Boundedness of a new kind of Toeplitz operator on 2π-periodic holomorphic functions on the upper halfplane

Mohammad Ali Ardalani[a,*], S. Haftbaradaran[a]

[a]*Department of Mathematics, Faculty of Science, University of Kurdistan, Sanandaj 66177-15175, Iran*

(Communicated by Shahram Saeidi)

Abstract

In the present paper, We introduce a new kind of Toeplitz operators on the spaces of 2π periodic holomorphic functions on the upper halfplane equipped with an integral norm similar to the norm of L^p spaces. We prove the boundedness of Toeplitz operators in the case of bounded symbols. Also, we state some open problems for unbounded symbols and other cases in which our spaces are not Hilbert spaces.

Keywords: Toeplitz operator, projection, 2π periodic holomorphic functions, upper halfplane.
2010 MSC: Primary 47B35; Secondary 46C05.

1. Introduction

We begin by recalling some definitions, notations and results of [1]. Let $G = \{\omega \in \mathbb{C} : Im\, \omega > 0\}$ denotes the upper halfplane. Also, let μ be a bounded positive non-atomic measure on $(0, \infty)$ with $\mu((0, \epsilon)) > 0$ for all $\epsilon > 0$. A function $f : G \longrightarrow \mathbb{C}$ is called 2π-periodic if $f(\omega + 2\pi) = f(\omega)$ for all $\omega \in G$. For example $f(\omega) = e^{i\omega}$ is a 2π-periodic function on G. For a holomorphic function $f : G \longrightarrow \mathbb{C}$ and $1 \leq p < \infty$ we put

$$\|f\|_{p,\mu}^p = \frac{1}{2\pi} \int_0^\infty \int_{-\pi}^\pi \mid f(x+it) \mid^p dx d\mu(t),$$

$$H_{2\pi,\mu}^p(G) = \{f : G \longrightarrow \mathbb{C} \text{ is } 2\pi \text{ periodic, } \|f\|_{p,\mu} < \infty\}$$

---

*Corresponding author

Email addresses: `m.ardalani@uok.ac.ir` (Mohammad Ali Ardalani), `s.haftbaradaran@sci.uok.ac.ir` (S. Haftbaradaran)

and

$$L^2(G) = \{f \mid \text{ is measurable and } \|f\|_{p,\mu} < \infty\}.$$

Here, we mean that f is measurable with respect to $dxd\mu$ as a product measure on G. We say measure μ satisfies condition (B_1) if there exists $n \in \mathbb{N}$ such that $\int_0^\infty e^{nt}d\mu(t) = \infty$ (See Example 1.2 of [1]).

It is well-known that:
(a) $H_{2\pi,\mu}^2(G)$ is isomorphic to the Hilbert space l_2(See Theorem 1.4 of [1]).
(b) For each $f \in H_{2\pi,\mu}^2(G)$ there exist α_k such that $f(\omega) = \sum_{k=-\infty}^{\infty} \alpha_k e^{ik\omega}$ where the series converges uniformly on compact subsets of G (See Lemma 2.4 of [1]).
(c) $L^2(G)$ is a Hilbert space with the usual inner product

$$\forall \ f,g \in L^2(G) \ \ (f,g) = \frac{1}{2\pi} \int_0^\infty \int_{-\pi}^\pi f(x+it)\overline{g(x+it)}dxd\mu(t).$$

2. Preliminaries

In this section we provide necessary tools which for obtaining main results of this paper in the next section. For readers which are not familiar with Toeplitz operator we refer the reader to [**?**]. As first step we prove the following classical result for our case.

Lemma 2.1. $H_{2\pi,\mu}^2(G)$ *is a closed subspace of $L^2(G)$ and therefore a Banach space.*

Proof . Suppose $\{f_n\}$ is a sequence in $H_{2\pi,\mu}^2(G)$ such that $\|f_n - f\|_{2,\mu} \to 0$ for some $f \in L^2(G)$. It is well-known that , there is at least a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that f_{n_k} converges pointwise to f. Since $\{f_{n_k}\}$'s are 2π periodic so f is 2π periodic too. Also, f_{n_k} are uniformly convergent to f on compact subsets on G. This implies that f is holomorphic on G. Thus $f \in H_{2\pi,\mu}^2(G)$ and we are done. $\square$

Now, we intend to present an orthonormal basis for the Hilbert space $H_{2\pi,\mu}^2(G)$, but firstly we need to prove that:

Lemma 2.2. *(i) If μ does not satisfy condition (B_1), then $\|e^{ik\omega}\|_{2,\mu}$ is finite for each $k \in \mathbb{Z}$.*
(ii) If μ satisfies condition (B_1), then for each $k \geq -n+1$ $\|e^{ik\omega}\|_{2,\mu}$ is finite.

Proof . (i): There is no $n \in \mathbb{N}$ with $\int_0^\infty e^{nt}d\mu(t) = +\infty$ (equivalently μ does not satisfy condition (B_1)) and $f(\omega) = \sum_{k=-\infty}^{\infty} \alpha_k e^{ik\omega}$. By our definition for $\omega = x + it$ we have

$$\begin{aligned}
\|e^{ik\omega}\|_{2,\mu}^2 &= \frac{1}{2\pi} \int_0^\infty \int_{-\pi}^\pi \mid e^{ik\omega} \mid^2 dxd\mu(t) \\
&= \frac{1}{2\pi} \int_0^\infty \int_{-\pi}^\pi \mid e^{ikx}e^{-kt} \mid^2 dxd\mu(t) \\
&= \frac{1}{2\pi} \int_0^\infty \int_{-\pi}^\pi \mid e^{-kt} \mid^2 dxd\mu(t) \\
&= \int_0^\infty e^{-2kt}d\mu(t)
\end{aligned}$$

If $k \geq 0$ then $e^{-2kt} \leq 1$ and $\|e^{ik\omega}\|_{2,\mu}^2 < \mu((0,\infty))$. Since μ is a bounded measure so $\|e^{ik\omega}\|_{2,\mu} < \infty$. If $k < 0$ then $-2k \in \mathbb{N}$. Since μ does not satisfy condition (B_1), again $\|e^{ik\omega}\|_{2,\mu} < \infty$.

(ii): Let μ satisfy condition (B_1) and n be the smallest integer such that $\int_0^\infty e^{nt}d\mu(t) = \infty$, then $f(\omega) = \sum_{k=-n+1}^\infty \alpha_k e^{ik\omega}$ (See Lemma 2.7 of [1]). By the proof of Lemma 2.5 of [1] we have

$$| \alpha_k | \, \|e^{ik\omega}\|_{2,\mu} = \|\alpha_k e^{ik\omega}\|_{2,\mu} \leq \|f\|_{2,\mu} < \infty. \tag{2.1}$$

Since $\alpha'_k s$ in relation (2.1) are not necessarily zero, so for each $k \geq -n+1$ $\|e^{ik\omega}\|_{2,\mu} < \infty$. Here note that $\alpha_k = 0$ for each $k \leq -n$. $\square$

Remark 2.3. *Lemma 2.2 enable us to define an orthonormal basis for $H_{2\pi,\mu}^2(G)$ independent of that μ satisfies or does not satisfy condition(B_1). Indeed it insure us for any $f \in H_{2\pi,\mu}^2(G)$ and each k which α_k is not zero in the representation of f $\|e^{ik\omega}\|_{2,\mu} < \infty$.*

By Lemma Lemma 2.4 of [1], it is clear the set $\{e^{ik\omega} : k \in \mathbb{Z}\}$ spans $H_{2\pi,\mu}^2(G)$ and since

$$(e^{ik\omega}, e^{ik'\omega}) = \frac{1}{2\pi} \int_0^\infty \int_{-\pi}^\pi (e^{ik\omega}(e^{ik'\omega}dxd\mu(t)$$
$$= \frac{1}{2\pi} \int_0^\infty \int_{-\pi}^\pi e^{i(k-k')x}e^{(k-k')t}dxd\mu(t)$$
$$= 0$$

whenever $k \neq k'$, so $e^{ik\omega}$s are mutually orthogonal. Therefore $e_k(\omega) = \frac{e^{ik\omega}}{\|e^{ik\omega}\|_{2,\mu}}$ is an orthonormal basis for $H_{2\pi,\mu}^2(G)$ and following definition make sense.

Definition 2.4. *The orthogonal projection $P : L^2(G) \longrightarrow H_{2\pi,\mu}^2(G)$ is defined by*

$$P(f) = \sum_{k=-\infty}^\infty (f, e_k(\omega))e_k(\omega)$$
$$= \frac{1}{2\pi} \sum_{k=-\infty}^\infty \int_0^\infty \int_{-\pi}^\pi f(x+it)\overline{e_k(x+it)}dxd\mu(t)e_k(x+it)$$

where $\omega = x + it \quad -\pi \leq x \leq \pi$ and $t > 0$.

In this step we must show that P in the above is really well defined i.e $P(f) \in H_{2\pi,\mu}^2(G)$ for each $f \in L^2(G)$. Equivalently, we must show that coefficients which appear in the representation of Definition 2.4 are the same as α_k's which come from Lemma 2.4 of [1].

Lemma 2.5. *The orthogonal projection P in Definition 2.4 is well defined.*

Proof. Firstly we compute coefficients in the representation of P in Definition 2.4. Since $\overline{e_k(x+it)} = \frac{e^{-ikx}e^{-kt}}{\|e^{ik\omega}\|_{2,\mu}}$,

$$P(f) = \frac{1}{2\pi} \sum_{k=-\infty}^\infty \frac{1}{\|e^{ik\omega}\|_{2,\mu}^2} \int_0^\infty \int_{-\pi}^\pi f(x+it)e^{-ikx}e^{-kt}dxd\mu(t)e^{ik\omega}.$$

This means in the representation of $P(f)$ our coefficients are of the following form

$$\beta_k = \frac{1}{2\pi\|e^{ik\omega}\|_{2,\mu}^2} \int_0^\infty \int_{-\pi}^\pi f(x+it)e^{-ikx}e^{-kt}dxd\mu(t). \tag{2.2}$$

By Lemma 2.4 of [1] coefficients which appear in the representation of an element $f \in H^2_{2\pi,\mu}(G)$ are α_k's with the following form

$$\alpha_k = \frac{1}{2\pi} \int_0^\infty \int_{-\pi}^\pi e^{-ikx+kt} f(x+it)dx \frac{d\mu(t)}{\mu((0,\infty))}. \tag{2.3}$$

(see the last assertion of Lemma 2.4 of [1]). Relations (2.2) and (2.3) are apparently different. In order to show that the orthogonal projection P is really well-defined, we must show that $\alpha_k = \beta_k$ for each $k \in \mathbb{Z}$. Evidently, $\alpha_k = \beta_k$ if and only if

$$\frac{1}{\|e^{ik\omega}\|_{2,\mu}} \frac{1}{2\pi} \int_0^\infty \int_{-\pi}^\pi f(x+it)e^{-ikx}e^{-kt}dxd\mu(t) = \|e^{ik\omega}\|_{2,\mu}\alpha_k \tag{2.4}$$

Since $f(\omega) = f(x+it) = e^{ik\omega}$ generates $H^2_{2\pi,\mu}(G)$ it is enough to prove relation (2.4) for $f(\omega)$. For $f(\omega) = e^{ik\omega}$ we have $\alpha_k = 1$. Hence the right hand side of (2.4) is equal to $\|e^{ik\omega}\|_{2,\mu}$, but the left hand side of (2.4) is equal to

$$\frac{1}{\|e^{ik\omega}\|_{2,\mu}} \frac{1}{2\pi} \int_0^\infty \int_{-\pi}^\pi e^{ikx}e^{-kt}e^{-ikx}e^{-kt}dxd\mu(t) = \frac{1}{\|e^{ik\omega}\|_{2,\mu}} \int_0^\infty e^{-2kt}d\mu(t)$$

$$= \frac{\|e^{ik\omega}\|^2_{2,\mu}}{\|e^{ik\omega}\|_{2,\mu}} = \|e^{ik\omega}\|_{2,\mu}.$$

which shows that $\alpha_k = \beta_k$ for each integer k. $\square$

Now, consider a measurable function f from G into $\mathbb{C}$. We intend to define a Toeplitz operator T_f (with symbol f), from $H^2_{2\pi,\mu}(G)$ into $H^2_{2\pi,\mu}(G)$. Usually, T_f is defined by $T_f(h) = P(fh)$ for each $h \in H^2_{2\pi,\mu}(G)$ if fh really belongs to $L^2(G)$ i.e $\|fh\|_{2,\mu} < \infty$. If $f \in H^\infty(G)$ (space of all bounded holomorphic functions on the upper halfplane) then

$$\|fh\|^2_{2,\mu} = \frac{1}{2\pi} \int_0^\infty \int_{-\pi}^\pi |f(x+it)|^2 |h(x+it)|^2 \, dxd\mu(t)$$

$$\leq M^2\|h\|^2_{2,\mu}$$

$$< \infty$$

where $\|f\|_\infty = \sup_{\omega \in G} |f(\omega)| = M$. Hence if $f \in H^\infty(G)$ then $fh \in L^2(G)$ for each $h \in H^2_{2\pi,\mu}(G)$. For example $f(\omega) = e^{i\omega} \in H^\infty(G)$. Hence we have the following definition.

Definition 2.6. *Suppose $f \in H^\infty(G)$. Toeplitz operator $T_f : H^2_{2\pi,\mu}(G) \longrightarrow H^2_{2\pi,\mu}(G)$ can be defined by $T_f(h) = P(fh)$ for each $f \in H^2_{2\pi,\mu}(G)$.*

Here we give an example of a function which is not in $H^\infty(G)$, while its multiplication by any element of $H^2_{2\pi,\mu}(G)$ is again an element $H^2_{2\pi,\mu}(G)$.

Example 2.7. *Put $f(\omega) = f(x+it) = e^{x+it} = e^\omega$. Since $|f(\omega)| = e^x$ so $f \notin H^\infty(G)$. But f is bounded on the strip $\{x+it : -\pi \leq x \leq \pi, \ t > 0\}$ and $e^{-\pi} \leq |f(\omega)| \leq e^\pi$. Thus*

$$\|fh\|^2_{2,\mu} \leq e^\pi\|h\|^2_{2,\mu} < \infty.$$

Therefore $fh \in L^2(G)$ for each $h \in H^2_{2\pi,\mu}(G)$.

Remark 2.8. *(i) As Example 2.7 shows assumption $f \in H^\infty(G)$ in Definition 2.6 is only a sufficient condition which guarantees T_f is well defined and it is not a necessary condition. Hence the set of functions f such that $fh \in L^2(G)$ for each $h \in H^2_{2\pi,\mu}(G)$ includes $H^\infty(G)$. Also Example 2.7 revalues that all the holomorphic functions which are bounded on the strip $\{x + it : -\pi \leq x \leq \pi, \ t > 0\}$ induces well defined Toeplitz operators. Therefore, characterizing the set of functions f such that $fh \in L^2(G)$ for each $h \in H^2_{2\pi,\mu}(G)$ is an open problem.*
(ii) Note that for our aim, we need not necessarily assume that $fh \in L^2(G)$ for each $h \in H^2_{2\pi,\mu}(G)$. If we apply the definition of P to fh we have

$$P(fh) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_0^\infty \int_{-\pi}^{\pi} f(x + it)h(x + it)\overline{e_k(x + it)}dxd\mu(t)e_k(x + it) \tag{2.5}$$

another alternative for defining a class of well defined Toeplitz operators is finding a condition on f such that the series of the right hand side of relation (2.6) converges uniformly on the compact subsets of G. In many classical texts on reflexive Bergaman spaces (see [2,3,4,5,6]) a priori assumption $f \in L^1$ (for our case $f \in L^1(G)$) is a usual sufficient condition in order to guarantee that series (2.6) converges uniformly on the compact subsets of G. Here

$$P(fh) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \frac{1}{\|e^{ik\omega}\|^2_{2,\mu}} \int_0^\infty \int_{-\pi}^{\pi} f(x + it)h(x + it)dxd\mu(t) \tag{2.6}$$

looking at the proof of part (i) of Lemma 2.2 we see that the estimation for $\|e^{ik\omega}\|_{2,\mu}$ does not depend on k. So there is not a straightforward way in order to show the convergence of the series of the reight hand side of (2.6). Maybe a better converges estimation (depending on k) for $\sum_{k=-\infty}^{\infty} \frac{1}{\|e^{ik\omega}\|^2_{2,\mu}}$ can be worked out.

3. Main result

Toeplitz operator can be viewed from another aspect as a composition of multiplication operator and projection operator. Indeed, if we consider $M_f : H^2_{2\pi,\mu}(G) \longrightarrow L^2(G)$ and $P : L^2(G) \longrightarrow H^2_{2\pi,\mu}(G)$ then $T_f(h) = (P \circ M_f)(h)$ for any $h \in H^2_{2\pi,\mu}(G)$. This motivates to characterize boundedness Toeplitz operator in terms of boundedness of multiplication operator and projection operator. In the following Theorem we characterize boundedness of multiplication operator.

Theorem 3.1. *Following statements are equivalent.*
(i) Multiplication operator $M_f : H^2_{2\pi,\mu}(G) \longrightarrow L^2(G)$ is bounded.
(ii) $f \in H^\infty(G)$.
Besides, if M_f is bounded then $\|M_f\| = \|f\|_\infty$.

Proof . $(i) \Rightarrow (ii)$: Since M_f is bounded so the adjoint operator M_f^* is bounded too and $\|M_f\| = \|M_f^*\|$. for any fixed ω in G the evolutional function $\delta_\omega : L^2(G) \longrightarrow \mathbb{C}$ defined by $\delta_\omega(h) = h(\omega)$ ($h \in L^2(G)$) belongs to : $L^2(G)^*$. We have

$$| f(\omega) | = \frac{\|M_{fi}^*(\delta_\omega)\|}{\|\delta_\omega\|} \leq \|M_f^*\| = \|M_f\| < \infty.$$

Since $\omega \in G$ is arbitrary so

$$\sup\{| f(\omega) | : \omega \in G\} = \|f\|_\infty \leq \|M_f\| < \infty.$$

$(ii) \Rightarrow (i)$: Computation which has done before Definition 2.6 shows that $\|M_f\| \leq \|f\|_\infty$ whenever $f \in H^\infty(G)$. $\square$

Theorem 3.2. *Let $f \in H^\infty(G)$ and projection P be a bounded operator. Then the Toeplitz operator of Definition 2.6 is bounded.*

Proof . Since $f \in H^\infty(G)$ Theorem 3.1 implies that the multiplication operator M_f is bounded. Now, T_f as a composition of bounded operators is bounded too. $\square$

The question of boundedness of Toeplitz operator for unbounded symbols is an open problem. However, we have obtained a partial result. Looking at Example 2.7 we have

Corollary 3.3. *Let f be a holomorphic function on G such that f is bounded on the strip $\{x + it : -\pi \leq x \leq \pi, \ t > 0\}$ (f is not necessarily in $H^\infty(G)$) and P be a bounded projection. Then the Toeplitz operator of Definition 2.6 is bounded.*

Open problems

1- We can try to define Toeplitz operators in a more general case on $H^p_{2\pi,\mu}(G)$ $p \neq 2$ and characterize its boundedness. Since in this situation $H^p_{2\pi,\mu}(G)$'s are not Hilbert spaces so we probably deal with a very big and complicated problem.

2- Characterizing of compactness of Teplitz operator of Definition2.6 as first step and then characterizing of compactness of Teplitz operators on $H^p_{2\pi,\mu}(G)$ $p \neq 2$.

References

[1] M.A. Ardalani, *On the isomorphism classification of spaces 2π periodic holomorphic functions on the upper halfplane*, J. Math. Anal. Appl. 459 (2018) 350–359.

[2] O. Constantin and J. A. Pelaez, *Boudndedness of the Bergman projection on L^p-spaces with expotential weights*, Bull. Sci. Math. 139 (2015) 245–268.

[3] M. Dostanic, *Unboundedness of the Bergman projection on L^p spaces with expotential weights*, Proc. Edinb. Math. Soc. 47 (2004) 111–117.

[4] M. Englis, *Toeplitz operators and weighted Bergman kernels*, J. Funct. Anal. 255 (6) (2008) 1419–1457.

[5] S. Grudsky, A. Karapetyants, N. Vasilevski, *Toeplitz operators on the unit ball of $\mathbb{C}^{\mathbb{N}}$ with radial symbols*, J. Oper. Theory 49 (2003) 325–346.

[6] W. Lusky and J. Taskinen, *Toeplitz operators on Bergman spaces and Hardy multipliers*, Studia. Math. 204 (2011) 137–154.

[7] K. Zhu, *Operator Theory in Function Spaces.* 2nd edition, Mathematical surves and monographs, Vol. 138, American Mathematical Society, Providence, RI, 2007.