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Solution of n-th order interval fuzzy differential equations using the backstepping method

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Abstract

There are two key points in this work as the main objectives. The first is how to convert n^{th} order fuzzy differential equation into a first-order system of fuzzy differential equations using the notion of upper and lower bounds of the fuzzy solution to constitute the so-called interval fuzzy solution. The second is to solve the obtained system from the first step using a powerful method (the backstepping method) to provide an asymptotically stable solution by applying direct methods of stability (Lyapunov direct method).

Keywords: Backstepping method, Fuzzy differential equations, Uncertainty interval, Control problems, Lyapunov functions, Quadratic from

1. Introduction

Modeling topics in science and engineering, as well as, in numerous physical systems, using fuzzy differential equations(FDEs)with uncertainty parameters may be so many considered as result, for modeling uncertainty, interval analysis and fuzzy set theory which have become important approaches in handling such problems. Sheldon, S. C., & Zadeh, L. A. in 1972 [27] introduced the concept of a fuzzy derivative while Dubois, D., & Prade, H. in 1978 [12] and then Puri, M. L., & Ralescu, D. A. in 2011 [22] developed the topic of FDEs which was followed by Kaleva O. in 1987 [19] who applied the extension principle and the most significant notions of fuzzy stability of linear and nonlinear systems and which difficult to study stability in fuzzy logic.

In the backstepping method, a construction procedure for Lyapunov functions were previously presendted, and several researchers as those in [2, 3, 4, 5] provided some numerical approaches for

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solving FDEs in which this solution has a wide range of applications in various fields of engineering mathematics and control theory [7, 28]. Looked at how to solve second-order fuzzy initial value problems, Bukley J.J. and Fenring I. in 2001 [9] investigated the n^{th}-order linear FDE, whereas Georgiou, D. N., Nieto, J. J., & Rodriguez-Lopez, R. in 2005 [14] examined n^{th}-order FDE and prove the existence and uniqueness of solutions to fuzzy initial value problems. Chakraverty, S., & Behera, D. in 2013 [10] solved two crisp systems of linear equations, this work proposes a new and straightforward method for solving of real-world linear fuzzy equations where the matrix of coefficients is considered to be real crisp numbers and then in 2012, the same authors in [8] considers polynomial parametric equation obtained from of fuzzy numbers, this article offered two innovative and simple solution methods for solving system of linear fuzzy equations with fuzzy coefficients and crisp variables. Kargar, R., Allahviranloo, T., Rostami-Malkhalifeh, M., & Jahanshaloo, G. R. in 2014 [28] proposed a new method for solving systems of fuzzy linear equations with crisp coefficients and fuzzy or interval non homogeneous term in which a realistic approach is described in detail, as well as, certain conditions for the existence of a fuzzy or interval solution of linear system. Jafari R. and Razvarz in 2018 [17] proposed the fuzzy sumudu transform approach to find an approximate solutions of FDEs and to describe the nature of such transforms, significant theorems are proposed and proved. Seilcala S. in 1987 [25] investigated and developed various analytical methods for solving some types of first order linear and FDEs with fuzzy initial conditions using the homotopy analytical approach and the approximate Padè method. Saber et al. in [23] obtains correct fuzzy approximate-analytical answers that were extremely close to fuzzy exact-analytical solutions [18] They are start with a generalized concept of higher-order differentiability for fuzzy functions were after that, inset by Khastan A. [18] to interpret Nth-order fuzzy differential equations new definitions for fuzzy differential equation solutions are presented [24] This research uses the associated integral forms to examine a large number of solutions to n-th-order FDEs in Banach space and also proves an existence and uniqueness theorem [15] In analytic methods are used by GnoD.N. and Shang D. to solve n-th order FDEs with fuzzy initial conditions utilizing the eigenvalue-eigenvector method, A variety of numerical scenarios are also used to clarify the approach. Kaleva O. in 1987 [19] used the n-th derivative theorem to generalize the fuzzy Laplace transformation for evaluating the n-th derivative of a fuzzy valued function, and it is also used in an analytical solution method for the solution of an n^{th} order fuzzy initial value problem under the strongly generalized differentiability concept.

In this paper a modified approach will be followed for stabilizing and solving systems of FODEs using a hybrid approach between the backstepping method and the method of solving systems of first order FDEs in terms of lower and upper solutions. This approach shows, its reliability and efficiency for studying and solving such types of problems.

2. Prelimiries

In this section, we shall go over some basic definitions and notations that we will need for this work.

Definition 2.1. *[8] A fuzzy number $\widetilde{u}$ is a fuzzy set with member ship $\mu_{\widetilde{u}} : \mathbb{R} \to I = [0, 1]$, where $\mathbb{R}$ is the set of real numbers which satisfies the following:*

1. *$\mu_{\widetilde{u}}$ is the upper semi continuous.*
2. *$\mu_{\widetilde{u}}$ is fuzzy convex; that is, $\mu_{\widetilde{u}}(\lambda x + (1-\lambda)y) \geq min\{\mu_{\widetilde{u}}(x), \mu_{\widetilde{u}}(y)\}$ for all $x, y \in \mathbb{R}, \lambda \in [0, 1]$.*
3. *$\mu_{\widetilde{u}}$ is normal; that is, there exist $x_0 \in R$, such that $\mu_{\widetilde{u}}(x_0) = 1$.*

4. The support set of $\mu_{\widetilde{u}}$ *is the set* $\{x \in R, u(x) > 0\}$*and its closure is compact.*

Definition 2.2. *[10] A fuzzy number u in parametric form is a pair* $(\underline{u}(r), \overline{u}(r))$*,* $0 \leq r \leq 1$*, which satisfies the following.*

1. $\underline{u}(r)$ *is a bounded monotonic increasing left continuous function.*
2. $\overline{u}(r)$ *is a bounded monotonic decreasing left continuous function.*
3. $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

Note 1. [11] Let $\widetilde{X} = [\underline{x}, \overline{x}]$, $\widetilde{Y} = [\underline{y}, \overline{y}]$, where $\underline{x}, \overline{x}, \underline{y}$ and $\overline{y}$ represent the lower and upper bounds intervals of fuzzy functions $\widetilde{X}$ and $\widetilde{Y}$ respectively, then the following arithmetic operations are fulfilled:

1. $\widetilde{X} = \widetilde{Y}$ if and only if $\underline{x} = \underline{y}$ and $\overline{x} = \overline{y}$.
2. $\widetilde{X} \mp \widetilde{Y} = [\underline{x} \mp \underline{y}, \overline{x} \mp \overline{y}]$.
3. If k is any real number, then $\begin{cases} [k\underline{x}, k\overline{x}] & if k \geq 0 \\ [k\overline{x}, k\underline{x}] & if k < 0 \end{cases}$

Now, it may be worth mention that FDEs may be modeled using interval fuzzy function method as it is followed in [11]. For this purpose, consider the linear interval-valued FDEs:

$$\widetilde{y'} = \widetilde{f}(t, y), \widetilde{y}(t_0) = \widetilde{y_0}$$

Where:

$$\widetilde{f}(t, y) = [\underline{f}(t, y), \overline{f}(t, y)] \; for \; \widetilde{y'} = [\underline{y'}, \overline{y'}], \widetilde{y_0} = [\underline{y_0}, \overline{y_0}]$$

Thus, by using the differential form in [19] to the last FODEs we arrive at:

$$min\{\underline{y'}, \overline{y'}\} = \underline{f}(t, y) \; and \; max\{\underline{y'}, \overline{y'}\} = \overline{f}(t, y)$$

In terms of the starting conditions, $\underline{y}(t_0) = \underline{y}_0, \overline{y}(t_0) = \overline{y}_0$, we obtain two situations as follows:

Case I: If $\underline{y}'(t) \leq \overline{y}'(t)$, then the outcome of a hypothetical differential equation are

$$\underline{y}'(t) = \underline{f}(t, y) \; and \; \overline{y}'(t) = \overline{f}(t, y)$$

with starting conditions $\underline{y}(t_0) = \underline{y}_0, \overline{y}(t_0) = \overline{y}_0$.

Case II: If $\underline{y}'(t) \geq \overline{y}'(t)$, Then the differential equations resulting are

$$\underline{y}'(t) = \overline{f}(t, y) \; and \; \overline{y}'(t) = \underline{f}(t, y)$$

with starting conditions $\underline{y}(t_0) = \underline{y}_0, \overline{y}(t_0) = \overline{y}_0$.

3. Problem statement

Consider the n^{th} – order linear FDE with constant coefficients:

$$\widetilde{y}^{(n)}(t) + C_{n-1}\widetilde{y}^{(n-1)}(t) + \cdots + C_1\widetilde{y}^{'}(t) + C_0\widetilde{y}(t) = \widetilde{R}(t) \tag{1}$$

with the following initial conditions:

$$\widetilde{y}(0) = \widetilde{b}_0, \widetilde{y}^{'}(0) = \widetilde{b}_1, \ldots, \widetilde{y}^{n-1}(0) = \widetilde{b}_{n-1}$$

where ci's, for all $i = 0, 1, \ldots, n-1$ are real constants, $\widetilde{b}_i^{\prime}s$ are fuzzy interval values, with interval terms of lower and upper bounds, and $\widetilde{R(t)}$ is any given function. Equation (1) can be written as:

$$[\underline{y}^{(n)}(t), \overline{y}^{(n)}(t)] + c_{n-1}[\underline{y}^{(n-1)}(t), \overline{y}^{(n-1)}(t)] + \cdots + c_1[\underline{y}^{'}(t), \overline{y}^{'}(t)] - c_0[\underline{y}(t), \overline{y}(t)] = [\underline{R}(t), \overline{R}(t)] \tag{2}$$

with the interval initial conditions:

$$[\underline{y}(0), \overline{y}(0)] = [\underline{b}_0, \overline{by}_0], [\underline{y}^{'}(0), \overline{y}^{'}(0)] = [\underline{b}_1, \overline{b}_1], \ldots, [\underline{y}^{n-1}(0), \overline{y}^{n-1}(0)] = [\underline{b}_{n-1}, \overline{b}_{n-1}]$$

Equation(2) has three different situations that come as a result of the connected differential equations with coefficients' sign. **Case 1.** If all the coefficients c i $'s$, $i = 0, 1, \ldots, n-1$ are positive then equation (2) may be divided into two equations using fuzzy interval analysis, for the lower and upper cases, as follows:

$$\underline{y}^{(n)}(t) + c_{n-1}\underline{y}^{(n-1)}(t) + \cdots + c_1\underline{y}^{'}(t) - c_0\underline{y}(t) = \underline{R}(t) \tag{3}$$

with the initial conditions, $\underline{y}(0) = \underline{b}_0, \underline{y}^{'}(0) = \underline{b}_1, \ldots, \underline{y}^{n-1}(0) = \underline{b}_{n-1}$.

and

$$\overline{y}^{(n)}(t) + c_{n-1}\overline{y}^{(n-1)}(t) + \cdots + c_1\overline{y}^{'}(t) - c_0\overline{y}(t) = \overline{R}(t) \tag{4}$$

with the initial conditions $\overline{y}(0) = \overline{b}_0, \overline{y}^{'}(0) = \overline{b}_1, \ldots, \overline{y}^{n-1}(0) = \overline{b}_{n-1}$. **Case 2.** If all the coefficients ci's are negative, then equation (2) is divided into two equations as follows:

$$\underline{y}^{(n)}(t) + c_{n-1}\overline{y}^{(n-1)}(t) + \cdots + c_1\overline{y}^{'}(t) - c_0\overline{y}(t) = \underline{R}(t) \tag{5}$$

with the initial conditions $\overline{y}(0) = \overline{b}_0, \overline{y}^{'}(0) = \overline{b}_1, \ldots, \overline{y}^{n-1}(0) = \overline{b}_{n-1}$.

and

$$\overline{y}^{(n)}(t) + c_{n-1}\underline{y}^{(n-1)}(t) + \cdots + c_1\underline{y}^{'}(t) - c_0\underline{y}(t) = \overline{R}(t) \tag{6}$$

with initial conditions, $\underline{y}(0) = \underline{b}_0, \underline{y}^{'}(0) = \underline{b}_1, \ldots, \underline{y}^{n-1}(0) = \underline{b}_{n-1}$.

Case 3. If some of the coefficients, for instance, are positive say $c_{n-1}, c_{n-2}, \ldots, c_{n-m}, m < n, m, n \in \mathbb{N}$ and some of them are negative, say $c_{n-m-1}, c_{n-m-2}, \ldots, c_1, c_0$ and so equation (2) is divided into two equations, as follows:

$$\overline{y}^{(n)}(t) + c_{n-m-1}\overline{y}^{(n-m-1)}(t) + \cdots + c_1\overline{y}^{'}(t) + c_0\overline{y}(t) = \underline{R}(t) \tag{7}$$

with initial conditions $\overline{y}(0) = \overline{b}_0, \overline{y}^{'}(0) = \overline{b}_1, \ldots, \overline{y}^{n-m-1}(0) = \overline{b}_{n-m-1}$.

$$\overline{y}^{(n)}(t) + c_{n-1}\overline{y}^{(n-1)}(t) + c_{n-2}\overline{y}^{(n-2)}(t) + \cdots + c_{n-m}\overline{y}^{n-m}(t) = \overline{R}(t) \tag{8}$$

with initial conditions $\overline{y}^{(n-m)}(0) = \overline{b}_{n-m}, \ldots, \overline{y}^{(n-2)}(0) = \overline{b}_{n-2}, \overline{y}^{(n-1)}(0) = \overline{b}_{n-1}$.

4. Transformation of the n^{th}-Order Linear FDEs in Linear System of 2n-Equations of the first order

Now, let us look at how to convert equation (1) into a system of the first-order equations and in order to apply then after the backstepping method to stabilize and solve the transformed system. Assume that all of the $c_i's$, $i = 0, 1, \dots, n$; are positive, then:

$$\widetilde{y}^{(n)}(t) + c_{n-1}\widetilde{y}^{(n-1)}(t) + \cdots + c_1\widetilde{y}'(t) + c_0\widetilde{y}(t) = \widetilde{R}(t)$$

Then upon rewriting the above equations (2) and applying case 1, we have by letting:

$$\begin{aligned} &\underline{y}_1 = \underline{y}, \quad and \quad \overline{y}_1 = \overline{y} \\ &\underline{y}_2 = \underline{y}' = \underline{y}_1' \quad and \quad \overline{y}_2 = \overline{y}' = \overline{y}_1' \end{aligned}$$

continue up to n variable, to get:

$$\begin{aligned} &\underline{y}_{n-1} = \underline{y}^{n-2} = \underline{y}'_{n-2} \quad and \quad \overline{y}_{n-1} = \overline{y}^{n-2} = \overline{y}'_{n-2} \\ &\underline{y}'_n = \underline{y}^{n-1} = \underline{y}'_{n-1} \quad and \quad \overline{y}_n = \overline{y}^{n-1} = \overline{y}_{n-1} \end{aligned}$$

Hence, the first order system of 2n-equations in terms of the lower and upper solutions $\underline{y}$ and $\overline{y}$ respectively will take the form:

$$\begin{aligned} &\underline{y}'_1 = \underline{y}_2 \\ &\overline{y}'_1 = \overline{y}_2 \\ &\vdots \\ &\underline{y}'_n = -c_{(n-1)}\overline{y}^n(t) - \cdots - c_1\overline{y}^2(t) - c_0\overline{y}^1(t) + \underline{R}(t) \\ &\overline{y}'_n = -c_{(n-1)}\underline{y}^n(t) - \cdots - c_1\underline{y}^2(t) - c_0\underline{y}^1(t) + \overline{R}(t) \end{aligned} \tag{9}$$

with initial conditions:

$$[\underline{y}_1(0), \overline{y}_1(0)] = [\underline{b}_0, \overline{b}_0], [\underline{y}_2(0), \overline{y}_2(0)] = [\underline{b}_1, \overline{b}_1], \dots, [\underline{y}_n(0), \overline{y}_n(0)] = [\underline{b}_{n-1}, \overline{b}_{n-1}]$$

Or in matrix form:

$$\begin{bmatrix} \underline{y}'_1 \\ \overline{y}'_1 \\ \vdots \\ \underline{y}'_n \\ \overline{y}'_n \end{bmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -c_0 & -c_1 & -c_2 & \cdot & \cdot & \cdot & \cdot & -c_{n-1} \\ -c_0 & -c_1 & -c_2 & \cdot & \cdot & \cdot & \cdot & -c_{n-1} \end{pmatrix} \begin{bmatrix} \underline{y}_1 \\ \overline{y}_1 \\ \underline{y}_2 \\ \overline{y}_2 \\ \vdots \\ \underline{y}_n \\ \overline{y}_n \end{bmatrix}$$

After this proposed transform, we can now apply the backstepping method to solve system (9) to get asymptotically stable solutions.

5. Applying the backstepping method for interval system of FDEs

Backstepping is a control theory technique invented in the 1990s by Petar V. Kokotovic and colleagues for generating stabilizing controllers for a certain type of nonlinear dynamical systems [20]. These systems are decomposed of subsystems radiating out from an irreducible subsystem that can be stabilized in a different method. Because of its recursive structure, the designer can start with a known-stable system and "back out" new controllers to stabilize each outer subsystem. The procedure is completed when the last external control is attained [21]. Backstepping method is a valuable and strong technique for constructing linear and nonlinear controllers. The main idea behind the backstepping method is to build controllers by treating some of the states as "virtual controls" and using them as intermediate control variables. The presence of such controllers in the nonlinear system that are input values in any subsystems or equations under investigation should be founded on the fact that these systems or equations are smooth [29, 26].

The semi-discretized backstepping control technique is used to stabilize the partial differential equation with fractional order $0 < q < 1$ for stability [16].

By turning nonlinear delay differential equations into systems of ordinary differential equations, the backstepping method for stabilizing and solving systems of ordinary and partial differential equations will be applied to systems of delay differential equations in this research [13].

The adaptive backstepping approach based on constructing Lyapunove functions in quadratic form that will be integrated with the so called α level sets in fuzzy set theory in this study, which is regarded as a component of the control theory of FDEs. The suggested method's dependability is demonstrated by solving two cases, both linear and nonlinear, in which the original system may be unstable [1].

Consider system (9) and assume for simplicity of notations:

$$\begin{aligned}
&x_1(t) = \underline{y}_1(t), x_2(t) = \overline{y}_1(t)\\
&x_3(t) = \underline{y}_2(t), x_4(t) = \overline{y}_2(t)\\
&x_5(t) = \underline{y}_3(t), x_6(t) = \overline{y}_3(t)\\
&\vdots\\
&x_{2n-1}(t) = \underline{y}_n(t), x_{2n}(t) = \overline{y}_n(t)
\end{aligned}$$

So, system (9) becomes;

$$\begin{aligned}
&x_1'(t) = x_3(t)\\
&x_2'(t) = x_4(t)\\
&x_3'(t) = x_5(t)\\
&\vdots\\
&x_{2n-1}'(t) = -c_{2n-1}x_{2n-1}(t) - \cdots - c_1x_4(t) - c_0x_2(t) + \underline{R}(x)\\
&x_{2n}'(t) = -c_{2n-1}x_{2n}(t) - \cdots - c_1x_3(t) - c_0x_1(t) + \overline{R}(x)
\end{aligned} \tag{10}$$

To apply the backstepping method on system (10) introduce the controller functions $u_i, i = 1, 2, \ldots, 2n$

as followes;

$$\begin{aligned}
x_1'(t) &= x_3(t) + u_1(x_1, x_2, \ldots, x_{2n}) \\
x_2'(t) &= x_4(t) + u_2(x_1, x_2, \ldots, x_{2n}) \\
x_3'(t) &= x_5(t) + u_3(x_1, x_2, \ldots, x_{2n}) \\
&\vdots \\
x_{2n-1}'(t) &= -c_{2n-1}x_{2n-1}(t) - \cdots - c_1x_4(t) - c_0x_2(t) + \underline{R}(x) + u_{2n-1}(x_1, x_2, \ldots, x_{2n}) \\
x_{2n}'(t) &= -c_{2n-1}x_{2n}(t) - \cdots - c_1x_3(t) - c_0x_1(t) + \overline{R}(x) + u_{2n}(x_1, x_2, \ldots, x_{2n})
\end{aligned} \tag{11}$$

where $t \in [0, 1]$ and the functions $u_i : C[0, 1] \to \mathbb{R}^+, i = 1, 2, \ldots, 2n$ are the controller input functions, and $x_1, x_2, \ldots, x_n$ are the system's state variables. If the control Lyapunov functions of system (11) are assumed to be of quadratic form, say:

$$V_1(z_1) = p_1z_1p_1, p_1 \in \mathbb{R}^+, z_1(t; r) \in \mathbb{R}^+$$

and for $i = 1, 2, \ldots, 2n$, define:

$$V_i(z_1, z_2, \ldots, z_i) = V_{i-1}(z_1, z_2, \ldots, z_i - 1) + z_ip_1z_i, p_i \in \mathbb{R}^+$$

where $z_1(t) = x_1(t), z_i(t) = x_i(t) - \alpha_{i-1}(x_i(t))\ forall\ i = 1, 2, \ldots, 2n$; and

$$\alpha_i(z_1, z_2, \ldots, z_i) = f_i(z_1, z_2, \ldots, z_i, x_{i+1}, \ldots, x_{2n}) + u_i(x_1, x_2, \ldots, x_{2n}) - z_i(t)$$

Then the closed loop system (11) will then become asymptotically stable and solvable due to the nonlinear controller functions $u_1, u_2, \ldots, u_{2n}$.

Then try to find $u_1, u_2, \ldots, u_{2n}$ that stabilizes system (11) and for this purpose. the following steps can be taken to obtain the above:

Step1: Consider the first equation of system (11)

$$x_1'(t) = x_3(t) + u_1(x_1, x_2, \ldots, x_{2n}) \tag{12}$$

Assume that, $z_1(t) = x_1(t)$ then if differentiating this assumption with respect to t implies to:

$$z_1'(t) = x_1'(t) = x_3(t) + u_1(z_1, x_2, \ldots, x_{2n})$$

The Lyapunov function may be taken in quadratic form as :

$$V_1(z_1(t)) = p_1z_1(t)p_1, \qquad p_1 \in \mathbb{R}, z_1(t) \in \mathbb{R}^+$$

When this Lyapunov function is differentiated with respect to time t, we get:

$$\begin{aligned}
V_1'(z_1(t)) &= \frac{dV_1}{dt} \\
&= \frac{dV_1}{dz_1}\frac{dz_1}{dt} \\
&= -z_1(t)q_1z_1(t), \quad q_1 \in \mathbb{R}^+
\end{aligned}$$

Clear that $V_1'(z_1(t)) < 0$ is obvious, and the first equation of system (11) is asymptotically stable according to the Lyapunov stability theory.

Step 2: Define the error term as follows:

$$z_2(t) = x_2(t) - \alpha_1(z_1(t)), \quad z_2(t) \in \mathbb{R}$$

Then the new subsystem of system (3) is:

$$\begin{aligned} z_1'(t) &= x_3(t) + u_1(z_1, x_2, \ldots, x_{2n}) \\ z_2'(t) &= x_4(t) + u_2(z_1, z_2, \ldots, x_{2n}) + \alpha_1'(z_1(t)) \end{aligned} \tag{13}$$

Assume that $x_3(t)$ is the virtual controller to be chosen, and that $\alpha_2(z_1(t), z_2(t))$ is to be selected to make the above subsystem (13) asymptotically stable. The Lyapunove function can be taken of the following from:

$$V_2(z_1(t), z_2(t)) = V_1(z_1(t)) + z_2(t)p_2 z_2(t), \qquad p_2 \in \mathbb{R}^+, z_2(t) \in \mathbb{R}$$

The time derivative of V_2 is:

$$V_2' = -z_1(t)q_1 z_1(t) - z_2(t)q_2 z_2(t), \qquad q_1, q_2 \in \mathbb{R}^+$$

Then the asymptotical stability of subsystem (13) is ensured.

We arrive at the following $2n$-step by continuing in the same direction until we get the last equation of system (11).

Step 2n: The error term should be defined as follows:

$$z_{2n}(t) = x_{2n}(t) - \alpha_{2n-1}(z_1(t), z_2(t), \ldots, z_{2n-1}(t))$$

The system then becomes:

$$\begin{aligned} z_1'(t) &= f_1(z_1, x_2, \ldots, x_{2n}) + u_1(z_1, x_2, \ldots, x_{2n}) \\ z_2'(t) &= f_2(z_1, z_2, \ldots, x_{2n}) + \alpha_1'(z_1(t)) + u_2(z_1, z_2, \ldots, x_{2n}) \\ &\vdots \\ z_{2n}'(t) &= f_{2n}(z_1, z_2, \ldots, z_{2n}) + \alpha_{2n-1}'(z_1, z_2, \ldots, z_{2n}) + u_{2n}(z_1, z_2, \ldots, x_{2n}) \end{aligned} \tag{14}$$

The Lyapunove function is taken to be:

$$V_{2n}(z_1, z_2, \ldots, z_n) = V_{2n-1}(z_1, z_2, \ldots, z_{2n-1}) + z_{2n}(t)p_{2n}z_{2n}(t), \quad p_{2n} \in \mathbb{R}^+, z_{2n}(t) \in \mathbb{R}$$

As a result, V_{2n} has the following derivative:

$$V_{2n}' = -z_1(t)q_1 z_1(t) - z_2(t)q_2 z_2(t) - \cdots - z_{2n}(t)q_{2n}z_{2n}(t), \quad q_1, q_2, \ldots, q_{2n} \in \mathbb{R}^+$$

As a conclusion, system (11) is asymptotically stable.

6. Applications of the backstepping method

This section will provide illustrative examples of how the proposed strategy given in section 4 can be applied covering different cases of FDEs.

Example 1: Consider the FDE: $\widetilde{y}'' - 5\widetilde{y}' + 4\widetilde{y} = 0$ subject to the interval initial conditions $\widetilde{y}(0) \cong [0.2, 0.8]$ and $\widetilde{y}(0) \cong [0.8, 1.2]$.

Retiring the above FDE as a system of the first order differential equations in terms of the lower and upper solutions by letting $\underline{y}_1 = \underline{y}$ and $\overline{y}_1 = \overline{y}$ then $\underline{y_1}' = \underline{y}' = \underline{y_2}$ and $\overline{y_1}' = \overline{y}' = \overline{y_2}$, $\underline{y_2}' = \underline{y}''$ and $\overline{y_2}' = \overline{y}''$. So:

$$\begin{aligned} \underline{y_2}' - 5\overline{y_2} + 4\underline{y_1} &= 0 \\ \overline{y_2}' - 5\underline{y_2} + 4\overline{y_1} &= 0 \end{aligned}$$

Or,

$$\begin{aligned} \underline{y_2}' &= 5\overline{y_2} - 4\underline{y_1} \\ \overline{y_2}' &= 5\underline{y_2} - 4\overline{y_1} \end{aligned}$$

The system is;

$$\begin{aligned} \underline{y_1'} &= \underline{y_2} \\ \overline{y_1}' &= \overline{y_2} \\ \underline{y_2'} &= 5\overline{y_2} - 4\underline{y_1} \\ \overline{y_2'} &= 5\underline{y_2} - 4\overline{y_1} \end{aligned} \tag{15}$$

So, in matrix form:

$$\begin{bmatrix} \underline{y'}_1 \\ \overline{y'}_1 \\ \underline{y'}_2 \\ \overline{y'}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 0 & 5 \\ 0 & -4 & 5 & 0 \end{bmatrix} \begin{bmatrix} \underline{y}_1 \\ \overline{y}_1 \\ \underline{y}_2 \\ \overline{y}_2 \end{bmatrix}$$

To simplify the notations, assume that;

$$x_1(t) = \underline{y}_1(t), x_2(t) = \overline{y}_1(t), x_3(t) = \underline{y}_2(t), x_4(t) = \overline{y}_2(t), \ with\ t \in [0,1]$$

In terms of the new variables, system (15) looks like this:

$$\begin{aligned} x_1'(t) &= x_3(t) \\ x_2'(t) &= x_4(t) \\ x_3'(t) &= -4x_1(t) + 5x_4(t) \\ x_4'(t) &= -4x_2(t) + 5x_3(t) \end{aligned} \tag{16}$$

Now, consider the control functions $u_i, i = 1,2,3,4$ in each equation in system (16) in order to use the backstepping method:

$$\begin{aligned} x_1'(t) &= x_3(t) + u_1(x_1, x_2, x_3, x_4) \\ x_2'(t) &= x_4(t) + u_2(x_1, x_2, x_3, x_4) \\ x_3'(t) &= -4x_1(t) + 5x_4(t) + u_3(x_1, x_2, x_3, x_4) \\ x_4'(t) &= -4x_2(t) + 5x_3(t) + u_4(x_1, x_2, x_3, x_4) \end{aligned} \tag{17}$$

Now, to asymptotically stabilize system (17), consider the following steps:

Step1: Consider the first question of system (17)

$$x_1'(t) = x_3(t) + u_1(x_1, x_2, x_3, x_4)$$

Let $z_1(t) = x_1(t)$, and differentiate with respect to t implies:

$$z_1'(t) = x_1'(t) = x_3(t) + u_1(x_1, x_2, x_3, x_4) \tag{18}$$

It is assumed that the Lyapunov function is $V_1(z_1) = \dfrac{1}{2} z_1^2(t)$ and its derivative is

$$\begin{aligned} V_1'(t) = \frac{dv_1}{dt} = \frac{dv_1}{dz_1}\frac{dz_1}{dt} &= z_1(t) z_1'(t; r) \\ &= z_1(t; r)[x_3(t; r) + u_1(x_1, x_2, x_3, x_4; r)] \end{aligned}$$

Assume $u_1(x_1, x_2, x_3, x_4; r) = -x_3(t; r) - z_1(t; r)$, Then $V_1' = -z_1^2(t; r) < 0$ and as a result, equation (18) is asymptotically stable.

Step 2: Letting $\alpha_1(z_1; r) = 0$ and $z_2(t; r) = x_2(t; r) - \alpha_1(z; r)$, Then equation(2) is:

$$z_2'(t; r) = x_2'(t; r) = x_4(t; r) + u_2(x_1, x_2, x_3, x_4; r) \tag{19}$$

The Lyapunov function is assumed based on the proposed approach as follows:

$$V_2(z_1, z_2) = V_1(z_1) + \frac{1}{2} z_1^2(t)$$

Then differentiating V_2 with respect to t implies to:

$$\begin{aligned} V_2' &= V_1' + z_2(t) z_2'(t) \\ &= -z_1^2(t) + z_2[x_4(t) + u_2(x_1, x_2, x_3, x_4)] \end{aligned}$$

assume $u_2(x_1, x_2, x_3, x_4) = -x_4(t) - 2z_2(t)$, then $V_2' = - - z_1^2(t) - 2z_2^2(t)$.

hich is also negative definite function, and thus equation (19) is asymptotically stable.

Step 3: Define the error term between $x_3(t)$ and $\alpha_2(z_1, z_2)$ as:

$$z_3(t) = x_3(t) - \alpha_2(z_1, z_2)$$

Then

$$z_3'(t) = x_3'(t) = -4x_1(t) + 5x_4(t) + u_3(x_1, x_2, x_3, x_4) \tag{20}$$

The Lyapunov function is: $\mathrm{V}_3(z_1, z_2, z_3) = V_2 + \dfrac{1}{2} z_3^2(t)$ and the differentiation with respect to t will give:

$$\begin{aligned} V_3' &= V_2' + z_3(t) z_3'(t) \\ &= -z_1^2(t) - 2z_2^2(t) + z_3(t)[-4z_1(t) + 5x_4(t) + u_3(x_1, x_2, x_3, x_4)] \end{aligned}$$

assume that $u_3(x_1, x_2, x_3, x_4) = 4z_1(t) - 5x_4(t) - 3z_3(t)$ then, $V_3' = -z_1^2(t) - 2z_2^2(t) - 2z_3^2(t)$.

which is negative definite function, then equation (20) is asymptotically stable.

Step 4: Define $z_4(t) = x_4(t) - \alpha_3(z_1, z_2, z_3)$, and the system is:

$$\begin{aligned}
z_1^{'}(t) &= z_3(t) + u_1(x_1, x_2, x_3, x_4) \\
z_2^{'}(t) &= z_4(t) + u_2(x_1, x_2, x_3, x_4) \\
z_3^{'}(t) &= -4z_1(t) + 5z_4(t) + u_3(x_1, x_2, x_3, x_4) \\
z_4^{'}(t) &= -4z_2(t) + 5z_3(t) + u_4(x_1, x_2, x_3, x_4)
\end{aligned}$$

The Lyapannov function is:

$$V_4(z_1, z_2, z_3, z_4) = V_3(z_1, z_2, z_3) + \frac{1}{2}z_4^2(t)$$

The derivative of V_4 is:

$$\begin{aligned}
V_4^{'} &= V_3^{'} + z_4(t)z_4^{'}(t) \\
&= -z_1^2(t) - 2z_2^2(t) - 3z_3^2(t) + z_4(t)[-4z_2(t) + 5z_3(t) + u_4(x_1, x_2, x_3, x_4)]
\end{aligned}$$

assume that $u_4(x_1, x_2, x_3, x_4) = 4z_2(t) - 5z_3(t) - 4z_4(t)$, then $V_4^{'} = -z_1^2(t) - 2z_2^2(t) - 3z_3^2(t) - 4z_4^2(t), \quad Z_i \in \mathbb{R}^+$.

equation (4) of system (17) is then asymptotically stable because it is a negative definite function. since, $\alpha_1(z_1) = \alpha_2(z_1, z_2) = \alpha_3(z_1, z_2, z_3) = 0$ and

$$\begin{aligned}
z_1(t) = x_1(t) \quad &, \quad z_2(t) = x_2(t) \\
z_3(t) = x_3(t) \quad &, \quad z_4(t) = x_4(t)
\end{aligned}$$

then as a result, the control functions are:

$$\begin{aligned}
u_1(x_1, x_2, x_3, x_4) &= -x_3(t) - x_1(t) \\
u_2(x_1, x_2, x_3, x_4) &= -x_4(t) - 2x_2(t) \\
u_3(x_1, x_2, x_3, x_4) &= 4x_1(t) - 5x_4(t) - 3x_3(t) \\
u_4(x_1, x_2, x_3, x_4) &= 4x_2(t) - 5x_3(t) - 4x_4(t)
\end{aligned}$$

we get the following transformed system by inserting u_1, u_2, u_3 and u_4 in system (17).

$$x_1^{'}(t) = -x_1(t)x_2^{'}(t) = -2x_2(t)x_3^{'}(t) = -3x_3(t)x_4^{'}(t) = -4x_4(t)$$

and, back to the hypotheses;

$$x_1(t) = \underline{y_1}(t), x_2(t) = \overline{y_1}(t), x_3(t) = \underline{y_2}(t), x_4(t) = \overline{y_2}(t),$$

So,

$$\begin{aligned}
\underline{y_1}^{'}(t) &= -\underline{y_1}(t) \\
\overline{y_1^{'}}(t) &= -2\overline{y_1}(t) \\
\underline{y_2^{'}}(t) &= -3\underline{y_2}(t) \\
\overline{y_2^{'}}(t) &= -4\overline{y_2}(t)
\end{aligned}$$

Then the following transformed system in matrix form is obtained:

$$\begin{bmatrix} \underline{y}'_1 \\ \overline{y}'_1 \\ \underline{y}'_2 \\ \overline{y}'_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} \underline{y}_1 \\ \overline{y}_1 \\ \underline{y}_2 \\ \overline{y}_2 \end{bmatrix}$$

Because the coefficient matrix is diagonal, its eigenvalues are $-1, -2, -3, -4$ which are distinct and negative. Thus the system becomes asymptotically stable. In addition, as a result of this, the solution with the initial conditions are:

$$\underline{y_1}(t;r) \cong 0.2e^{-t}, \underline{y_2}(t;r) = 0.8e^{-3t}, \overline{y_1}(t;r) = 0.8e^{-2t}, \overline{y_2}(t;r) = 1.2e^{-4t}$$

and the control functions are given by:

$$\begin{aligned} u_1(t) &= -\underline{y_2}(t) - \underline{y_1}(t) \\ u_2(t) &= -\overline{y_2}(t) - 2\overline{y_1}(t) \\ u_3(t) &= 4\underline{y_1}(t) - 9\overline{y_2}(t) \\ u_4(t) &= 4\overline{y_1}(t) - 8\underline{y_2}(t) \end{aligned}$$

The fuzzy solutions are:

$$\begin{aligned} \widetilde{y_1}(t;r) &= [\underline{y_1}(t;r), \overline{y_1}(t;r)] = [0.2e^{-t}, 0.8e^{-2t}] \\ \widetilde{y_2}(t;r) &= [\underline{y_2}(t;r), \overline{y_2}(t;r)] = [0.8e^{-3t}, 1.2e^{-4t}] \end{aligned}$$

Figures 1 and 2 presents the fuzzy solutions $\widetilde{y_1}$ and $\widetilde{y_2}$, respectively in terms of lower and upper solutions.

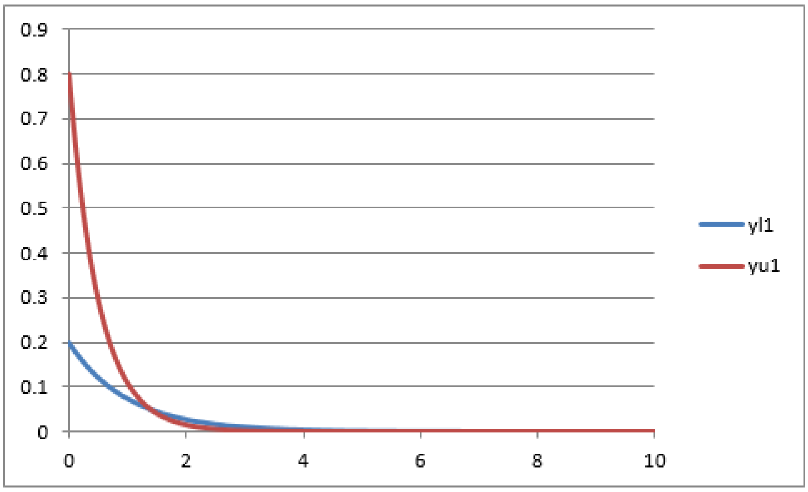

Figure 1: The lower and upper solutions of $\widetilde{y_1}$.

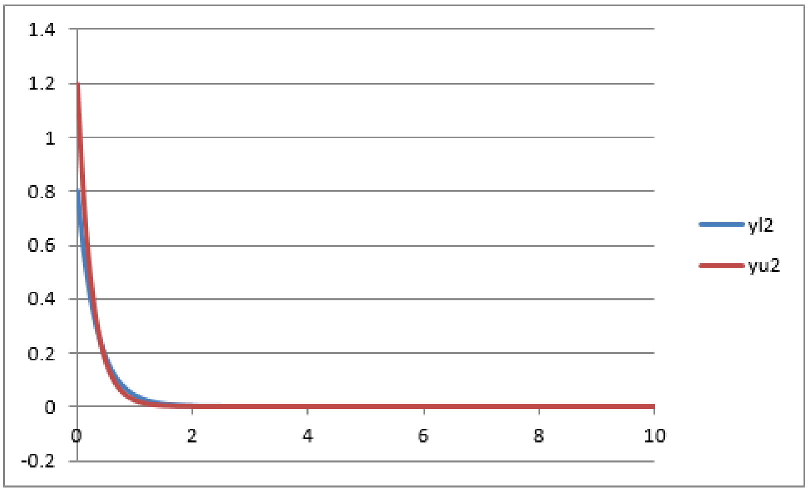

Figure 2: The lower and upper solutions of $\widetilde{y}_2$.

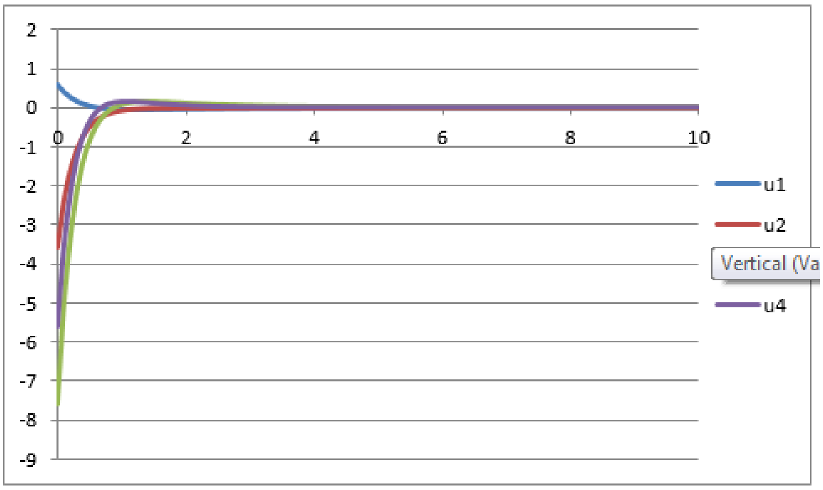

Figure 3: The controller functions u_1, u_2, u_3, u_4.

Comparing with the unstable solutions given in [11], one may see the efficiency and applicability of the approach.

$$\underline{y}(x) = \frac{1}{3}(2sinh(x) + cosh(4x)) - \frac{2}{15}e^{-x}$$
$$\overline{y}(x) = \frac{1}{3}(2cosh(x) + sinh(4x)) + \frac{2}{15}e^{-x}$$

Example 2: Consider the fuzzy differential equation below

$$\widetilde{y'''} - 2\widetilde{y''} - 3\widetilde{y'} = 0 \quad (24)$$

with initial values given as fuzzy intervals.

$$\widetilde{y}(0) \cong [\underline{y}, \overline{y}] = [3, 5], \widetilde{y'}(0) \cong [\underline{y'}, \overline{y'}] = [-3, -1], \widetilde{y''}(0) \cong [\underline{y''}, \overline{y''}] = [8, 10]$$

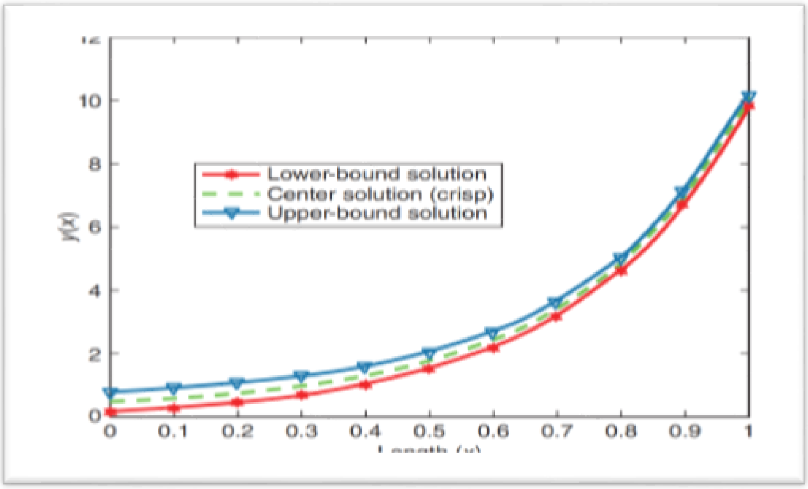

Figure 4: The example (5.1) lower and upper solutions in [11]

In order to stabilize and solve this equation using the backstepping method, Let $\underline{y}_1 = \underline{y}$, $\overline{y_1} = \overline{y}$, $\underline{y_1}' = \underline{y}' = \underline{y_2}, \overline{y_1'} = \overline{y'} = \overline{y_2}$, $\underline{y_2}' = \underline{y}'' = \underline{y_3}$ and $\overline{y_2}' = \overline{y}'' = \overline{y_3}$, Thus:

$$
\begin{aligned}
&\underline{y_3}' - 2\overline{y_3} - 3\overline{y_2} = 0\\
&\overline{y_3'} - 2\underline{y_3} - 3\underline{y_2} = 0
\end{aligned}
$$

Or,

$$
\begin{aligned}
&\underline{y_3}' = 2\overline{y_3} + 3\overline{y_2}\\
&\overline{y_3'} = 2\underline{y_3} + 3\underline{y_2}
\end{aligned}
$$

The system is;

$$
\begin{aligned}
&\underline{y_1'} = \underline{y_2}\\
&\overline{y_1'} = \overline{y_2}\\
&\underline{y_2'}(t;r) = \underline{y_3}(t;r)\\
&\overline{y_2'}(t) = \overline{y_3}\\
&\underline{y_3'} = 2\overline{y_3}(t) + 3\overline{y_2}(t)\\
&\overline{y_3'} = 2\underline{y_3}(t) + 3\underline{y_2}(t)
\end{aligned}
\tag{25}
$$

Or the matrix form:

$$
\begin{bmatrix} \underline{y'}_1 \\ \overline{y'}_1 \\ \underline{y'}_2 \\ \overline{y'}_2 \\ \underline{y'}_3 \\ \overline{y'}_3 \end{bmatrix}
=
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 3 & 0 & 2\\
0 & 0 & 3 & 0 & 2 & 0
\end{bmatrix}
\begin{bmatrix} \underline{y}_1 \\ \overline{y}_1 \\ \underline{y}_2 \\ \overline{y}_2 \\ \underline{y}_3 \\ \overline{y}_3 \end{bmatrix}
$$

To simplify the notations assume that:

$$x_1(t) = \underline{y}_1(t), x_2(t) = \overline{y}_1(t), x_3(t) = \underline{y}_2(t), x_4(t) = \overline{y}_2(t), x_5(t) = \underline{y}_3(t), x_6(t) = \overline{y}_3(t) \quad t \in [0,1]$$

In terms of the new variables, the system (25) is as follows:

$$\begin{aligned}
x_1'(t) &= x_3(t) \\
x_2'(t) &= x_4(t) \\
x_3'(t) &= x_5(t) \\
x_4'(t) &= x_6(t) \\
x_5'(t) &= 3x_4(t) + 2x_6(t) \\
x_6'(t) &= 3x_3(t) + 2x_5(t)
\end{aligned} \tag{27}$$

Consider the control functions $u_i, i = 1,2,3,4,5,6.$ in each equation in system (27) as follows to apply the back stepping method:

$$\begin{aligned}
x_1'(t) &= x_3(t) + u_1(x_1,x_2,x_3,x_4,x_5,x_6) \\
x_2'(t) &= x_4(t) + u_2(x_1,x_2,x_3,x_4,x_5,x_6) \\
x_3'(t) &= x_5(t) + u_3(x_1,x_2,x_3,x_4,x_5,x_6) \\
x_4'(t) &= x_6(t) + u_4(x_1,x_2,x_3,x_4,x_5,x_6) \\
x_5'(t) &= 3x_4(t) + 2x_6(t) + u_5(x_1,x_2,x_3,x_4,x_5,x_6) \\
x_6'(t) &= 3x_3(t) + 2x_5(t) + u_6(x_1,x_2,x_3,x_4,x_5,x_6)
\end{aligned} \tag{28}$$

Step1: Consider the system's first equation (28)

$$x_1'(t) = x_3(t) + u_1(x_1,x_2,x_3,x_4,x_5,x_6)$$

Let $z_1(t) = x_1(t)$, Then differentiating with respect to t implies:

$$z_1'(t) = x_1'(t) = x_3(t) + u_1(x_1,x_2,x_3,x_4,x_5,x_6) \tag{29}$$

It is assumed that the Lyapunov function is $V_1(z_1) = \frac{1}{2}z_1^2(t)$ and its derivative is

$$\begin{aligned}
V_1'(t) = \frac{dv_1}{dt} = \frac{dv_1}{dz_1}\frac{dz_1}{dt} &= z_1(t)z_1'(t;r) \\
&= z_1(t;r)[x_3(t;r) + u_1(x_1,x_2,x_3,x_4,x_5,x_6)]
\end{aligned}$$

Assume $u_1(x_1,x_2,x_3,x_4,x_5,x_6) = -x_3(t) - 2z_1(t)$, Then $V_1' = -2z_1^2(t) < 0$ and as a result, equation (29) is asymptotically stable.

Step 2: Letting $\alpha_1(z) = 0$ and $z_2(t) = x_2(t) - \alpha_1(z)$, then equation (2) is;

$$z_2'(t) = x_2'(t) = x_4(t) + u_2(x_1,x_2,x_3,x_4,x_5,x_6) \tag{30}$$

Based on the presented technique, the Lyapunov function is supposed to be as follows:

$$V_2(z_1,z_2) = V_1(z_1) + \frac{1}{2}z_1^2(t)$$

Then differentiating V_2 with respect to t implies to:

$$\begin{aligned} V_2^{'} &= V_1^{'} + z_2(t)z_2^{'}(t) \\ &= -2z_1^2(t) + z_2(t)[x_4(t) + u_2(x_1, x_2, x_3, x_4, x_5, x_6)] \end{aligned}$$

Assume $u_2(x_1, x_2, x_3, x_4, x_5, x_6) = -x_4(t) - 2z_2(t)$, then $V_2^{'} = -2z_1^2(t) - 4z_2^2(t)$ as a result, equation (30) is asymptotically stable.

Step 3: Define the error term between $x_3(t)$ and $\alpha_2(z_1, z_2)$ as $z_3(t) = x_3(t) - \alpha_2(z_1, z_2)$, Then

$$z_3^{'}(t) = x_3^{'}(t) = x_5(t) + u_3(x_1, x_2, x_3, x_4, x_5, x_6) \tag{31}$$

The Lyapunov function is: $V_3(z_1, z_2, z_3) = V_2 + \frac{1}{2}z_3^2(t)$ and the differentiation is:

$$\begin{aligned} V_3^{'} &= V_2^{'} + z_3(t)z_3^{'}(t) \\ &= -2z_1^2(t) - 4z_2^2(t) + z_3(t)[x_5(t) + u_3(x_1, x_2, x_3, x_4, x_5, x_6)] \end{aligned}$$

assume that $u_3(x_1, x_2, x_3, x_4, x_5, x_6) = -x_5(t) - 6z_3(t)$ then, $V_3^{'} = -2z_1^2(t) - 2z_2^2(t) - 2z_3^2(t)$ which is negative definition function, then equation (31) is asymptotically stable.

Step 4: Define $z_4(t) = x_4(t) - \alpha_3(z_1, z_2, z_3)$, Then ;

$$z_4^{'}(t) = x_4^{'}(t) = x_6(t) + u_4(x_1, x_2, x_3, x_4, x_5, x_6) \tag{32}$$

The Lyapannov function is $V_4(z_1, z_2, z_3, z_4) = V_3(z_1, z_2, z_3) + \frac{1}{2}z_4^2(t)$

The derivative of V_4 is:

$$\begin{aligned} V_4^{'} &= V_3^{'} + z_4(t)z_4^{'}(t) \\ &= -2z_1^2(t) - 4z_2^2(t) - 6z_3^2(t) + z_4(t)[x_6(t) + u_4(x_1, x_2, x_3, x_4, x_5, x_6)] \end{aligned}$$

Assume that $u_4(x_1, x_2, x_3, x_4, x_5, x_6) = -x_6(t) - 8z_4(t)$, then $V_4^{'} = -2z_1^2(t) - 4z_2^2(t) - 6z_3^2(t) - 8z_4^2(t), \quad Z_i \in \mathbb{R}^+$.

Which of the following is a negative definite function as a result, equation (32) has asymptotically stable solutions.

Step 5: Define $z_5(t) = x_5(t) - \alpha_4(z_1, z_2, z_3, z_4)$, Then ;

$$z_5^{'}(t) = x_5^{'}(t) = 3x_4(t) + 2x_6(t) + u_5(x_1, x_2, x_3, x_4, x_5, x_6) \tag{33}$$

The Lyapannov function is $V_5(z_1, z_2, z_3, z_4, z_5) = V_4(z_1, z_2, z_3, z_4) + \frac{1}{2}z_5^2(t)$.

The derivative of V_5 is:

$$\begin{aligned} V_5^{'} &= V_4^{'} + z_5(t)z_5^{'}(t) \\ &= -2z_1^2(t) - 4z_2^2(t) - 6z_3^2(t) - 8z_4^2 + z_5(t)[3x_4(t) + 2x_6(t) + u_5(x_1, x_2, x_3, x_4, x_5, x_6)] \end{aligned}$$

Assume that $u_5(x_1, x_2, x_3, x_4, x_5, x_6) = -2x_6(t) - 3x_4(t) - 3z_5(t)$, then $V_4^{'} = -2z_1^2(t) - 4z_2^2(t) - 6z_3^2(t) - 8z_4^2(t) - 3z_5^2(t) < 0, \quad Z_i \in \mathbb{R}^+$.

which of the following is a negative definite function as a result, equation (33) has asymptotically stable solutions.

Step 6: Define $z_6(t) = x_6(t) - \alpha_4(z_1, z_2, z_3, z_4, z_5)$, Then ;

$$z_6^{'}(t) = x_6^{'}(t) = 3x_3(t) + 2x_5(t) + u_6(x_1, x_2, x_3, x_4, x_5, x_6) \tag{34}$$

The Lyapannov function is $V_6(z_1, z_2, z_3, z_4, z_5) = V_5(z_1, z_2, z_3, z_4) + \dfrac{1}{2}z_6^2(t)$.

The derivative of V_6 is:

$$\begin{aligned} V_6^{'} &= V_5^{'} + z_6(t)z_6^{'}(t) \\ &= -2z_1^2(t) - 4z_2^2(t) - 6z_3^2(t) - 8z_4^2 - 3z_5^2 + z_6(t)[3x_3(t) + 2x_5(t) + u_6(x_1, x_2, x_3, x_4, x_5, x_6)] \end{aligned}$$

Assume that $u_6(x_1, x_2, x_3, x_4, x_5, x_6) = -3x_3(t) - 2x_5(t) - z_6(t)$, then $V_4^{'} = -2z_1^2(t) - 4z_2^2(t) - 6z_3^2(t) - 8z_4^2(t) - 3z_5^2(t) - z_6^2(t) < 0, \quad Z_i \in \mathbb{R}^+$.

which of the following is a negative definite function as a result, equation (34) has asymptotically stable solutions.

since, $\alpha_1(z_1) = \alpha_2(z_1, z_2) = \alpha_3(z_1, z_2, z_3) = \alpha_4(z_1, z_2, z_3, z_4) = \alpha_5(z_1, z_2, z_3, z_4, z_5) = 0$ and

$$\begin{aligned} z_1(t) &= x_1(t) \quad , \quad z_2(t) = x_2(t) \\ z_3(t) &= x_3(t) \quad , \quad z_4(t) = x_4(t) \\ z_5(t) &= x_5(t) \quad , \quad z_6(t) = x_6(t) \end{aligned}$$

Then

$$\begin{aligned} u_1(x_1, x_2, x_3, x_4, x_5, x_6) &= -x_3(t) - 2x_1(t) \\ u_2(x_1, x_2, x_3, x_4, x_5, x_6) &= -x_4(t) - 4x_2(t) \\ u_3(x_1, x_2, x_3, x_4, x_5, x_6) &= -x_5(t) - 6x_3(t) \\ u_4(x_1, x_2, x_3, x_4, x_5, x_6) &= -x_6(t) - 8x_4(t) \\ u_5(x_1, x_2, x_3, x_4, x_5, x_6) &== -2x_6(t) - 3x_4(t) - 3x_5(t) \\ u_6(x_1, x_2, x_3, x_4, x_5, x_6) &== -2x_5(t) - 3x_3(t) - x_6(t) \end{aligned}$$

After substituting $u_1, u_2, u_3, u_4, u_5, u_6$ in system (28) we get the following transformed system.

$$x_1^{'}(t) = -2x_1(t)x_2^{'}(t) = -4x_2(t)x_3^{'}(t) = -6x_3(t)x_4^{'}(t) = -8x_4(t)x_5^{'}(t) = -3x_5(t)x_6^{'}(t) = -x_6(t)$$

And ,back to the hypotheses;

$$\begin{aligned} x_1(t) &= \underline{y_1}(t), \\ x_2(t) &= \overline{y_1}(t), \\ x_3(t) &= \underline{y_2}(t), \\ x_4(t) &= \overline{y_2}(t), \\ x_5(t) &= \underline{y_3}(t), \\ x_6(t) &= \overline{y_3}(t), \end{aligned}$$

So,

$$\underline{y_1}'(t) = -2\underline{y_1}(t)$$
$$\overline{y_1'}(t) = -4\overline{y_1}(t)$$
$$\underline{y_2'}(t) = -6\underline{y_2}(t)$$
$$\overline{y_2'}(t) = -8\overline{y_2}(t)$$
$$\underline{y_3}'(t) = -3\underline{y_3}(t)$$
$$\overline{y_3'}(t) = -\overline{y_3}(t)$$

Or in the matrix form:

$$\begin{bmatrix} \underline{y'}_1 \\ \overline{y'}_1 \\ \underline{y'}_2 \\ \overline{y'}_2 \\ \underline{y'}_3 \\ \overline{y'}_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 & 0 \\ 0 & 0 & 0 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \underline{y}_1 \\ \overline{y}_1 \\ \underline{y}_2 \\ \overline{y}_2 \\ \underline{y}_3 \\ \overline{y}_3 \end{bmatrix}$$

The eigenvalues of the coefficient matrix are distinct and negative $-2, -4, -6, -8, -3, -1$ since it is diagonal. Then comparing with the eigenvalues in [5] which are found to be 0,-1,3, which means that the solution is unstable. Thus, it is conclude that our solutions are asymptotically stable as a result, the solution is as follows:

$$\widetilde{y_1}(t) = [\underline{y_1}(t), \overline{y_1}(t)] = [3e^{-2t}, 5e^{-4t}],$$
$$\widetilde{y_2}(t) = [\underline{y_2}(t), \overline{y_2}(t)] = [-3e^{-6t}, -e^{-8t}],$$
$$\widetilde{y_3}(t) = [\underline{y_3}(t), \overline{y_3}(t)] = [8e^{-3t}, 10e^{-t}],$$

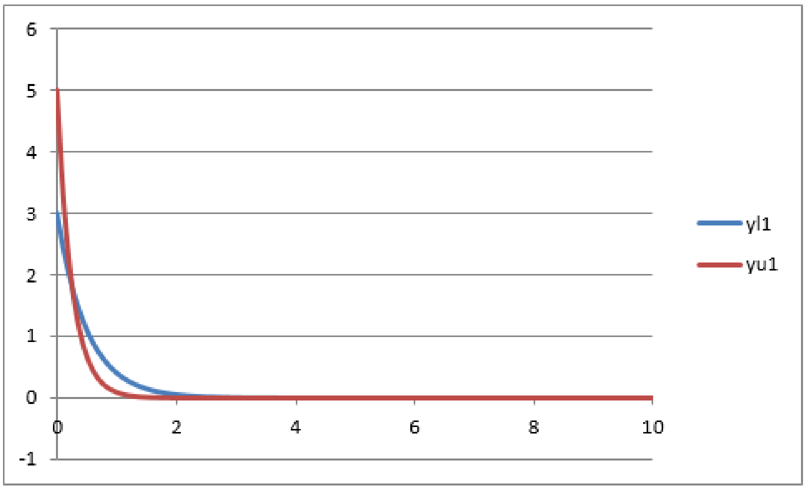

Figure 5: The lower and upper solutions of y_1

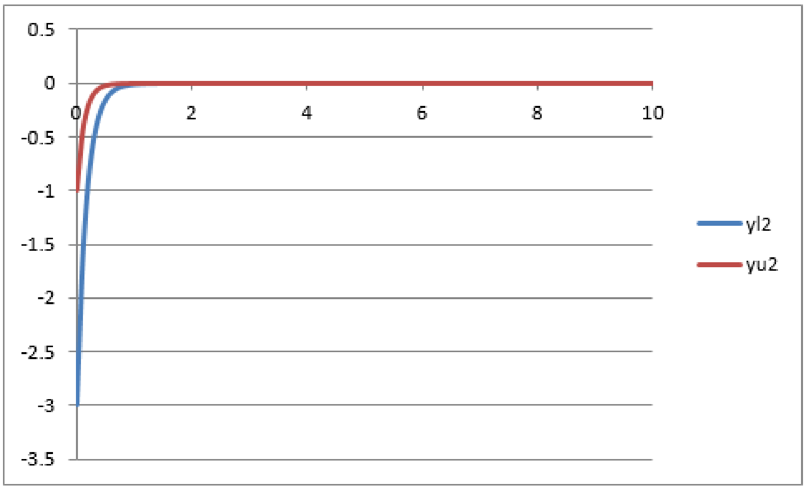

Figure 6: The lower and upper solutions of y_2

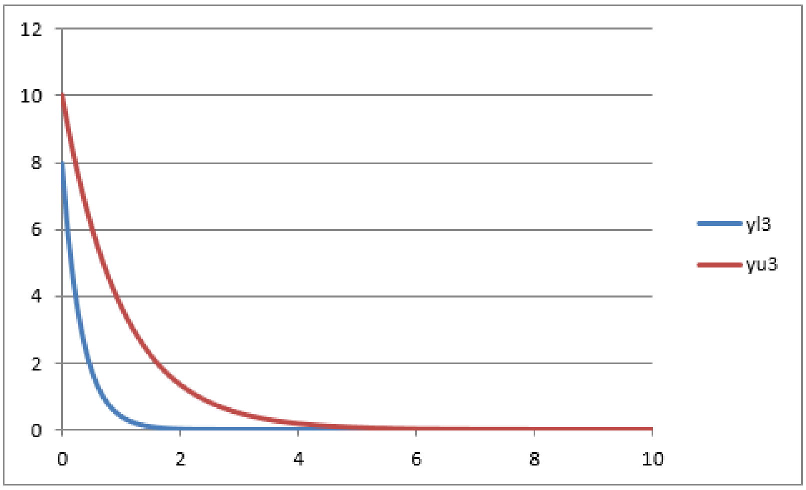

Figure 7: The lower and upper solutions of y_3

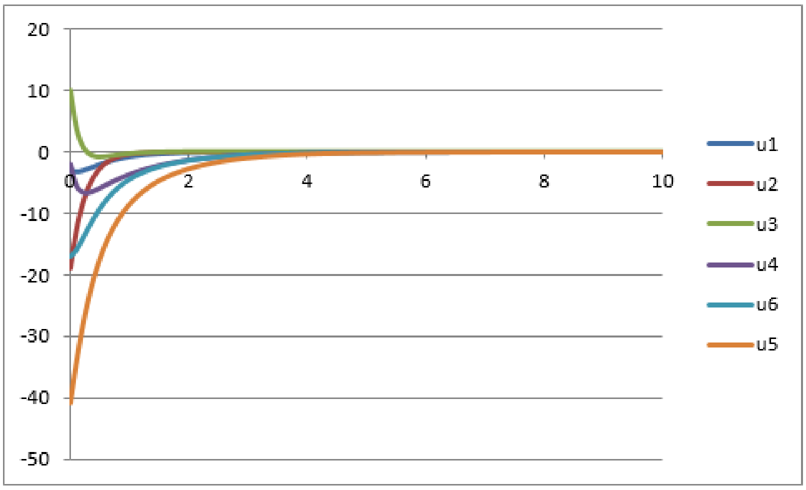

Figure 8: The controller functions $u_1, u_2, u_3, u_4, u_5, u_6$

7. Conclusions

We have shown in this paper that the FDEs can be transformed into a linear system of differential equation of the first order, and we have demonstrated that the procedure of the backstepping method can be applied to FDEs after converting it into a system of equations provided that the method of the solution of an nth order FDEs and find the asymptotically stable solution, for future research we suggest try to solve the nonhomogeneous FDEs of the 3^{rd} order

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