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ING-induced topology from tritopological space on a locally finite graph

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Abstract

The aim of this article is to associate a tritopological space with a locally finite graph, and induce a new type of topology from three well-known topologies on the same locally finite graph. These are the three tritopologies proposed recently to associate topological spaces with undirected graphs, the first (Independent Topology), the second is (Non-Incidence Topology) and the third is (Graphic Topology). Then some results and properties of these tritopological spaces and new induced topologies were investigated. Giving a fundamental step toward studying some properties of locally finite graphs by their corresponding tritopological spaces is our motivation.

Keywords: locally finite graphs, Tritopological spaces, Independent Topology, Non-Incidence Topology, Graphic Topology
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1. Introduction

In Mathematics, the field of graph theory have a very long history, one of the most used branch of graph theory is a topological graph theory. Which is the relation between graph theory and topological theory existed before and used many times by some researchers to deduce a topology from a given or certin graph. Some of them makes new models defined on the set of vertices $\mathbb{V}$ of the graph $\mathbb{G}$ only and others made it on the set of edges $\mathbb{E}$. They studies the graphs as a topologies and

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have been applied in almost every scientific field. Many excellent basics of graph theory, topological graph theory and some main applications may be found in the references [3, 9, 8, 14, 11, 12, 13].

In general graphs can be divided in two types; directed and undirected graphs. To an undirected graph some researchers associate a topological spaces as fellow;

In 2013 [1], Jafarian et al. associate a topology which named Graphic Topology with the vertex set $\mathbb{V}$ of a locally finite graph $\mathbb{G}$ without isolated vertex, and they defined a sub-basis family for a graphic topology as a sets of all vertices adjacent to the vertex v. Also there is some researchers associate a topology with the vertex set $\mathbb{V}$ of a graph $\mathbb{G}$.

All the previous works of topology on graph $\mathbb{G}$ was associated with a set of vertices $\mathbb{V}$ without isolated vertex, these topologies are not appropriate to be associated with graphs that have an isolated vertices. Therefore, these reasons motivate the authors in [6], [7] to associate a topology on the vertex set $\mathbb{V}$ of any undirected graph $\mathbb{G}$ (not only simple graph or locally finite graph) and which may contain one isolated vertex or more by introducing a topology which was named (Independent Topology) [6] in 2020, and in [7] the authors associated a topology on the vertex set $\mathbb{V}$ of any simple graph $\mathbb{G}$ by introducing a new sub-basis family defined as a sets of all vertices non-incident with the edge $\mathcal{E}$ (non-end vertices of the edge $\mathcal{E}$) to induce the new topology which was named (Non-Incidence Topology) in 2021, and they presented some main results and properties of simple graphs by their corresponding topologies.

Simply a tritopological space is a non-empty set $\mathbb{X}$ which is associated with three arbitrary topologies, this definition was initiated in 2000 by Kovar [10]. In 2004, Asmhan was dealt with it in detail and introduced the definition of certain type of open set in tritopological spaces. And provided the tritopological theory by all topological structures of that open set. Also in 2019, she was first initiated the fuzzy soft tritopological theory [5].

Our motivation or target in this paper is to associate a tritopological space with locally finite graph, i.e. three different topologies associate from the same graph. These three different topologies are the three mentioned above, the first (Independent Topology), the second is (Non-Incident Topology), and the third is (Graphic Topology). And we give a fundamental step toward studying some properties of locally finite graphs by their corresponding tritopological spaces.

So, we have two goals for this paper: Firstly, we introduce a definition of new tritopological space created from three different topologies associated with locally finite graphs. Secondly, we proposed a novel model of topology induced from these three different topologies associated with locally finite graphs.

In Section 2 of the article we give some fundamental definitions and preliminaries of graph theory, topology, tritopology and the three well-known topologies associated with graphs which we need. Section 3 is dedicated to main results of tritopological spaces on locally finite graphs, we define our new ING-induced topological space on locally finite graphs. Section 4 is devoted to definition and some preliminaries results of ING-Alexandroff topology. In last Section 5, conclusions of this new topology of tritopological spaces on locally finite graphs.

2. Preliminaries

In this section we give some fundamental definitions and preliminaries of tritopology and graph theory. All these definitions are standard, and can be found for example in references [9, 8, 14, 1, 6, 7, 4, 2].

Usually the graph is a pair $\mathbb{G} = (\mathbb{V}, \mathbb{E})$, for more exactly a graph $\mathbb{G}$ consist of a non-empty set $\mathbb{V}$ of vertices (or nodes), and a set $\mathbb{E}$ of edges (or arcs). If $\mathcal{E}$ is an edge in $\mathbb{G}$ we can write $\mathcal{E} = vu$ ($\mathcal{E}$ is join each vertex v and u), where v and u are vertices in $\mathbb{V}$, then (v and u) are said adjacent vertices and incident with the edge $\mathcal{E}$. If there is no vertex adjacent with a vertex v, then v is said isolated vertex. the degree of the vertex v denoted by $d(v)$) is the number of the edges where v incident with $\mathcal{E}$, and $\mathcal{D}(\mathbb{G})$ is the maximum degree of vertices in $\mathbb{G}$. A vertex of degree 0 is isolated. An independent set in a graph $\mathbb{G}$ is a set of non-adjacent vertices. The graph $\mathbb{G}$ is finite if the number of the vertices in $\mathbb{G}$ also the number of the edges in $\mathbb{G}$ is finite, then; otherwise it is an infinite graph. If any vertex can be reached from any other vertex in $\mathbb{G}$ by travelling along the edges, then $\mathbb{G}$ is called connected graph and is called disconnected otherwise.

We use notations $\mathbb{K}_n, \mathbb{K}_{m,n}, \mathbb{P}_n$ and $\mathbb{C}_n$ for a complete graph with n vertices, the complete bipartite graph when partite sets $\mathcal{A}, \mathcal{B}$ have sizes m and n, the path on n vertices and the cycle on n vertices, respectively.

A topology $\mathbb{T}$ on a set $\mathbb{X}$ is a combination of subsets of $\mathbb{X}$, called open, such that the union of the members of any subset of $\mathbb{T}$ is a member of $\mathbb{T}$, the intersection of the members of any finite subset of $\mathbb{T}$ is a member of $\mathbb{T}$, and both empty φ set and $\mathbb{X}$ are in $\mathbb{T}$. The ordered pair $(\mathbb{X}, \mathbb{T})$ is called a topological space. When the topology $\mathbb{T} = P(\mathbb{X})$ on $\mathbb{X}$ is called discrete topology while the topology $\mathbb{T} = \{\mathbb{X}, \varphi\}$ on $\mathbb{X}$ is called indiscrete (or trivial) topology. A topology in which arbitrary intersection of open sets is open set called an Alexandroff space.

A tritopological space is simply a non-empty set $\mathbb{X}$ which is associated with three arbitrary topologies $\mathbb{T}_1, \mathbb{T}_2$ and $\mathbb{T}_3$ on $\mathbb{X}$ as $(X, \mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3)$.

Now, in this paper we use three topological spaces associated with graphs which mentioned in the introduction above, i.e. Independent Topology, Graphic Topology and Non-Incidence Topology, which defined as fellows;

Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ be an undirected graph, and Let $S_I = \{I_v : v \in \mathbb{V}\}$ such that I_v is the set of all vertices not adjacent to v. $\mathbb{G}$ which may contain one isolated vertex or more, we have $\mathbb{V} = \cup_{v\in\mathbb{V}} I_v$ Hence S_I forms a sub-basis for a topology $\mathcal{T}_I$ on $\mathbb{V}$, and $\mathcal{T}_I$ called Independent Topology of $\mathbb{G}$.

Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ be a locally finite graph, i.e. a graph in which every vertex has finitely many adjacent vertices (a simple graph without isolated vertex). Define S_G as follows: $S_G = \{A_v | v \in \mathbb{V}\}$ such that A_v is the set of all vertices adjacent to v. Since $\mathbb{G}$ has no isolated vertex, we have $\mathbb{V} = \cup_{v\in\mathbb{V}} A_v$. Hence S_G forms a sub-basis for a topology $\mathcal{T}_G$ on $\mathbb{V}$, and $\mathcal{T}_G$ called Graphic Topology of $\mathbb{G}$.

Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ be an undirected graph, $NI_{\mathcal{E}}$ is the set of all vertices non-incident with the edge $\mathcal{E}$ (non-end vertices of the edge $\mathcal{E}$) with condition $|\ \mathbb{E}\ | \geq 3$.

Define $\mathcal{S}_N$ as follows $\mathcal{S}_N = \{NI_{\mathcal{E}} \mid \mathcal{E} \in \mathbb{E}\}$, we have $\mathbb{V} = \bigcup_{\mathcal{E}\in\mathbb{E}} NI_{\mathcal{E}}$, then $\mathcal{S}_N$ forms a sub-basis for a topology $\mathcal{T}_N$ on $\mathbb{V}$, called non-incidence (non-end vertices) topology of $\mathbb{G}$.

3. Tritopological spaces and ING-induced topological space on locally finite graphs

There are three topological spaces associated with graphs, Non-Incidence Topology [7], Independent Topology [6] and Graphic Topology [1], as defined above.

Notice that the Graphic Topology defined on locally finite graph, Non-Incidence Topology with condition $|\ \mathbb{E}\ | \geq 3$. Then all graphs throughout this paper are locally finite simple graphs with condition $|\ \mathbb{E}\ | \geq 3$.

Remark 3.1. 1. *Any topological space can be a tritopological space.*
Let $(\mathbb{V}, \mathcal{T}_{NI})$ be a Non-Incidence topological space, then $(\mathbb{V}, \mathcal{T}_N, \mathcal{T}_N, \mathcal{T}_N)$ is a tritopological space.
2. *The three topologies $\mathcal{T}_N, \mathcal{T}_I$ and $\mathcal{T}_G$ on $\mathbb{V}$ give the tritopological space $(\mathbb{V}, \mathcal{T}_I, \mathcal{T}_N, \mathcal{T}_G)$.*

Remark 3.2. *Any tritopological space is not necessary to be topological space, but any tritopological space induces a topological space in many ways. For example, if we take the intersection of any*

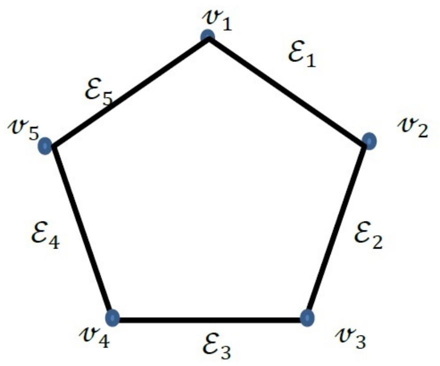

Figure 1:

number of topologies, then we get a topological space.
Now, we will associated the set of vertices $\mathbb{V}$ *with the three above topologies* $\mathcal{T}_N, \mathcal{T}_I$ *and* $\mathcal{T}_G$*, to be a tritopological space and written as* $(\mathbb{V}, \mathcal{T}_I, \mathcal{T}_N, \mathcal{T}_G)$.

Definition 3.3. *Let* $(\mathbb{V}, \mathcal{T}_I, \mathcal{T}_N, \mathcal{T}_G)$ *be a tritopological space, then* $(\mathbb{V}, \mathcal{T}_{ING})$ *where* $\mathcal{T}_{ING} = \mathcal{T}_I \bigcap \mathcal{T}_N \bigcap \mathcal{T}_G$ *is called the ING-induced topological space on locally finite graphs.*

Definition 3.4. *Let* $(\mathbb{V}, \mathcal{T}_I, \mathcal{T}_N, \mathcal{T}_G)$ *be a tritopological space, and let* $\mathcal{H} \subseteq \mathbb{V}$. $\mathcal{H}$ *is called an* $\mathcal{T}_{ING}$*-tri-open set in* $\mathbb{V}$ *if* $\mathcal{H}$ *is open in the ING-induced topology. (i.e* $\mathcal{H} \in \mathcal{T}_I \bigcap \mathcal{T}_N \bigcap \mathcal{T}_G$*).*

Definition 3.5. *Let* $(\mathbb{V}, \mathcal{T}_I, \mathcal{T}_N, \mathcal{T}_G)$ *be a tritopological space. Let* $\mathcal{F} \subseteq \mathbb{V}$. $\mathcal{F}$ *is called an* $\mathcal{T}_{ING}$*-tri-open set in* $\mathbb{V}$ *if* $\mathcal{F}$ *is closed in the ING-induced topology.*

Remark 3.6. *The Independent Topology* $\mathcal{T}_I$ *of a graph* $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ *is discrete if and only if* $I_u \nsubseteq I_v$ *and* $I_v \nsubseteq I_u$ *for every distinct pair of vertices* $v, u \in \mathbb{V}$ *[6], from [1] the Graphic Topology* $\mathcal{T}_G$ *is discrete if and only if* $A_u \nsubseteq A_v$ *and* $A_v \nsubseteq A_u$ *for every distinct pair of vertices* $v, u \in \mathbb{V}$*, and by [7] the Non-Incidence Topology* $\mathcal{T}_I$ *is discrete if* $d(v) > 2$ *for every* $v \in \mathbb{V}$*. Therefore, if a graph* $\mathbb{G}$ *satisfies the stipulations above, then* $\mathcal{T}_I, \mathcal{T}_N$ *and* $\mathcal{T}_G$ *are identical topologies (all are discrete topologies).*

Remark 3.7. *It is easy to see that the ING-induced topological space* $(\mathbb{V}, \mathcal{T}_{ING})$ *of a cycle* $\mathbb{C}_n$ *when* $n \geq 5$ *represent a discrete topological space, because the Graphic Topology is discrete topology in* $n \geq 3$ *(but not discrete just in* $n = 4$*), this note not mention in the paper [1], Independent Topology is discrete topological space in* $n \geq 4$ *[6], but we notice that Non-Incidence Topology is discrete topological space in* $n \geq 3$ *[7].*

Example 3.8. *Let* $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ *be a cycle* C_5 *as in Figure 1 such that:*
$\mathbb{V} = \{v_1, v_2, v_3, v_4, v_5\}, \mathbb{E} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5\}$. *Then,*
The Graphic Topology
$S_G = \{\varphi, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_1, v_3\}, \{v_2, v_5\}, \{v_2, v_4\}, \{v_3, v_5\}, \{v_1, v_4\}\}$

$$\begin{aligned}\mathcal{T}_G = \{&\varphi, \mathbb{V}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_1, v_3\}, \{v_2, v_5\}, \{v_2, v_4\}, \{v_3, v_5\}, \{v_1, v_4\}, \{v_1, v_2\}\\ &, \{v_1, v_5\}, \{v_1, v_2, v_4\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_5\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_1, v_2, v_3\}\\ &, \{v_2, v_3, v_5\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_2, v_3, v_5\}, \{v_1, v_3, v_4\}, \{v_4, v_5\}, \{v_2, v_4, v_5\}, \{v_1, v_3, v_4\}\\ &, \{v_3, v_4, v_5\}, \{v_1, v_3, v_5\}, \{v_1, v_4, v_5\}, \{v_2, v_5, v_1, v_3\}, \{v_2, v_4, v_3, v_5\}, \{v_3, v_5, v_2\}, \{v_1, v_2, v_4, v_5\}\\ &, \{v_1, v_2, v_3, v_4\}, \{v_5, v_1, v_3\}\{v_4, v_1, v_3\}, \{v_2, v_3, v_4, v_5\}, \{v_1, v_3, v_4, v_5\}\}\end{aligned}$$

The Independent Topology
$S_I = \{\varphi, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_1, v_5\}, \{v_1, v_2\}, \{v_2, v_3\}\}$

$$\begin{aligned}\mathcal{T}_I = \{&\varphi, \mathbb{V}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_1, v_3\}, \{v_2, v_5\}, \{v_2, v_4\}, \{v_3, v_5\}, \{v_1, v_4\}, \{v_1, v_2\}\\ &, \{v_1, v_5\}, \{v_1, v_2, v_4\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_5\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_1, v_2, v_3\}\\ &, \{v_2, v_3, v_5\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_2, v_3, v_5\}, \{v_1, v_3, v_4\}, \{v_4, v_5\}, \{v_2, v_4, v_5\}, \{v_1, v_3, v_4\}\\ &, \{v_3, v_4, v_5\}, \{v_1, v_3, v_5\}, \{v_1, v_4, v_5\}, \{v_2, v_5, v_1, v_3\}, \{v_2, v_4, v_3, v_5\}, \{v_3, v_5, v_2\}, \{v_1, v_2, v_4, v_5\}\\ &, \{v_1, v_2, v_3, v_4\}, \{v_5, v_1, v_3\}\{v_4, v_1, v_3\}, \{v_2, v_3, v_4, v_5\}, \{v_1, v_3, v_4, v_5\}\}\end{aligned}$$

The Non-Incidence Topology

$$\begin{aligned}S_N = \{&\varphi, \{v_3, v_4, v_5\}, \{v_1, v_3, v_5\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_2, v_4, v_5\}, \{v_3, v_5\}\\ &, \{v_3\}, \{v_4\}, \{v_5\}, \{v_1, v_3\}, \{v_1\}, \{v_2, v_4\}, \{v_1, v_2\}, \{v_2\}, \{v_4, v_5\}\}\end{aligned}$$

$$\begin{aligned}\mathcal{T}_N = \{&\varphi, \mathbb{V}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_1, v_3\}, \{v_2, v_5\}, \{v_2, v_4\}, \{v_3, v_5\}, \{v_1, v_4\}, \{v_1, v_2\}\\ &, \{v_1, v_5\}, \{v_1, v_2, v_4\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_5\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_1, v_2, v_3\}\\ &, \{v_2, v_3, v_5\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_2, v_3, v_5\}, \{v_1, v_3, v_4\}, \{v_4, v_5\}, \{v_2, v_4, v_5\}, \{v_1, v_3, v_4\}\\ &, \{v_3, v_4, v_5\}, \{v_1, v_3, v_5\}, \{v_1, v_4, v_5\}, \{v_2, v_5, v_1, v_3\}, \{v_2, v_4, v_3, v_5\}, \{v_3, v_5, v_2\}, \{v_1, v_2, v_4, v_5\}\\ &, \{v_1, v_2, v_3, v_4\}, \{v_5, v_1, v_3\}\{v_4, v_1, v_3\}, \{v_2, v_3, v_4, v_5\}, \{v_1, v_3, v_4, v_5\}\}\end{aligned}$$

Then, $\mathcal{T}_{ING} = \mathcal{T}_I \cap \mathcal{T}_N \cap \mathcal{T}_G$, and the ING-induced topology is:

$$\begin{aligned}\mathcal{T}_{ING} = \{&\varphi, \mathbb{V}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_1, v_3\}, \{v_2, v_5\}, \{v_2, v_4\}, \{v_3, v_5\}, \{v_1, v_4\}, \{v_1, v_2\}\\ &, \{v_1, v_5\}, \{v_1, v_2, v_4\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_5\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_1, v_2, v_3\}\\ &, \{v_2, v_3, v_5\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_2, v_3, v_5\}, \{v_1, v_3, v_4\}, \{v_4, v_5\}, \{v_2, v_4, v_5\}, \{v_1, v_3, v_4\}\\ &, \{v_3, v_4, v_5\}, \{v_1, v_3, v_5\}, \{v_1, v_4, v_5\}, \{v_2, v_5, v_1, v_3\}, \{v_2, v_4, v_3, v_5\}, \{v_3, v_5, v_2\}, \{v_1, v_2, v_4, v_5\}\\ &, \{v_1, v_2, v_3, v_4\}, \{v_5, v_1, v_3\}\{v_4, v_1, v_3\}, \{v_2, v_3, v_4, v_5\}, \{v_1, v_3, v_4, v_5\}\}\end{aligned}$$

It is clear that the ING-induced topological space represent a discrete topological space.

Remark 3.9. *The ING-induced topological space $(\mathbb{V}, \mathcal{T}_{ING})$ of a of a path $\mathbb{P}_n$ when $n \geq 3$ is not discrete. because, the Graphic Topology of $\mathbb{P}_n$ is not discrete [1]. And Independent Topology of $\mathbb{P}_n$ is not discrete [6]. The Non- Incidence Topology of the path $\mathbb{P}_n$ is not discrete [7].*

Example 3.10. *Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ be a path $\mathbb{P}_4$ as in Figure(2) such that:*
$\mathbb{V} = \{v_1, v_2, v_3, v_4\}, \mathbb{E} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$. *Then,*
The Graphic Topology
$S_G = \{\varphi, \{v_2\}, \{v_3\}, \{v_1, v_3\}, \{v_2, v_4\}\}$
$\mathcal{T}_G = \{\varphi, \mathbb{V}, \{v_2\}, \{v_3\}, \{v_1, v_3\}, \{v_2, v_4\}, \{v_2, v_3\}, \{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}\}$
The Independent Topology
$S_I = \{\varphi, \{v_1\}, \{v_4\}, \{v_3, v_4\}, \{v_1, v_2\}\}$

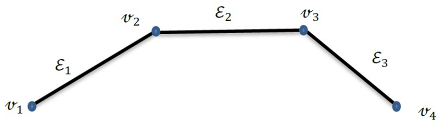

Figure 2:

$\mathcal{T}_I = \{\varphi, \mathbb{V}, \{v_1\}, \{v_4\}, \{v_3, v_4\}, \{v_1, v_2\}, \{v_1, v_3, v_4\}, \{v_1, v_2, v_4\}\}$
The Non-Incidence Topology
$S_N = \{\varphi, \{v_1\}, \{v_4\}, \{v_3, v_4\}, \{v_1, v_4\}, \{v_1, v_2\}\}$
$\mathcal{T}_N = \{\varphi, \mathbb{V}, \{v_1\}, \{v_4\}, \{v_3, v_4\}, \{v_1, v_4\}, \{v_1, v_2\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}\}$
Then, $\mathcal{T}_{ING} = \mathcal{T}_I \cap \mathcal{T}_N \cap \mathcal{T}_G$ *and the ING-induced topology* $\mathcal{T}_{ING} = \{\varphi, \mathbb{V}\}$

Remark 3.11. *The ING-induced topological space* $(\mathbb{V}, \mathcal{T}_{ING})$ *of a complete bipartite* $\mathbb{K}_{n,m}$ *is not discrete. Because, the Graphic Topology is equal to* $\{\varphi, \mathbb{V}, \mathcal{A}, \mathcal{B}\}$*, where* $\mathcal{A}$ *and* $\mathcal{B}$ *are partite sets of* $\mathbb{K}_{n,m}$*, but discrete topology in the Independent Topology and the Non-Incident Topology. Then the ING-induced topological space of complete bipartite* $\mathbb{K}_{n,m}$ *is not discrete and is equal to* $\{\varphi, \mathbb{V}, \mathcal{A}, \mathcal{B}\}$*.*

Remark 3.12. *The ING-induced topological space* $(\mathbb{V}, \mathcal{T}_{ING})$ *of incomplete bipartite* $\mathbb{K}_{n,m}$ *is not discrete. Because, the Graphic Topology is equal to* $\{\varphi, \mathbb{V}, \mathcal{A}, \mathcal{B}\}$*, where* $\mathcal{A}$ *and* $\mathcal{B}$ *are partite sets of* $\mathbb{K}_{n,m}$*, and not discrete topology in the Independent Topology and the Non-Incident Topology. Then the ING-induced topological space of incomplete bipartite* $\mathbb{K}_{n,m}$ *is not discrete.*

Remark 3.13. *The ING-induced topological space* $(\mathbb{V}, \mathcal{T}_{ING})$ *of tree is not discrete. Because, the Graphic Topology, Independent Topology and Non-Incidence Topology and is not discrete.*

Remark 3.14. *The ING-induced topological space* $(\mathbb{V}, \mathcal{T}_{ING})$ *of a graph has two components is not discrete. Because, the Graphic Topology, Independent Topology and Non-Incidence Topology is not discrete. And ING-induced topology graph has two components is as in the Independent Topology, i.e.* $\mathcal{T}_{ING} = \{\varphi, \mathbb{V}, \{vertices\ of\ first\ component\}, \{vertices\ of\ second\ component\}\}$
Also we noticed that when graph has more two components is not discrete.

4. ING -Alexandroff Tritopological Space

Alexandroff spaces became much more important because of their use in field of digital topology. Alexandroff space is a topological space, in which arbitrary intersection of open sets is open (or arbitrary union of closed sets is closed) equivalently, we say that each singleton has minimal neighborhood base. A tritopological space is simply a non-empty set X which is associated with three arbitrary topologies $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ on X as $(\mathbb{X}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$.
In this section, we mean by a tritopological space $(\mathbb{V}, \mathcal{T}_I, \mathcal{T}_N, \mathcal{T}_G)$ is an Alexandroff tritopological space, satisfy the stronger condition namely, arbitrary intersection of members of sub-basis are open in their topologies as defined below.

Definition 4.1. *The Graphic Topology, Independent Topology and Non-Incidence Topology on* $\mathbb{V}$ *forms the tritopological space* $(\mathbb{V}, \mathcal{T}_I, \mathcal{T}_N, \mathcal{T}_G)$*. This tritopological Space is called ING-Alexandroff tritopological space if and only if arbitrary intersection of members of* S_I, S_I *and* S_G *are open in* $\mathcal{T}_I, \mathcal{T}_N$ *and* $\mathcal{T}_G$ *respectively on* $\mathbb{V}$*, where* S_I *is the sub-basis for an Independent Topology* $\mathcal{T}_I, S_N$ *is the sub-basis for an Non-Incidence Topology* $\mathcal{T}_N$ *and* S_G *is the sub-basis for a Graphic Topology* $\mathcal{T}_G$*.*

Remark 4.2. *The Graphic topological space* $(\mathbb{V}, \mathcal{T}_G)$ *is an Alexandroff space [1], Independent topological space* $(\mathbb{V}, \mathcal{T}_I)$ *is an Alexandroff space [6], The Non-Incidence Topological space* $(\mathbb{V}, \mathcal{T}_N)$ *is an Alexandroff space [7].*

Proposition 4.3. *Let* $G = (\mathbb{V}, E)$ *be a simple graph. And* $(\mathbb{V}, \mathcal{T}_I, \mathcal{T}_N, \mathcal{T}_G)$ *be a tritopological space, then* $(\mathbb{V}, \mathcal{T}_{ING})$ *is an ING –Alexandroff topological space, where* $\mathcal{T}_{ING}$ *is the ING-induced topology from* $\mathcal{T}_I, \mathcal{T}_N$ *and* $\mathcal{T}_G$ *(i.e.* $\mathcal{T}_{ING} = \mathcal{T}_I \cap \mathcal{T}_N \cap \mathcal{T}_G$*).*

Proof . we have to prove that the arbitrary intersection of members of S_G, S_N and S_I is open in $\mathcal{T}_{ING}$. Let $M \subseteq \mathbb{V}$. If $v \in \cap_{u \in M} S_u$ (When S_u the arbitrary intersection of members of S_G, S_I and S_N) then $v \in S_u$ for each $u \in M$. Hence $u \in S_v$ for each $u \in M$ and so $M \subseteq S_v$, S_v and M are finite sets. This means that if M is infinite, then $\cap_{u \in M} S_u$ is empty,but if M is finite, then $\cap_{u \in M} S_u$ is the intersection of finitely many open sets and hence $\cap_{u \in M} S_u$ is open. □

5. Conclusions

In this paper a synthesis between tritopological theory and graph theory has been made. A tritopological space with any locally finite graph has been associated. Then some results and properties of this tritopological space have been studied in details. Furthermore, a fundamental step toward studying some results and properties of locally finite graphs by their corresponding tritopological spaces has been displayed.

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