01213	9.30245	0.00027	0.0138	0.06658	-0.00005
Dift13	0.025	0.01205	9.30203	0.00016	0.01369	0.06633	-0.00014
	0.975	0.01221	9.30284	0.00037	0.01393	0.06681	0.00005
	Mean	-4.52828	-1.49926	-0.00092	-0.17436	0.01116	-10.2919
Dift21	0.025	-4.52862	-1.49968	-0.00122	-0.17443	0.01106	-10.292
	0.975	-4.52795	-1.49889	-0.00062	-0.17429	0.01124	-10.2918
		H_1	H_2	H_3	H_{12}	H_{13}	H_{23}
	mean	0.55189	0.71851	0.75067	0.53004	0.57861	0.50963
Hurst index	0.025	0.55163	0.7183	0.75047	0.53001	0.57823	0.50948
	0.975	0.5521	0.71875	0.75085	0.53008	0.57901	0.50979
		Series1	Series2	Series3	mean		
MSE		0.0006972986	0.0002888781	0.00180868	0.000931618		

MSE (mean square error)

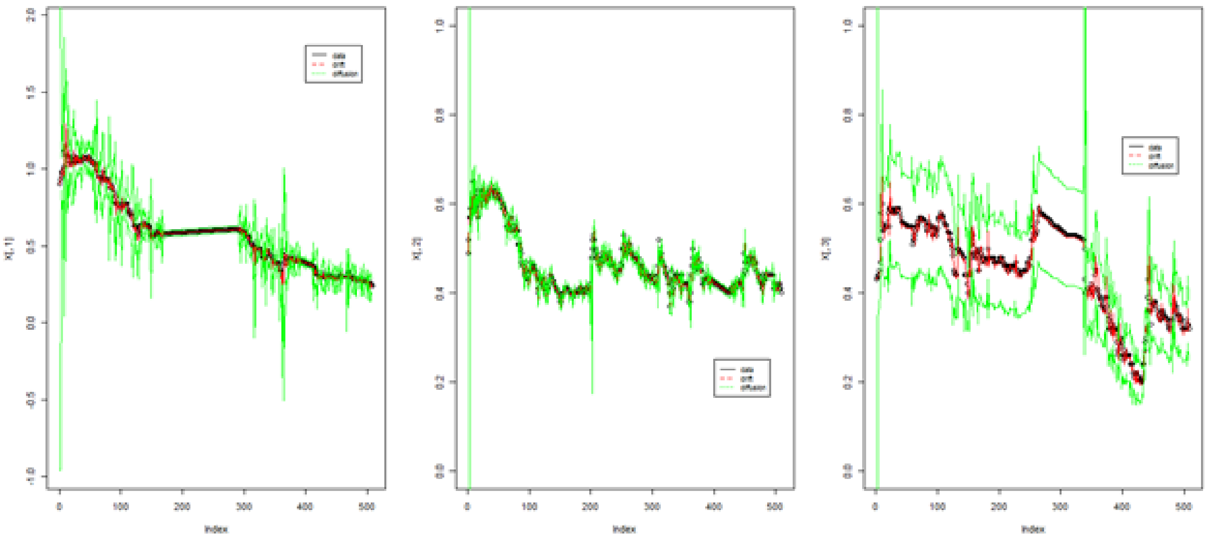


Figure 14: Plot of fitted models by multivariate maximum likelihood for three series, data (black points), fitted drift (red line), and fitted diffusion (green line).

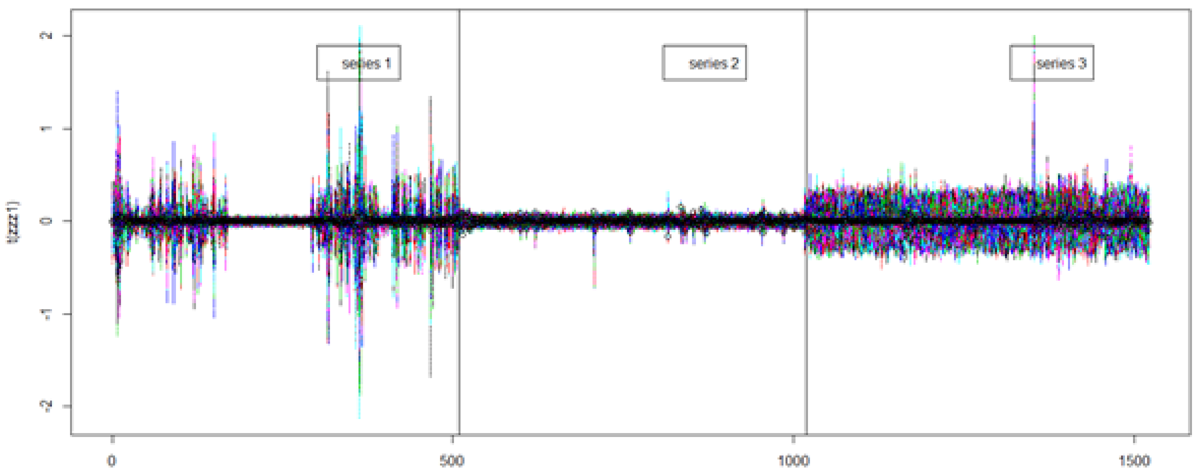
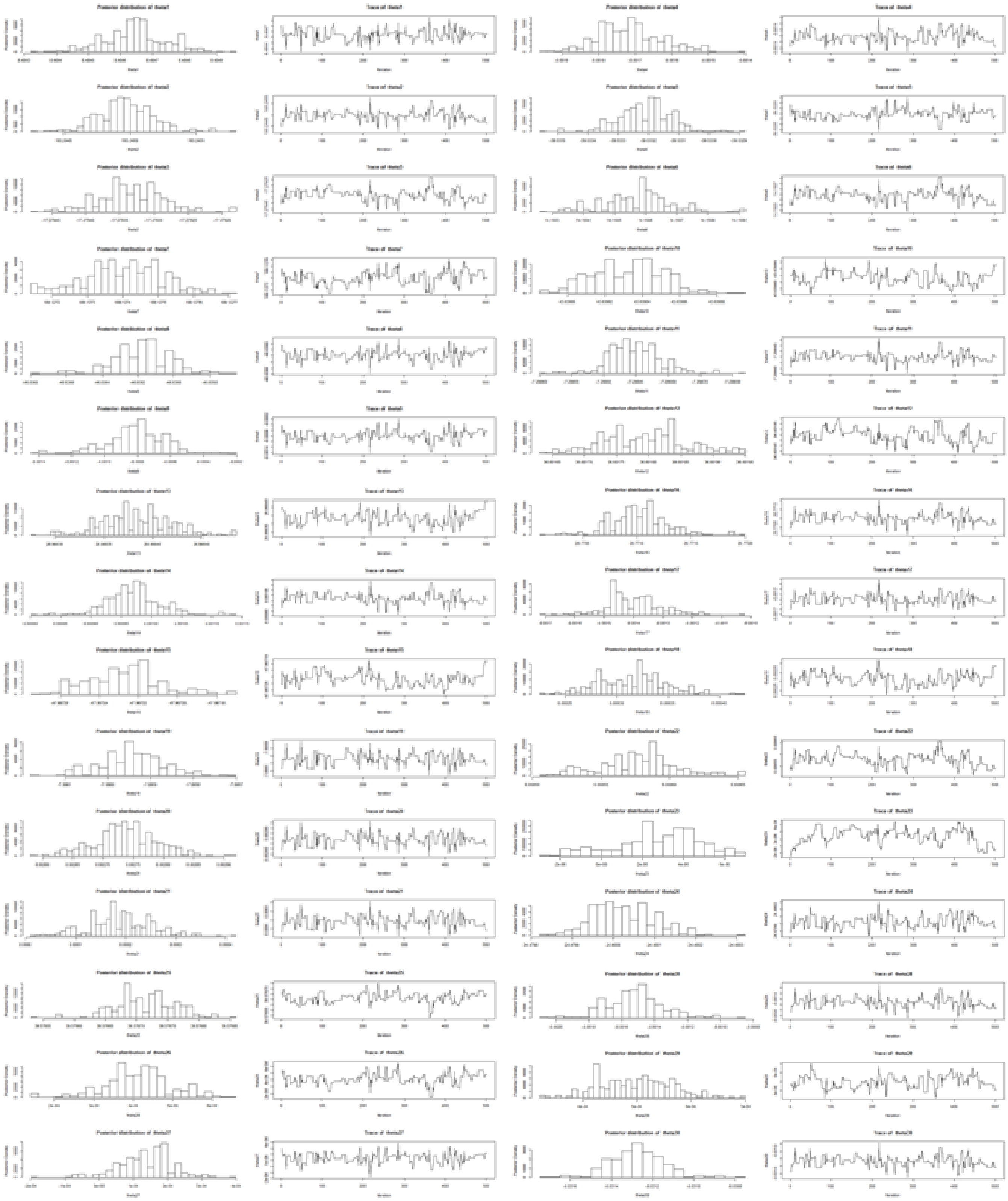


Figure 15: Plot of 1000 generated samples from estimated model by Bayesian multivariate maximum likelihood for three returns, data (black points)..

7.1. Adaptive Metropolis Hasting algorithm

In this section we will use the Adaptive Metropolis Hasting algorithm as explained in section 1.7 , and result was as following



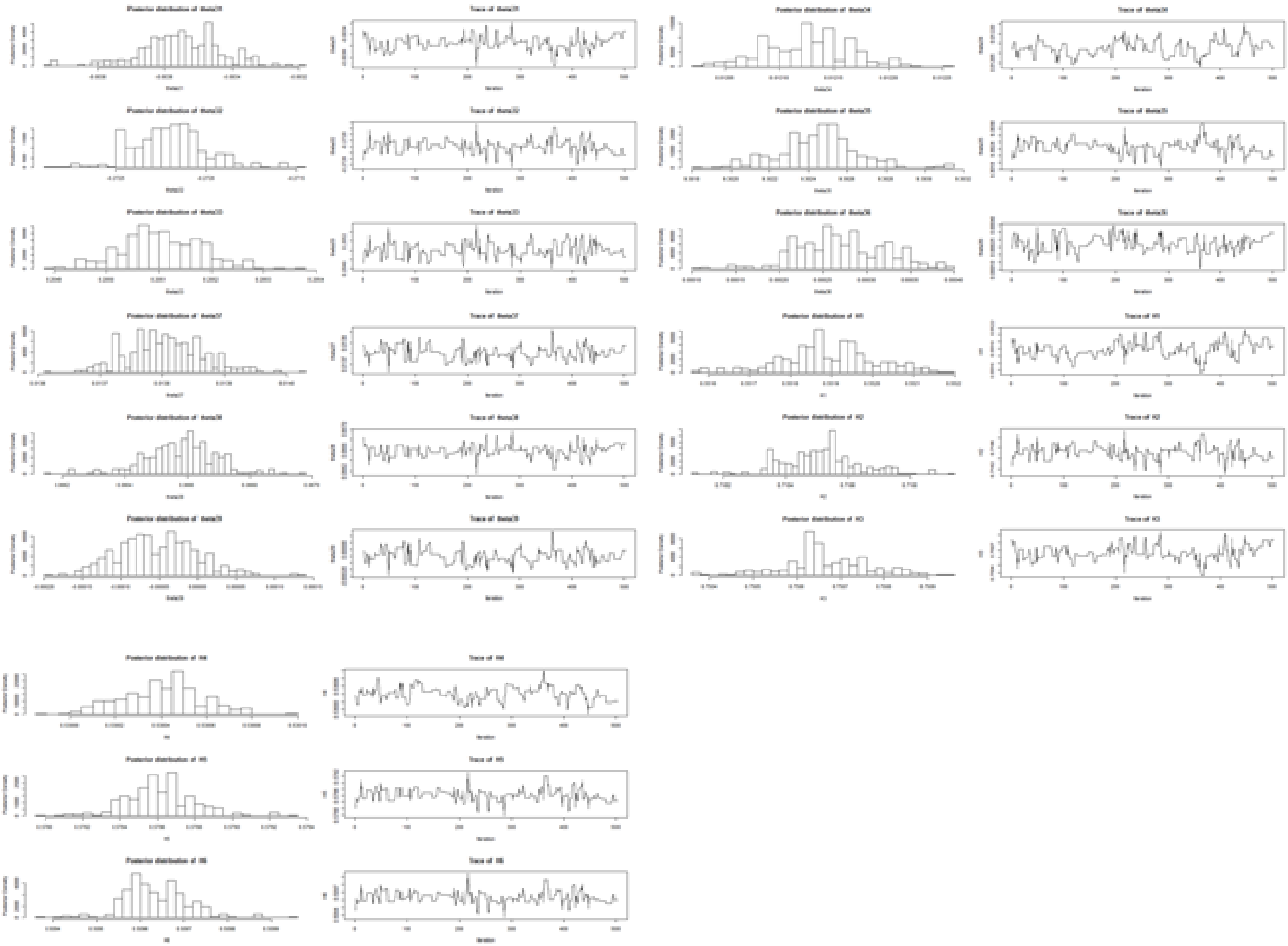


Figure 16: Plot of histogram and stationary series of Metropolis Hasting sampling method for model parameters

7.2. The Multivariate FBM

In this section we will apply the multivariate analysis of fractional Brownian motion as posed in section 4 , the result is viewed in Table 3

Table 3: The estimate of Hurst indices by multivariate analysis

H_i	σ_i	ρ_{ij}	η_{ij}
0.4955512	0.0020709730	0.2744565	2.754457
0.4738972	0.0002274438	0.1553428	2.297712
0.5672862	0.0002039744	0.1531577	3.687176

7.3. Portfolio Building

After we find the model drift vector and diffusion matrix we will use them to select the best strategy to invest in these three stocks , we use the method explained in Section 5 , and the portfolio is

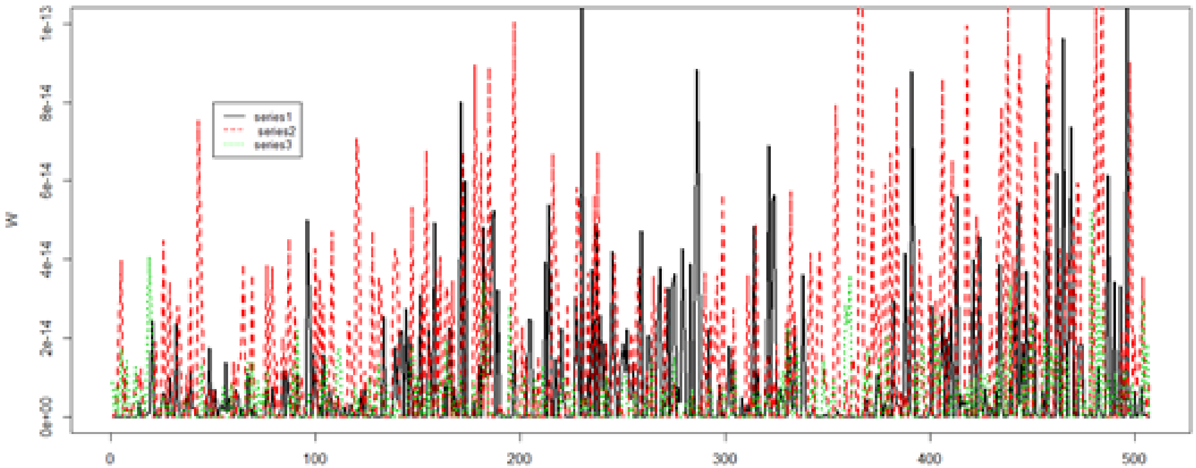


Figure 17: Portfolio strategy for sell and buy normalized quantities for each stock series for all period

We want to point out that the sum of all weight in the figure 17 add up to 1, this means that this strategy for all period, and not for one time of variables in the portfolio, for example if we want to invest in one week period we must distribute the capital on stocks and period but not on stocks only.

8. Conclusion

Bayesian Fractional maximum likelihood suggested method for stochastic differential equations are a very efficient tool to represent the financial phenomena because it reflect the dynamical behavior with long memory that is an intrinsic characteristic of financial phenomenon. The suggested method superimposed other models in capturing the minute details in the data. The drift and diffusion are very important quantities in many application especially in the financial portfolio building. The model specified shows many features in the data to be used to predict the future values and so build the portfolio. That is much benefited for the investor to overcome the risk of stock prices and to achieve a profit. The parameter of the model is estimated numerically by Metropolis Hasting method to maximize the logarithm of the posterior distribution using R program. The results in Table2 that Hurst indices are higher than 0.5 , and this means there is a strong long memory behavior in the three series. We see that from Figure 13, that the diffusion is very high (green line) in series 1, because of Hurst index is 0.55189, and diffusion in series 3 is low because of high Hurst index. This reflects the long memory existence that will decrease the uncertainty and the prediction will be more accurate. The cross Hurst indices reflect the cross long memory correlation between different variables. As we conclude from Table 2, there is a cross long memory between three series. By comparison with traditional methods like quasi multivariate maximum likelihood, we see from table 1 and table 2 that the mean of MSE for maximum multivariate likelihood is smaller.

9. Appendix

To construct the matrix of diffusion with the autocorrelation matrix of Fractional Brownian motion, the vectorization of variables must include Kronecker product of variance-covariance matrix as in SURE (Seemingly Unrelated Regression Equations) model as below [2] :

$\Sigma \otimes I_n$

if

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 \end{bmatrix}$$

$vec(\Sigma * N(0,1)) = N(0, \Sigma \otimes I_3)$

$$\Sigma \otimes I_3 = \begin{bmatrix} \sigma_1^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \sigma_{12} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \sigma_{13} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \sigma_{21} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \sigma_2^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \sigma_{23} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \sigma_{31} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \sigma_{32} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \sigma_3^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_1^2 & 0 \\ 0 & 0 & \sigma_1^2 \end{bmatrix} & \begin{bmatrix} \sigma_{12} & 0 & 0 \\ 0 & \sigma_{12} & 0 \\ 0 & 0 & \sigma_{12} \end{bmatrix} & \begin{bmatrix} \sigma_{13} & 0 & 0 \\ 0 & \sigma_{13} & 0 \\ 0 & 0 & \sigma_{13} \end{bmatrix} \\ \begin{bmatrix} \sigma_{21} & 0 & 0 \\ 0 & \sigma_{21} & 0 \\ 0 & 0 & \sigma_{21} \end{bmatrix} & \begin{bmatrix} \sigma_2^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_2^2 \end{bmatrix} & \begin{bmatrix} \sigma_{23} & 0 & 0 \\ 0 & \sigma_{23} & 0 \\ 0 & 0 & \sigma_{23} \end{bmatrix} \\ \begin{bmatrix} \sigma_{31} & 0 & 0 \\ 0 & \sigma_{31} & 0 \\ 0 & 0 & \sigma_{31} \end{bmatrix} & \begin{bmatrix} \sigma_{32} & 0 & 0 \\ 0 & \sigma_{32} & 0 \\ 0 & 0 & \sigma_{32} \end{bmatrix} & \begin{bmatrix} \sigma_3^2 & 0 & 0 \\ 0 & \sigma_3^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_1^2 & 0 \\ 0 & 0 & \sigma_1^2 \end{bmatrix} & \begin{bmatrix} \sigma_1\rho_{12}\sigma_2 & 0 & 0 \\ 0 & \sigma_1\rho_{12}\sigma_2 & 0 \\ 0 & 0 & \sigma_1\rho_{12}\sigma_2 \end{bmatrix} & \begin{bmatrix} \sigma_1\rho_{13}\sigma_3 & 0 & 0 \\ 0 & \sigma_1\rho_{13}\sigma_3 & 0 \\ 0 & 0 & \sigma_1\rho_{13}\sigma_3 \end{bmatrix} \\ \begin{bmatrix} \sigma_1\rho_{12}\sigma_2 & 0 & 0 \\ 0 & \sigma_1\rho_{12}\sigma_2 & 0 \\ 0 & 0 & \sigma_1\rho_{12}\sigma_2 \end{bmatrix} & \begin{bmatrix} \sigma_2^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_2^2 \end{bmatrix} & \begin{bmatrix} \sigma_2\rho_{23}\sigma_3 & 0 & 0 \\ 0 & \sigma_2\rho_{23}\sigma_3 & 0 \\ 0 & 0 & \sigma_2\rho_{23}\sigma_3 \end{bmatrix} \\ \begin{bmatrix} \sigma_1\rho_{13}\sigma_3 & 0 & 0 \\ 0 & \sigma_1\rho_{13}\sigma_3 & 0 \\ 0 & 0 & \sigma_1\rho_{13}\sigma_3 \end{bmatrix} & \begin{bmatrix} \sigma_2\rho_{23}\sigma_3 & 0 & 0 \\ 0 & \sigma_2\rho_{23}\sigma_3 & 0 \\ 0 & 0 & \sigma_2\rho_{23}\sigma_3 \end{bmatrix} & \begin{bmatrix} \sigma_3^2 & 0 & 0 \\ 0 & \sigma_3^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix}\sigma_1&0&0\\0&\sigma_1&0\\0&0&\sigma_1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\sigma_1&0&0\\0&\sigma_1&0\\0&0&\sigma_1\end{bmatrix} & \begin{bmatrix}\sigma_1&0&0\\0&\sigma_1&0\\0&0&\sigma_1\end{bmatrix}\begin{bmatrix}\rho_{12}&0&0\\0&\rho_{12}&0\\0&0&\rho_{12}\end{bmatrix}\begin{bmatrix}\sigma_2&0&0\\0&\sigma_2&0\\0&0&\sigma_2\end{bmatrix} & \begin{bmatrix}\sigma_1&0&0\\0&\sigma_1&0\\0&0&\sigma_1\end{bmatrix}\begin{bmatrix}\rho_{13}&0&0\\0&\rho_{13}&0\\0&0&\rho_{13}\end{bmatrix}\begin{bmatrix}\sigma_3&0&0\\0&\sigma_3&0\\0&0&\sigma_3\end{bmatrix} \\ \begin{bmatrix}\sigma_1&0&0\\0&\sigma_1&0\\0&0&\sigma_1\end{bmatrix}\begin{bmatrix}\rho_{12}&0&0\\0&\rho_{12}&0\\0&0&\rho_{12}\end{bmatrix}\begin{bmatrix}\sigma_2&0&0\\0&\sigma_2&0\\0&0&\sigma_2\end{bmatrix} & \begin{bmatrix}\sigma_2&0&0\\0&\sigma_2&0\\0&0&\sigma_2\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\sigma_2&0&0\\0&\sigma_2&0\\0&0&\sigma_2\end{bmatrix} & \begin{bmatrix}\sigma_2&0&0\\0&\sigma_2&0\\0&0&\sigma_2\end{bmatrix}\begin{bmatrix}\rho_{23}&0&0\\0&\rho_{23}&0\\0&0&\rho_{23}\end{bmatrix}\begin{bmatrix}\sigma_3&0&0\\0&\sigma_3&0\\0&0&\sigma_3\end{bmatrix} \\ \begin{bmatrix}\sigma_1&0&0\\0&\sigma_1&0\\0&0&\sigma_1\end{bmatrix}\begin{bmatrix}\rho_{13}&0&0\\0&\rho_{13}&0\\0&0&\rho_{13}\end{bmatrix}\begin{bmatrix}\sigma_3&0&0\\0&\sigma_3&0\\0&0&\sigma_3\end{bmatrix} & \begin{bmatrix}\sigma_2&0&0\\0&\sigma_2&0\\0&0&\sigma_2\end{bmatrix}\begin{bmatrix}\rho_{23}&0&0\\0&\rho_{23}&0\\0&0&\rho_{23}\end{bmatrix}\begin{bmatrix}\sigma_3&0&0\\0&\sigma_3&0\\0&0&\sigma_3\end{bmatrix} & \begin{bmatrix}\sigma_3&0&0\\0&\sigma_3&0\\0&0&\sigma_3\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\sigma_3&0&0\\0&\sigma_3&0\\0&0&\sigma_3\end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_{11}^{1/2}\Sigma_{11}^{1/2} & \Sigma_{11}^{1/2}\Gamma_{12}\Sigma_{22}^{1/2} & \Sigma_{11}^{1/2}\Gamma_{13}\Sigma_{33}^{1/2} \\ \Sigma_{11}^{1/2}\Gamma_{12}\Sigma_{22}^{1/2} & \Sigma_{22}^{1/2}\Sigma_{22}^{1/2} & \Sigma_{22}^{1/2}\Gamma_{23}\Sigma_{33}^{1/2} \\ \Sigma_{11}^{1/2}\Gamma_{13}\Sigma_{33}^{1/2} & \Sigma_{22}^{1/2}\Gamma_{23}\Sigma_{33}^{1/2} & \Sigma_{33}^{1/2}\Sigma_{33}^{1/2} \end{bmatrix}$$

where $\Gamma_{ij} = \begin{bmatrix}\rho_{ij}&0&0\\0&\rho_{ij}&0\\0&0&\rho_{ij}\end{bmatrix}$

Here, I_n matrix represents the autocorrelation matrix with $\phi = 0$. So, if $\phi \neq 0$, we will have:

$$\Omega = \begin{bmatrix}1&\phi&\phi^2\\\phi&1&\phi\\\phi^2&\phi&1\end{bmatrix}$$

$$vec(\Sigma * N(0,\Omega)) = N(0, \Sigma \otimes \Omega)$$

$$\Sigma \otimes \Omega = \begin{bmatrix} \sigma_1^2\begin{bmatrix}1&\phi&\phi^2\\\phi&1&\phi\\\phi^2&\phi&1\end{bmatrix} & \sigma_{12}\begin{bmatrix}1&\phi&\phi^2\\\phi&1&\phi\\\phi^2&\phi&1\end{bmatrix} & \sigma_{13}\begin{bmatrix}1&\phi&\phi^2\\\phi&1&\phi\\\phi^2&\phi&1\end{bmatrix} \\ \sigma_{21}\begin{bmatrix}1&\phi&\phi^2\\\phi&1&\phi\\\phi^2&\phi&1\end{bmatrix} & \sigma_2^2\begin{bmatrix}1&\phi&\phi^2\\\phi&1&\phi\\\phi^2&\phi&1\end{bmatrix} & \sigma_{23}\begin{bmatrix}1&\phi&\phi^2\\\phi&1&\phi\\\phi^2&\phi&1\end{bmatrix} \\ \sigma_{31}\begin{bmatrix}1&\phi&\phi^2\\\phi&1&\phi\\\phi^2&\phi&1\end{bmatrix} & \sigma_{32}\begin{bmatrix}1&\phi&\phi^2\\\phi&1&\phi\\\phi^2&\phi&1\end{bmatrix} & \sigma_3^2\begin{bmatrix}1&\phi&\phi^2\\\phi&1&\phi\\\phi^2&\phi&1\end{bmatrix} \end{bmatrix}$$

Now, if we suppose that every observation generated from different mean and variance, we have

$$\Sigma = \begin{bmatrix}\Sigma_{11}&\Sigma_{12}&\Sigma_{13}\\\Sigma_{21}&\Sigma_{22}&\Sigma_{23}\\\Sigma_{31}&\Sigma_{32}&\Sigma_{33}\end{bmatrix} \quad \text{where } \Sigma \text{ is positive definite}$$

$$= \begin{bmatrix} \Sigma_{11} & \Sigma_{11}^{1/2}\Gamma_{12}\Sigma_{22}^{1/2} & \Sigma_{11}^{1/2}\Gamma_{13}\Sigma_{33}^{1/2} \\ \Sigma_{11}^{1/2}\Gamma_{12}\Sigma_{22}^{1/2} & \Sigma_{22} & \Sigma_{22}^{1/2}\Gamma_{23}\Sigma_{33}^{1/2} \\ \Sigma_{11}^{1/2}\Gamma_{13}\Sigma_{33}^{1/2} & \Sigma_{22}^{1/2}\Gamma_{23}\Sigma_{33}^{1/2} & \Sigma_{33} \end{bmatrix}$$

$$vec(\Sigma * N(0,\Omega)) = N(0, \Sigma \otimes \Omega)$$

$$\Sigma\otimes\Omega=\begin{bmatrix} \Sigma_{11}^{1/2}\begin{bmatrix}1&\phi&\phi^2\\ \phi&1&\phi\\ \phi^2&\phi&1\end{bmatrix}\Sigma_{11}^{1/2} & \Sigma_{11}^{1/2}\begin{bmatrix}\rho_{12}&\phi&\phi^2\\ \phi&\rho_{12}&\phi\\ \phi^2&\phi&\rho_{12}\end{bmatrix}\Sigma_{22}^{1/2} & \Sigma_{11}^{1/2}\begin{bmatrix}\rho_{13}&\phi&\phi^2\\ \phi&\rho_{13}&\phi\\ \phi^2&\phi&\rho_{13}\end{bmatrix}\Sigma_{33}^{1/2}\\ \Sigma_{22}^{1/2}\begin{bmatrix}\rho_{21}&\phi&\phi^2\\ \phi&\rho_{21}&\phi\\ \phi^2&\phi&\rho_{21}\end{bmatrix}\Sigma_{11}^{1/2} & \Sigma_{22}^{1/2}\begin{bmatrix}1&\phi&\phi^2\\ \phi&1&\phi\\ \phi^2&\phi&1\end{bmatrix}\Sigma_{22}^{1/2} & \Sigma_{22}^{1/2}\begin{bmatrix}\rho_{23}&\phi&\phi^2\\ \phi&\rho_{23}&\phi\\ \phi^2&\phi&\rho_{23}\end{bmatrix}\Sigma_{33}^{1/2}\\ \Sigma_{33}^{1/2}\begin{bmatrix}\rho_{31}&\phi&\phi^2\\ \phi&\rho_{31}&\phi\\ \phi^2&\phi&\rho_{31}\end{bmatrix}\Sigma_{11}^{1/2} & \Sigma_{33}^{1/2}\begin{bmatrix}\rho_{32}&\phi&\phi^2\\ \phi&\rho_{32}&\phi\\ \phi^2&\phi&\rho_{32}\end{bmatrix}\Sigma_{22}^{1/2} & \Sigma_{33}^{1/2}\begin{bmatrix}1&\phi&\phi^2\\ \phi&1&\phi\\ \phi^2&\phi&1\end{bmatrix}\Sigma_{33}^{1/2} \end{bmatrix}$$

Where ϕ is the autocorrelation and ρ is the correlation between two different variables

$$\Sigma\otimes\Omega=\begin{bmatrix} \Sigma_{11}^{1/2}\Omega\Sigma_{11}^{1/2} & \Sigma_{11}^{1/2}\left(\Gamma_{12}.\,\Omega\right)\Sigma_{22}^{1/2} & \Sigma_{11}^{1/2}\left(\Gamma_{13}.\,\Omega\right)\Sigma_{33}^{1/2}\\ \Sigma_{11}^{1/2}\left(\Gamma_{12}.\,\Omega\right)\Sigma_{22}^{1/2} & \Sigma_{22}^{1/2}\Omega\Sigma_{22}^{1/2} & \Sigma_{22}^{1/2}\left(\Gamma_{23}.\,\Omega\right)\Sigma_{33}^{1/2}\\ \Sigma_{11}^{1/2}\left(\Gamma_{13}.\,\Omega\right)\Sigma_{33}^{1/2} & \Sigma_{22}^{1/2}\left(\Gamma_{23}.\,\Omega\right)\Sigma_{33}^{1/2} & \Sigma_{33}^{1/2}\Omega\Sigma_{33}^{1/2} \end{bmatrix}$$

But the different variables cannot have the same autocorrelation, so the matrix will become:

$$\Sigma\otimes\Omega=\begin{bmatrix} \Sigma_{11}^{1/2}\Omega_{11}\Sigma_{11}^{1/2} & \Sigma_{11}^{1/2}\left(\Gamma_{12}.\,\Omega_{12}\right)\Sigma_{22}^{1/2} & \Sigma_{11}^{1/2}\left(\Gamma_{13}.\,\Omega_{13}\right)\Sigma_{33}^{1/2}\\ \Sigma_{11}^{1/2}\left(\Gamma_{12}.\,\Omega_{21}\right)\Sigma_{22}^{1/2} & \Sigma_{22}^{1/2}\,\Omega_{22}\Sigma_{22}^{1/2} & \Sigma_{22}^{1/2}\left(\Gamma_{23}.\,\Omega_{23}\right)\Sigma_{33}^{1/2}\\ \Sigma_{11}^{1/2}\left(\Gamma_{13}.\,\Omega_{31}\right)\Sigma_{33}^{1/2} & \Sigma_{22}^{1/2}\left(\Gamma_{23}.\,\Omega_{32}\right)\Sigma_{33}^{1/2} & \Sigma_{33}^{1/2}\,\Omega_{33}\Sigma_{33}^{1/2} \end{bmatrix}.$$

Now, we substitute ϕ with long memory dependence characterized by Hurst index. We get the following

$$\Sigma\otimes\Omega_H=\begin{bmatrix} \Sigma_{11}^{1/2}\Omega_{H_1}\Sigma_{11}^{1/2} & \Sigma_{11}^{1/2}\left(\Gamma_{12}.\,\Omega_{H_{12}}\right)\Sigma_{22}^{1/2} & \Sigma_{11}^{1/2}\left(\Gamma_{13}.\,\Omega_{H_{13}}\right)\Sigma_{33}^{1/2}\\ \Sigma_{11}^{1/2}\left(\Gamma_{12}.\,\Omega_{H_{12}}\right)\Sigma_{22}^{1/2} & \Sigma_{22}^{1/2}\,\Omega_{H_2}\Sigma_{22}^{1/2} & \Sigma_{22}^{1/2}\left(\Gamma_{23}.\,\Omega_{H_{23}}\right)\Sigma_{33}^{1/2}\\ \Sigma_{11}^{1/2}\left(\Gamma_{13}.\,\Omega_{H_{13}}\right)\Sigma_{33}^{1/2} & \Sigma_{22}^{1/2}\left(\Gamma_{23}.\,\Omega_{H_{23}}\right)\Sigma_{33}^{1/2} & \Sigma_{33}^{1/2}\,\Omega_{H_3}\Sigma_{33}^{1/2} \end{bmatrix}$$

Where Γ represents a diagonal matrix of correlation coefficients and $(\Gamma.\,\Omega)$ represents an element by element multiplication of two matrices

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