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A class of harmonic univalent functions defined by the q-derivative operator

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Abstract

In this paper, a class of harmonic univalent functions has been studied by using q-analogue of the derivative operator for complex harmonic functions. We have obtained a sufficient condition, a representation theorem for this harmonic univalent functions class and some other geometric properties.

Keywords: Univalent function, Harmonic function, Sense-preserving, q-difference operator.

1. Introduction

Let Υ denote the class of functions h that are analytic in unit disk

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}$$

with the condition $h(0) = h'(0) - 1 = 0$. Each complex–valued harmonic functions of the form $f = a + ib$, where a and b are real-valued harmonic functions in Δ can written as function $f = h + \overline{g}$. The Jacobian of the function $f = h + \overline{g}$ is given by [8]

$$J_f(z) = \left|h^{'}(z)\right|^2 - \left|g^{'}(z)\right|^2.$$

Every harmonic function $f = h + \overline{g}$ is locally univalent and sense-preserving in Δ if and only if J_f(z)>0 in another meaning $\left|h^{'}(z)\right| > \left|g^{'}(z)\right|$[11].

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Let $\mathscr{HR}$ be denote a class of harmonic functions $f = h + \overline{g}$ such that h is the analytic part and g is the co-analytic part and of the form

$$f(z) = z + \sum_{m=2}^{\infty} c_m z^m + \overline{\sum_{m=1}^{\infty} d_m z^m} \tag{1.1}$$

where $h(z) = z + \sum_{m=2}^{\infty} c_m z^m$ and $g(z) = \sum_{m=1}^{\infty} d_m z^m$.
If $g(z) = 0 \forall z \in \Delta$, then the class $\mathscr{HR}$ is diminutive to the class Υ of normalized and analytic functions which are univalent in Δ [7, 10, 11].

We can denoted by $S\mathscr{H}$ to the class of functions $f = h + \overline{g}$ which are harmonic, univalent and sense-preserving in Δ. In 1984 Clunie and Sheil-Small inspected the class $S\mathscr{H}$ as well as its geometric subclasses and obtained some coefficient bounds.

Furthermore, the theory of q-calculus has motivated the researchers due to its applications in the field of physical sciences, some applications were given by [4, 14] about q-calculus by preface the q-analogues of derivative operator ∂_q which is defined for $q \in (0, 1)$ by

$$\partial_q h(z) = \frac{h(z) - h(qz)}{(1-q)z} = 1 + \sum_{m=2}^{\infty} [m]_q c_m z^{m-1}, \quad q \neq 1, z \neq 0$$

where $h(z) = z + \sum_{m=2}^{\infty} c_m z^m, z \in \Delta$. Clearly, we have $\lim_{q \to 1^-} \partial_q h(z) = h'(z)$, as long as that the ordinary derivative $h'(z)$ exists.

Now, the q-number $[\kappa]_q$ has defined as follows

$$[\kappa]_q = \begin{cases} \frac{1-q^\kappa}{1-q}, & \kappa \in \mathbb{C} \\ \sum_{\ell=0}^{n-1} q^\ell, & \kappa = n \in \mathbb{N} \end{cases}$$

see [13] and q-factorial $[m]_q!$ has defined by

$$[m]_q! = \begin{cases} \prod_{\kappa=1}^{m} [\kappa]_q, & m \in \mathbb{N} \\ 1, & m = 0 \end{cases}$$

Some others applications of q-calculus are studied by [1, 3]. Other interesting works on harmonic functions can be traced in [6, 9, 12].

For $0 \leq \beta < 1$, a function $f = h + \overline{g}$ of the form (1.1) is said to be in the class $S^*\mathscr{H}(\beta), C\mathscr{H}(\beta)$ of normalized harmonic starlike functions and convex functions of order β respectively, in Δ if satisfies

$$S^*\mathscr{H}(\beta) = \left\{ f \;:\; Re\left(\frac{z\partial_q f(z)}{f(z)}\right) \geq \beta, \quad z \in \Delta \right\},$$

and

$$C\mathscr{H}(\beta) = \left\{ f \;:\; Re\left(1 + \frac{z\partial_q(\partial_q f(z))}{\partial_q f(z)}\right) \geq \beta, \quad z \in \Delta \right\}.$$

Definition 1.1. *[13] The q-analog of the derivative operator for the harmonic function f=h + $\overline{g}$ given by* (1.1) *can defined as*

$$\mathscr{B}_{\alpha,\lambda,q}^{\mu,s} f(z) = \mathscr{B}_{\alpha,\lambda,q}^{\mu,s} h(z) + (-1)^s \overline{\mathscr{B}_{\alpha,\lambda,q}^{\mu,s} g(z)},$$

where

$$\mathscr{B}^{\mu,s}_{\alpha,\lambda,q}h\,(z) = z + \sum_{m=2}^{\infty} \phi_m(\alpha,\mu,\lambda,s,q)c_m z^m,$$
$$\mathscr{B}^{\mu,s}_{\alpha,\lambda,q}g\,(z) = \sum_{m=1}^{\infty} \phi_m(\alpha,\mu,\lambda,s,q)d_m z^m,$$

and

$$\phi_m(\alpha,\mu,\lambda,s,q) = [m]_q^s \left(\frac{[\lambda+1]_{m-1}}{[m-1]_q!} \left\{ 1 + \alpha([m]_q - 1) \right\} \right)^{\mu}, \quad (\alpha,\lambda,\mu,s \in \mathbb{N}_0) \tag{1.2}$$

Remark 1.2. *[5] For $s = \alpha = 0$, $\mu = 1$, we obtain the q-Ruscheweyh derivative for harmonic functions.*

Remark 1.3. *For $s = 0$, $\mu = 1$ and $q \to 1^-$, we have the operator for harmonic functions studied by.[2]*

Remark 1.4. *[5] For $\mu = 0$, we get the operator of q-Salagean for harmonic functions .*

Definition 1.5. *Let $\mathfrak{H}\mathscr{H}^{\mu,s}_q(\alpha,\sigma,\lambda,\beta)$ denote the class of complex-valued and sense-preserving harmonic univalent functions of the form (1.1) which holds the next condition*

$$Re\left(\frac{z\partial_q \mathscr{B}^{\mu,s}_{\alpha,\lambda,q}f\,(z) + \sigma z^2 \partial_q(\partial_q \mathscr{B}^{\mu,s}_{\alpha,\lambda,q}f\,(z))}{(1-\sigma)\,\mathscr{B}^{\mu,s}_{\alpha,\lambda,q}f\,(z) + \sigma z \partial_q \mathscr{B}^{\mu,s}_{\alpha,\lambda,q}f\,(z)} \right) \geq \beta, \tag{1.3}$$

where $\beta \in [0,1)$, $\sigma \in [0,1]$ and $q \in (0,1)$.

By appropriately specializing the parameter σ , we can have several known subclasses .For example, if σ=0, we have a class of harmonic functions which was studied by [13]. On the other hand, when σ=1, we get a class of convex harmonic functions of order β.

We further denote by $\overline{\mathfrak{H}\mathscr{H}}^{\mu,s}_q(\alpha,\sigma,\lambda,\beta)$ the subclass of the class $\mathfrak{H}\mathscr{H}^{\mu,s}_q(\alpha,\sigma,\lambda,\beta)$ consists f_s such that the function $f_s = h_s + \overline{g_s}$ is of the following form

$$h_s(z) = z - \sum_{m=2}^{\infty} |c_m|\, z^m \quad and \quad g_s\,(z) = (-1)^s \sum_{m=1}^{\infty} |d_m|\, z^m, \quad |d_1| < 1) \tag{1.4}$$

such that $\overline{\mathfrak{H}\mathscr{H}}^{\mu,s}_q(\alpha,\sigma,\lambda,\beta) = \mathfrak{H}\mathscr{H}^{\mu,s}_q(\alpha,\sigma,\lambda,\beta) \cap \overline{S\mathscr{H}}$, where $\overline{S\mathscr{H}}$ denote the subclass of $S\mathscr{H}$ consisting of functions of the form f= h + $\overline{g}$.
Clearly, if $0 \leq \beta_1 \leq \beta < 1$, then $\mathfrak{H}\mathscr{H}^{\mu,s}_q(\alpha,\sigma,\lambda,\beta) \subset \mathfrak{H}\mathscr{H}^{\mu,s}_q(\alpha,\sigma,\lambda,\beta_1)$.

2. Coefficient Estimates

First we begin by proving some sharp inequalities for coefficient in the next theorem.

Theorem 2.1. *Let $f = h + \overline{g}$ be defined as in equation (1.1). Also suppose that*

$$\sum_{m=2}^{\infty}\left[\Psi\left(ø,\beta,m,\phi_m\right)|c_m| + \varphi\left(ø,\beta,m,\phi_m\right)|d_m|\right] \le 1 - \left(\frac{\beta - 2\beta ø + 1}{1-\beta}\right)\phi_1 |d_1|, \qquad (2.1)$$

where
$\Psi\left(ø,\beta,m,\phi_m\right) = \frac{\phi_m\left[\beta(1-ø) - [m]_q^2 ø + [m]_q ø - [m]_q + [m]_q ø\beta\right]}{1-\beta}$, *and* $\varphi\left(ø,\beta,m,\phi_m\right) = \frac{\phi_m\left[\beta(1-ø) - [m]_q^2 ø + [m]_q ø + [m]_q - [m]_q ø\beta\right]}{1-\beta}$,
where $\phi_m(\alpha,\mu,\lambda,s,q)$ given by (1.2) with $c_1 = 1$. Then f is sense-preserving, harmonic univalent in Δ and $f \in \mathfrak{H}\mathscr{H}_q^{\mu,s}(\alpha, ø, \lambda, \beta)$.

Proof . *Let f be as in (1.1) and holds the condition (2.1). Then f is sense-preserving in Δ if f holds $|\partial_q h(z)| > |\partial_q g(z)|$ so, we have*

$$\begin{aligned}
|\partial_q h(z)| &= \left|1 + \sum_{m=2}^{\infty}[m]_q c_m z^{m-1}\right| \ge 1 - \sum_{m=2}^{\infty}[m]_q |c_m| |z|^{m-1} > 1 - \sum_{m=2}^{\infty}[m]_q |c_m| \\
&\ge 1 - \sum_{m=2}^{\infty}|c_m| \Psi\left(ø,\beta,m,\phi_m\right) > \sum_{m=1}^{\infty}|d_m| \varphi\left(ø,\beta,m,\phi_m\right) > \sum_{m=1}^{\infty}[m]_q |d_m| \\
&> \sum_{m=1}^{\infty}[m]_q |d_m| |z|^{m-1} > |\partial_q g(z)|.
\end{aligned}$$

So that

$$h'(z) = \lim_{q\to 1^-}|\partial_q h(z)| > \lim_{q\to 1^-}|\partial_q g(z)| = g'(z).$$

To show that f is univalent in Δ , for $0 < |z_1| \le |z_2| < 1$, we obtian

$$\begin{aligned}
\left|\frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)}\right| &\ge 1 - \left|\frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)}\right| = 1 - \left|\frac{\sum_{m=1}^{\infty} d_m (z_1^m - z_2^m)}{(z_1 - z_2) + \sum_{m=2}^{\infty} c_m (z_1^m - z_2^m)}\right| \\
&> 1 - \frac{\sum_{m=1}^{\infty}[m]_q |d_m|}{1 - \sum_{m=2}^{\infty}[m]_q |c_m|} \ge 1 - \frac{\sum_{m=1}^{\infty}\varphi\left(ø,\beta,m,\phi_m\right)|d_m|}{1 - \sum_{m=2}^{\infty}\Psi\left(ø,\beta,m,\phi_m\right)|c_m|}.
\end{aligned}$$

Using the condition (2.1), then the last expression is non-negative.

Finally, to show that $f \in \mathfrak{H}\mathscr{H}_q^{\mu,s}(\alpha, ø, \lambda, \beta)$ we may show if (2.1) holds then (1.3) is satisfied. From (1.3), we can write

$$Re\left(\frac{z + \sum_{m=2}^{\infty}[m]_q \phi_m c_m z^m (1 + [m]_q ø - ø) + (-1)^s \sum_{m=1}^{\infty}[m]_q \phi_m d_m z^m \left(-1 - ø + [m]_q ø\right)}{z + \sum_{m=2}^{\infty}\phi_m c_m z^m \left(1 - ø + [m]_q ø\right) + (-1)^s \sum_{m=1}^{\infty}\phi_m d_m z^m \left(1 - ø - [m]_q ø\right)}\right) =$$
$$Re\left(\frac{U(z)}{W(z)}\right),$$

where $U(z) = z + \sum_{m=2}^{\infty}[m]_q \phi_m c_m z^m (1 + [m]_q ø - ø) + (-1)^s \sum_{m=1}^{\infty}[m]_q \phi_m d_m z^m \left(-1 - ø + [m]_q ø\right)$
and $W(z) = z + \sum_{m=2}^{\infty}\phi_m c_m z^m \left(1 - ø + [m]_q ø\right) + (-1)^s \sum_{m=1}^{\infty}\phi_m d_m z^m \left(1 - ø - [m]_q ø\right)$

Using the fact Re(M) $\geq \beta \Longleftrightarrow |1-\beta+M| \geq |1+\beta-M|$. *It suffices to show that*

$$\left|1-\beta+\frac{U(z)}{W(z)}\right|-\left|1+\beta-\frac{U(z)}{W(z)}\right|$$

$$\geq 2(1-\beta)|z|-\sum_{m=2}^{\infty}\phi_m(-[m]_q^2\varnothing-[m]_q+[m]_q\varnothing\beta+\varnothing+\beta-\beta\varnothing-1)|c_m||z|^m$$

$$-\sum_{m=1}^{\infty}\phi_m(-[m]_q^2\varnothing+[m]_q-[m]_q\varnothing\beta+\varnothing+\beta-\beta\varnothing+2[m]_q\varnothing-1)|d_m||z|^m$$

$$-\sum_{m=2}^{\infty}\phi_m(-[m]_q^2\varnothing-[m]_q+[m]_q\varnothing\beta-\varnothing+\beta-\beta\varnothing+2[m]_q\varnothing+1)|c_m||z|^m$$

$$-\sum_{m=1}^{\infty}\phi_m(-[m]_q^2\varnothing+[m]_q-[m]_q\varnothing\beta-\varnothing+\beta-\beta\varnothing+1)|d_m||z|^m$$

$$=2(1-\beta)|z|\left[1-\sum_{m=2}^{\infty}\phi_m(\frac{\left[\beta(1-\varnothing)-[m]_q^2\varnothing+[m]_q\varnothing-[m]_q+[m]_q\varnothing\beta\right]}{1-\beta})|c_m||z|^{m-1}\right.$$

$$\left.-\sum_{m=1}^{\infty}\phi_m(\frac{\left[\beta(1-\varnothing)-[m]_q^2\varnothing+[m]_q\varnothing+[m]_q-[m]_q\varnothing\beta\right]}{1-\beta})|d_m||z|^{m-1}\right]$$

$$>2(1-\beta)\left[1-\left(\frac{\beta-2\beta\varnothing+1}{1-\beta}\right)\phi_1|d_1|-\left(\sum_{m=2}^{\infty}\left[\Psi(\varnothing,\beta,m,\phi_m)|c_m|+\varphi(\varnothing,\beta,m,\phi_m)|d_m|\right]\right).\right]$$

The last expression is non-negative by (2.1), *hence* $f\in\mathfrak{H}\mathscr{H}_q^{\mu,s}(\alpha,\varnothing,\lambda,\beta)$. *The proof is complete.* □

From the different choices for parameters, we derive new results as following.
If $\mu=0$ in Theorem 2.1, we get the next result.

Corollary 2.2. *Let* $f=h+\overline{g}\in S\mathscr{H}$ *given by* (1.1) *and holds*

$$\sum_{m=2}^{\infty}\left[\Psi(\varnothing,\beta,m,\phi_m)|c_m|+\varphi(\varnothing,\beta,m,\phi_m)|d_m|\right]\leq 1-\left(\frac{\beta-2\beta\varnothing+1}{1-\beta}\right)|d_1|,$$

$\Psi(\varnothing,\beta,m,\phi_m)=\frac{\phi_m\left[\beta(1-\varnothing)-[m]_q^2\varnothing+[m]_q\varnothing-[m]_q+[m]_q\varnothing\beta\right]}{1-\beta}$, *and*
$\varphi(\varnothing,\beta,m,\phi_m)=\frac{\phi_m\left[\beta(1-\varnothing)-[m]_q^2\varnothing+[m]_q\varnothing+[m]_q-[m]_q\varnothing\beta\right]}{1-\beta}$, *where* $\phi_m(\alpha,\mu,\lambda,s,q)$ *given by* (1.2), $\varnothing\in$ *[0,1]* , $\beta\in$ *[0,1)* *and* $q\in(0,1)$. *Then* $f\in\mathfrak{H}\mathscr{H}_q^{s}(\alpha,\varnothing,\lambda,\beta)$.
when $q\to 1^-$, *then Corollary* (2.2) *derive to new result as follows*

Corollary 2.3. *Let f(z)=h(z)* $+\overline{g(z)}\in S\mathscr{H}$ *given by* (1.1) *and holds*

$$\sum_{m=2}^{\infty}\left[\Psi(\varnothing,\beta,m,\phi_m)|c_m|+\varphi(\varnothing,\beta,m,\phi_m)|d_m|\right]\leq 1-\left(\frac{\beta-2\beta\varnothing+1}{1-\beta}\right)|d_1|,\ \textit{where}$$

$\Psi(\varnothing,\beta,m,\phi_m)=\frac{\phi_m\left[\beta(1-\varnothing)-m^2\varnothing+m\varnothing-m+m\varnothing\beta\right]}{1-\beta}$ *and* $\varphi(\varnothing,\beta,m,\phi_m)=\frac{\phi_m\left[\beta(1-\varnothing)-m^2\varnothing+m\varnothing+m-m\varnothing\beta\right]}{1-\beta}$, $\phi_m(\alpha,\mu,\lambda,s,q)$ *given by* (1.2), $\varnothing\in$ *[0,1]* , $\beta\in$ *[0,1)*. *Then* $f\in\mathfrak{H}\mathscr{H}^{s}(\alpha,\varnothing,\lambda,\beta)$. *When* $\varnothing=0$ *in Corollary 2.3, then we have the next Corollary.*

Corollary 2.4. *Let f(z)=h(z) + $\overline{g(z)} \in S\mathscr{H}$ given by* (1.1) *and holds*

$$\sum_{m=2}^{\infty}\left[\frac{\phi_m\left[\beta-m\right]}{1-\beta}\left|c_m\right|+\frac{\phi_m\left[\beta+m\right]}{1-\beta}\left|d_m\right|\right]\leq 1-\left(\frac{1+\beta}{1-\beta}\right)\left|d_1\right|,$$

where $\phi_m(\alpha,\mu,\lambda,s,q)$ given by (1.2), *$\beta\in[0,1)$. Then $f\in\mathfrak{H}\mathscr{H}^{s}(\alpha,\lambda,\beta)$.*
The condition (2.1) *is also essential for functions belong to $\overline{\mathfrak{H}\mathscr{H}_q^{\mu,s}}(\alpha,\varrho,\lambda,\beta)$, this is what clarified in the following theorem.*

Theorem 2.5. *Let $f=h+\overline{g}$ with h and g given by* (1.1) *and $f_s=h_s+\overline{g_s}$ with h_s and g_s given by* (1.4). *Then $f_s\in\overline{\mathfrak{H}\mathscr{H}_q^{\mu,s}}(\alpha,\varrho,\lambda,\beta)$ if and only if the condition* (2.1) *is satisfied .*
Proof . *Since $\overline{\mathfrak{H}\mathscr{H}_q^{\mu,s}}(\alpha,\varrho,\lambda,\beta)\subset\mathfrak{H}\mathscr{H}_q^{\mu,s}(\alpha,\varrho,\lambda,\beta)$. Then the " if " part follows from Theorem 2.1 noting that if the functions h and g in $f=h+\overline{g}\in\mathfrak{H}\mathscr{H}_q^{\mu,s}(\alpha,\varrho,\lambda,\beta)$ are given in* (1.4) *then $f\in\overline{\mathfrak{H}\mathscr{H}_q^{\mu,s}}(\alpha,\varrho,\lambda,\beta)$. For the "only if" part, we show (by contradiction) that $f\notin\overline{\mathfrak{H}\mathscr{H}_q^{\mu,s}}(\alpha,\varrho,\lambda,\beta)$ if the condition* (2.1) *does not hold.*

Thus, we write

$$Re\left(\frac{z+\sum_{m=2}^{\infty}[m]_q\,\phi_m c_m z^m(1+[m]_q\varrho-\varrho+(-1)^s\sum_{m=1}^{\infty}[m]_q\,\phi_m d_m z^m\left(-1-\varrho+[m]_q\varrho\right)}{z+\sum_{m=2}^{\infty}\phi_m c_m z^m\left(1-\varrho+[m]_q\varrho\right)+(-1)^s\sum_{m=1}^{\infty}\phi_m d_m z^m\left(1-\varrho-[m]_q\varrho\right)}\right)\geq\beta,$$

or equivalent to

$$Re\left(\frac{z+\sum_{m=2}^{\infty}[m]_q\,\phi_m c_m z^m(1+[m]_q\varrho-\varrho+(-1)^s\sum_{m=1}^{\infty}[m]_q\,\phi_m d_m z^m\left(-1-\varrho+[m]_q\varrho\right)}{z+\sum_{m=2}^{\infty}\phi_m c_m z^m\left(1-\varrho+[m]_q\varrho\right)+(-1)^s\sum_{m=1}^{\infty}\phi_m d_m z^m\left(1-\varrho-[m]_q\varrho\right)}\right)-\beta\geq 0,$$

That is

$$Re\left(\left[(1-\beta)z+\sum_{m=2}^{\infty}\phi_m\left[\beta(1-\varrho)-[m]_q^2\varrho+[m]_q\varrho-[m]_q+[m]_q\varrho\beta\right]\right.\right.$$
$$\left.\left|c_m\right|z^m+(-1)^s\sum_{m=1}^{\infty}\phi_m\left[\beta(1-\varrho)-[m]_q^2\varrho+[m]_q\varrho+[m]_q-[m]_q\varrho\beta\right]\left|d_m\right|z^m\right]$$
$$\left.\left[z+\sum_{m=2}^{\infty}\phi_m\left|c_m\right|z^m\left(1-\varrho+[m]_q\varrho\right)+(-1)^s\sum_{m=1}^{\infty}\phi_m\left|d_m\right|z^m\left(1-\varrho-[m]_q\varrho\right)\right]^{-1}\right)\geq 0.$$

The above condition satisfies for all values of z .By choosing the values of z on the positive real axis ($0\leq z=r<1$) *we get*

$$\begin{aligned}&Re\left(\left[(1-\beta)+\sum_{m=2}^{\infty}\phi_m\left[\beta(1-\varrho)-[m]_q^2\varrho+[m]_q\varrho-[m]_q+[m]_q\varrho\beta\right]\left|c_m\right|r^{m-1}\right.\right.\\&\left.+\sum_{m=1}^{\infty}\phi_m\left[\beta(1-\varrho)-[m]_q^2\varrho+[m]_q\varrho+[m]_q-[m]_q\varrho\beta\right]\left|d_m\right|r^{m-1}\right]\\&\left.\left[1+\sum_{m=2}^{\infty}\phi_m c_m r^{m-1}\left(1-\varrho+[m]_q\varrho\right)+\sum_{m=1}^{\infty}\phi_m d_m r^{m-1}\left(1-\varrho-[m]_q\varrho\right)\right]^{-1}\right)\geq 0\end{aligned}\tag{2.2}$$

We note that if the condition (2.1) *does not satisfy, then numerator in* (2.2) *is negative . This contradicts with $f\in\overline{\mathfrak{H}\mathscr{H}_q^{\mu,s}}(\alpha,\varrho,\lambda,\beta)$. Hence, the proof is complete.* □

3. Convolution

In this section, we show that $\overline{\mathfrak{H}\mathscr{H}_q^{\mu,s}}(\alpha, \sigma, \lambda, \beta)$ is closed under convolution for $f_s(z)$= z-$\sum_{m=2}^{\infty}|c_m| z^m + (-1)^s \sum_{m=1}^{\infty}|d_m| z^m$, and H_s(z)= z-$\sum_{m=2}^{\infty}|p_m| z^m + (-1)^s \sum_{m=1}^{\infty}|l_m| z^m$, the convolution is given by

$$(f_s * H_s)(z) = f_s(z) * H_s(z) = z - \sum_{m=2}^{\infty}|c_m p_m| z^m + (-1)^s \sum_{m=1}^{\infty}|d_m l_m| z^m$$

Theorem 3.1. *For 0 $\leq \beta_1 \leq \beta < 1$, let $f_s \in \overline{\mathfrak{H}\mathscr{H}_q^{\mu,s}}(\alpha, \sigma, \lambda, \beta)$ and $H_s \in \overline{\mathfrak{H}\mathscr{H}_q^{\mu,s}}(\alpha, \sigma, \lambda, \beta_1)$. Then*

$$f_s * H_s \in \overline{\mathfrak{H}\mathscr{H}_q^{\mu,s}}(\alpha, \sigma, \lambda, \beta) \subset \overline{\mathfrak{H}\mathscr{H}_q^{\mu,s}}(\alpha, \sigma, \lambda, \beta_1)$$

Proof . *To show that the coefficients of $f_s * H_s$ satisfy the condition* (2.1)*, for $H_s \in \overline{\mathfrak{H}\mathscr{H}_q^{\mu,s}}(\alpha, \sigma, \lambda, \beta_1)$ we note that $|p_m| < 1$ and $|l_m| < 1$, we consider $f_s * H_s$ as follows*

$$\sum_{m=2}^{\infty} \frac{\phi_m \left[\beta_1 (1-\sigma) - [m]_q^2 \sigma + [m]_q \sigma - [m]_q + [m]_q \sigma \beta_1\right]}{1-\beta_1} |c_m| |p_m| +$$
$$\sum_{m=1}^{\infty} \frac{\phi_m \left[\beta_1 (1-\sigma) - [m]_q^2 \sigma + [m]_q \sigma + [m]_q - [m]_q \sigma \beta_1\right]}{1-\beta_1} |d_m| |l_m|$$
$$\leq \sum_{m=2}^{\infty} \frac{\phi_m \left[\beta_1 (1-\sigma) - [m]_q^2 \sigma + [m]_q \sigma - [m]_q + [m]_q \sigma \beta_1\right]}{1-\beta_1} |c_m| +$$
$$\sum_{m=1}^{\infty} \frac{\phi_m \left[\beta_1 (1-\sigma) - [m]_q^2 \sigma + [m]_q \sigma + [m]_q - [m]_q \sigma \beta_1\right]}{1-\beta_1} |d_m|$$
$$\leq \sum_{m=2}^{\infty} \Psi(\sigma, \beta, m, \phi_m) |c_m| + \sum_{m=1}^{\infty} \varphi(\sigma, \beta, m, \phi_m) |d_m| \leq 1,$$

Since $0 \leq \beta_1 \leq \beta < 1$ *and* $f_s \in \overline{\mathfrak{H}\mathscr{H}_q^{\mu,s}}(\alpha, \sigma, \lambda, \beta)$. *Therefore*

$$f_s * H_s \in \overline{\mathfrak{H}\mathscr{H}_q^{\mu,s}}(\alpha, \sigma, \lambda, \beta) \subset \overline{\mathfrak{H}\mathscr{H}_q^{\mu,s}}(\alpha, \sigma, \lambda, \beta_1).$$

□

4. Neighborhood

Suppose that M_s(z)= = z-$\sum_{m=2}^{\infty}|A_m| z^m + (-1)^s \sum_{m=1}^{\infty}|B_m| \overline{z}^m$, we call the set

$$N_T(f) = \{M_s : \sum_{m=1}^{\infty} [m]_q (|c_m - A_m| + |d_m - B_m|) \leq T\} \tag{4.1}$$

is the T-neighborhood of f.

From (4.1), we get

$$\sum_{m=1}^{\infty} [m]_q (|c_m - A_m| + |d_m - B_m|) = |d_1 - B_1| + \sum_{m=2}^{\infty} [m]_q (|c_m - A_m| + |d_m - B_m|) \leq T \tag{4.2}$$

Theorem 4.1. *Let $f_s \in \overline{\mathfrak{H}\mathscr{H}_q^{\mu,s}}(\alpha, \sigma, \lambda, \beta)$ and $T \leq \beta$. If $M_s \in N_T(f)$, then M_s is harmonic starlike function.*

Proof . *Assume that $M_s \in N_T(f)$, then we have*

$$\sum_{m=2}^{\infty} [m]_q (|A_m| + |B_m|) + |B_1| \leq \sum_{m=2}^{\infty} [m]_q (|c_m - A_m| + |d_m - B_m|) +$$
$$\sum_{m=2}^{\infty} [m]_q (|c_m|) + [m]_q (|d_m|) + |B_1 - d_1| + |d_1|$$
$$\leq \sum_{m=2}^{\infty} [\Psi(\sigma,\beta,m,\phi_m)(|c_m - A_m|) + \varphi(\sigma,\beta,m,\phi_m)(|d_m - B_m|)] + |B_1 - d_1| + |d_1| +$$
$$\sum_{m=2}^{\infty} [\Psi(\sigma,\beta,m,\phi_m)|c_m| + \varphi(\sigma,\beta,m,\phi_m)|d_m|] \leq T + |d_1| + (1 - \beta - |d_1|) \leq 1.$$

Hence, $M_s(z)$ is a harmonic starlike function. □

5. Extreme points

In this section, the extreme points of the closed convex hull denoted by clco $\overline{\mathfrak{H}\mathscr{H}_q^{\mu,s}}(\alpha, \sigma, \lambda, \beta)$ are obtained.

Theorem 5.1. *Let f_s be given by (1.4). Then $f_s \in \overline{\mathfrak{H}\mathscr{H}_q^{\mu,s}}(\alpha, \sigma, \lambda, \beta)$ if and only if $f_s(z) = \sum_{m=1}^{\infty} (x_m h_m(z) + y_m g_{sm}(z))$, where*

$$h_1(z) = z, \quad h_m(z) = z - \frac{1}{\Psi(\sigma,\beta,m,\phi_m)} z^m, \qquad m = 2, 3, \ldots \quad \text{and}$$
$$g_{sm}(z) = z + (-1)^s \frac{1}{\varphi(\sigma,\beta,m,\phi_m)} \overline{z}^m, \qquad m = 1, 2, \ldots$$
$$x_m \geq 0, \quad y_m \geq 0, \qquad x_1 = 1 - \sum_{m=2}^{\infty} x_m - \sum_{m=1}^{\infty} y_m.$$

Specially, the extreme points of $\overline{\mathfrak{H}\mathscr{H}_q^{\mu,s}}(\alpha, \sigma, \lambda, \beta)$ are $\{h_m\}$ and $\{g_{sm}\}$.

Proof . *Assume that f_s can be written as*

$$f_s(z) = \sum_{m=1}^{\infty} (x_m h_m(z) + y_m g_{sm}(z)) = \sum_{m=1}^{\infty} (x_m + y_m) z - \sum_{m=2}^{\infty} \frac{1}{\Psi(\sigma,\beta,m,\phi_m)} x_m z^m$$
$$+ (-1)^s \sum_{m=1}^{\infty} \frac{1}{\varphi(\sigma,\beta,m,\phi_m)} y_m \overline{z}^m,$$

then

$$\sum_{m=2}^{\infty} \Psi(\sigma,\beta,m,\phi_m) \left(\frac{1}{\Psi(\sigma,\beta,m,\phi_m)} x_m \right) + \sum_{m=1}^{\infty} \varphi(\sigma,\beta,m,\phi_m) \left(\frac{1}{\varphi(\sigma,\beta,m,\phi_m)} y_m \right)$$
$$= \sum_{m=2}^{\infty} x_m + \sum_{m=1}^{\infty} y_m = 1 - x_1 \leq 1$$

Hence $f_s(z) \in clco\ \overline{\mathfrak{H}\mathscr{H}}_q^{\mu,s}(\alpha, \sigma, \lambda, \beta)$.

Conversely, suppose that $f_s \in clco\ \overline{\mathfrak{H}\mathscr{H}}_q^{\mu,s}(\alpha, \sigma, \lambda, \beta)$ *and* $x_1 = 1 - \sum_{m=2}^{\infty} x_m - \sum_{m=1}^{\infty} y_m$. *Let* $x_m = \Psi(\sigma, \beta, m, \phi_m)\, c_m, (m = 2, 3, \ldots)$ *and* $y_m = \varphi(\sigma, \beta, m, \phi_m)\, d_m, \quad (m = 1, 2, \ldots)$, *we get*

$$f_s(z) = z - \sum_{m=2}^{\infty} c_m z^m + \ (-1)^s \sum_{m=1}^{\infty} d_m \overline{z}^m = \ z - \sum_{m=2}^{\infty} \frac{1}{\Psi(\sigma, \beta, m, \phi_m)} x_m z^m$$

$$+ (-1)^s \sum_{m=1}^{\infty} \frac{1}{\varphi(\sigma, \beta, m, \phi_m)} y_m \overline{z}^m = z - \sum_{m=2}^{\infty} (z - h_m(z))\, x_m - \sum_{m=1}^{\infty} (z - g_{sm}(z))\, y_m$$

$$= (1 - \sum_{m=2}^{\infty} x_m - \sum_{m=1}^{\infty} y_m) z + \sum_{m=2}^{\infty} x_m h_m(z) + \sum_{m=1}^{\infty} y_m g_{sm}(z) = \sum_{m=1}^{\infty} \Big(x_m h_m(z) + y_m g_{sm}(z) \Big).$$

□

6. The Distortion

The following theorem gives the distortion bounds in $\overline{\mathfrak{H}\mathscr{H}}_q^{\mu,s}(\alpha, \sigma, \lambda, \beta)$.

Theorem 6.1. *Let* $f \in \overline{\mathfrak{H}\mathscr{H}}_q^{\mu,s}(\alpha, \sigma, \lambda, \beta)$. *Then*

$$|f(z)| \geq (1 - |d_1|)\, r - \frac{1}{\phi_2} \left(\frac{1-\beta}{\mathscr{G}} - \frac{1+\beta}{\mathscr{G}} |d_1| \right) r^2$$

and

$$|f(z)| \leq (1 + |d_1|)\, r + \frac{1}{\phi_2} \left(\frac{1-\beta}{\mathscr{G}} - \frac{1+\beta}{\mathscr{G}} |d_1| \right) r^2$$

where $\phi_m(\alpha, \mu, \lambda, s, q)$ *given by* (1.2) *with* $c_1 = 1$ *and* $\mathscr{G} = \Big[\beta(1 - \sigma) - [2]_q^2 \sigma + [2]_q \sigma - [2]_q + [2]_q \sigma \beta \Big]$

Proof . *To prove the left-hand, we assume that* $f \in \overline{\mathfrak{H}\mathscr{H}}_q^{\mu,s}(\alpha, \sigma, \lambda, \beta)$ *then*

$$|f(z)| \geq (1 - |d_1|)\, r - \sum_{m=2}^{\infty} (|c_m| + |d_m|)\, r^m \geq (1 - |d_1|)\, r - \sum_{m=2}^{\infty} (|c_m| + |d_m|)\, r^2$$

$$\geq (1 - |d_1|)\, r - \frac{1-\beta}{\phi_2 \mathscr{G}} \sum_{m=2}^{\infty} \frac{\phi_2 \mathscr{G}}{1-\beta} (|c_m| + |d_m|)\, r^2$$

$$\geq (1 - |d_1|)\, r - \frac{1-\beta}{\phi_2 \mathscr{G}} \left(1 - \frac{1+\beta}{1-\beta} |d_1| \right) r^2 = (1 - |d_1|)\, r - \frac{1}{\phi_2} \left(\frac{1-\beta}{\mathscr{G}} - \frac{1+\beta}{\mathscr{G}} |d_1| \right) r^2.$$

In the same way, the right –hand is proven. □

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