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On solving Bratu's type equation by perturbation method

Safaa Ali Salem[a], Thair Younis Thanoon[a,*]

[a] *College of Computer Science and Mathematics, University of Mosul, Mosul, Iraq*

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Abstract

In this paper, the perturbation method is employed to obtain an approximate solution of some examples of the Bratu equation by choosing the different values of ε and comparison with the exact solutions. It can be seen that the perturbation method is an alternative technique to be considered in solving many practical problems involving differential equations.

Keywords: Non-linear differential equation; Perturbation method, Bratu's type equation, Approximate solution.
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1. Introduction

Bratu's equation is a nonlinear differential equation that has many applications in mathematics, physics, engineering and other sciences [2,4]. Moreover, Bratu equation is formulated in the form of a non-linear problem with initial or boundary conditions. In this paper, the study will deal with the Bratu type problem in one-dimensional with the initial conditions, which are as follows:

$$\begin{aligned} &\frac{d^2u}{dx^2} + \lambda e^u = 0, \quad 0 < x < 1, \quad \lambda > 0 \\ &u(0) = u'(0) = 0 \end{aligned} \tag{1.1}$$

This problem derives its importance from the first thermal combustion theory, which was created by the simplification of the solid fuel ignition model. Moreover, appeared in the Chandrasekhar

*Corresponding author
Email addresses: safaa.csp97@student.uomosul.edu.iq (Safaa Ali Salem), thairthanoon@uomosul.edu.iq (Thair Younis Thanoon)

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model of the expansion of the universe. It stimulates a thermal reaction process in a rigid material where the process depends on a balance between chemically generated heat and heat transfer by conduction [16, 17],. The exact solution was known, which facilitates the application of tests in different methods by comparing with the approximate solutions which showed the accuracy and efficiency of the methods [3].

The Perturbation Method (PM) is a well-known technique that one of the first to be used to solve many types of nonlinear problems. Poisson has been invented the approach of perturbation method and expanded by Poincare [12]. In early of nineteenth century the approach of perturbation method was developed, until the later of nineteenth century never any one used perturbation method solve nonlinear differential equations. Celestial mechanics, fluid mechanics, and aerodynamics were the areas where the most work was expended [7].

This type of equation has been solved by many researchers using different methods. Including the finite difference method, the variable frequency method, and the Adomian polynomial decomposition method for solving the value of the 1D plane Limits of the Bratu, boundary value problem [16]. In [14], utilized sinc-collocation method for solving Bratu's problem which was already presented. Chebyshev wavelets with collocation method has been employed for solving 1D Bratu problem [6]. Analytic approximate solutions using homotopy analysis has been presented by [11]. In [15], variable iteration method with three terms to expand the nonlinear part of the solution Bratu boundary value problem. Furthermore, used a non-polynomial spline method for solving bratu's equation [18], and in [13] study a good survey of the properties and different treatments of 1D and 2D Bratu problems, using finite difference and nonstandard finite difference for solving this equation with a simple starting function for this object. New improved variational homotopy perturbation method for bratu-type problems as did other scholars [1,5,8,9] .

In this study, the asymptotic expansion perturbation method applied on Bratu-type problem which was not presented in this form in previous work by the researchers. This is done by the effect of the non-linear part with a small coefficient called the perturbation coefficient and then expressing each solution in a power series with respect to obtaining an approximate solution after that. The results are compared with the exact solution.

2. Overview of the Perturbation Method

Let the Bratu's type perturbated equation of 1D be in the form

$$\frac{d^2u}{dx^2} + \lambda e^{\varepsilon u} = 0 \tag{2.1}$$

Assuming that the nonlinear term in (2.1) is a small perturbation and that the solution for (2.1) can be expressed as a power series in the small parameter:

$$u(x) = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots \tag{2.2}$$

It is feasible to create a series of differential equations that may be integrated recursively to determine the values for the functions by substituting (2.2) into (2.1) and equating terms with equal powers of ε,

$$u_0(x), u_1(x), u_2(x), \dots .$$

3. ILLUSTRATIVE EXAMPLES

Example 1. Consider the Bratu-type initial value problem (1.1), by choosing $\lambda = -2$ yields:

$$\begin{aligned} &u'' - 2e^{u} = 0, \quad 0 < x < 1 \\ &u(0) = u'(0) = 0 \end{aligned} \tag{3.1}$$

In [16], the exact solution of equation (3.1) which is given as follows:

$$u(x) = x^2 + \frac{x^4}{6} + \frac{2x^6}{45} + \frac{17x^8}{1260} + \dots \tag{3.2}$$

To solve equation (3.1) by perturbation method insert the perturbation parameter yields,

$$\begin{aligned} &u'' - 2e^{\varepsilon u} = 0 \\ &u(0) = u'(0) = 0 \end{aligned} \tag{3.3}$$

By substituting (2.2) into (3.3) we have

$$\begin{aligned} &u'' - 2e^{\varepsilon u_0} = 0 \\ &u_0'' + \varepsilon u_1'' + \varepsilon^2 u_2'' - 2\left(1 + \varepsilon u_0 + \varepsilon^2 \frac{u_0^2}{2} + \dots\right) = 0 \end{aligned} \tag{3.4}$$

Require that the terms of the same order are equal one by one

$$\left(\varepsilon^0\right) u_0'' - 2 = 0 \qquad u_0(0) = u_0'(0) = 0 \tag{3.5}$$

$$\left(\varepsilon^1\right) u_1'' - 2u_0 = 0 \qquad u_1(0) = u_1'(0) = 0 \tag{3.6}$$

$$\left(\varepsilon^2\right) u_2'' - u_0^2 = 0 \qquad u_2(0) = u_2'(0) = 0 \tag{3.7}$$

The solution of equations (3.5),(3.6) and (3.7) yields:

$$u_0(x) = x^2, u_1(x) = \frac{x^4}{6}, u_2(x) = \frac{x^6}{30} \tag{3.8}$$

By substituting the values of u_0, u_1, u_2 in (2.2) we get the approximate solution for (3.3), as it is shown:-

$$u(x) = x^2 + \varepsilon\frac{x^4}{6} + \varepsilon^2\frac{x^6}{30}$$

The numerical solution by perturbation method with deferent values of ε will be illustrate in tables and figures $1, 2$ and 3.

Table 1: Comparison between the exact solution and the numerical solution using perturbation method with $\varepsilon = 1$ results with absolute errors.

x	Exact solutions u(x)	Numerical $u_\varepsilon(x)$ solution by using PM	Absolute error
0.1	0.01001666667	0.01001667784	1.117×10^{-8}
0.2	0.04026666667	0.04026739302	$7.2635\ \times10^{-7}$
0.3	0.09135000000	0.09135849054	0.00000849054
0.4	0.1642666667	0.1643160787	0.0000494120
0.5	0.2604166667	0.2606135293	0.0001968626
0.6	0.3816000000	0.3822183771	0.0006183771
0.7	0.5300166667	0.5316670207	0.0016503540
0.8	0.7082666667	0.7121780216	0.0039113549
0.9	0.9193500000	0.9278172048	0.0084672048

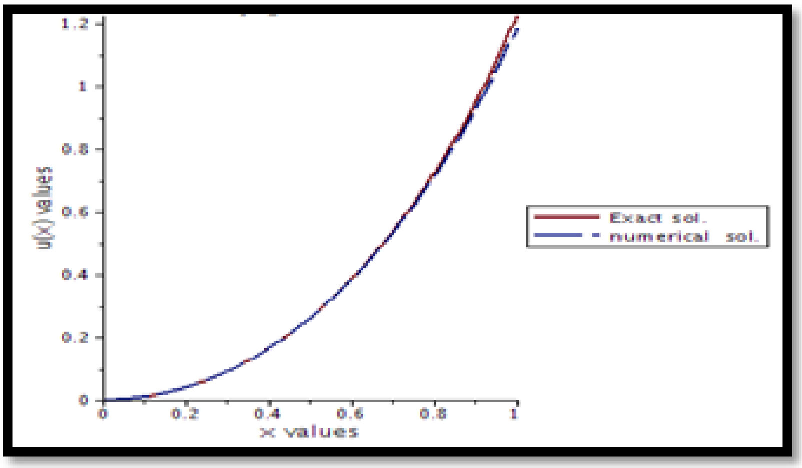

Figure 1: Comparison between graph of exact solution and numerical solution with $\varepsilon = 1$

Table 2: Comparison between the exact solution and the numerical solution using perturbation method with $\varepsilon = 0.5$ results with absolute errors.

x	Exact solutions u(x)	Numerical $u_\varepsilon(x)$ solution by using PM	Absolute error
0.1	0.01001666667	0.01000835000	0.00000831667
0.2	0.04013440000	0.04013440000	0.00013226667
0.3	0.09135000000	0.09068715000	0.09135000000
0.4	0.1642666667	0.1622016000	0.0020650667
0.5	0.2604166667	0.2554687500	0.0049479167
0.6	0.3816000000	0.3715776000	0.0100224000
0.7	0.5300166667	0.5119691500	0.0180475167
0.8	0.7082666667	0.6785024000	0.0297642667
0.9	0.9193500000	0.8735323500	0.0458176500

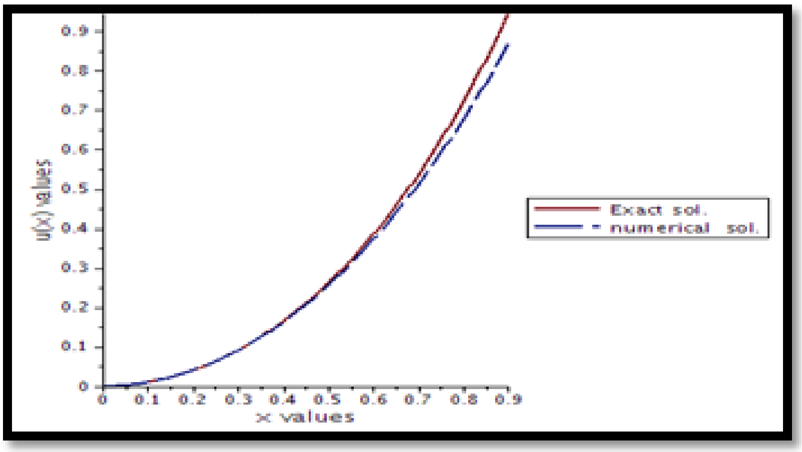

Figure 2: Comparison between graph of exact solution and numerical solution with $\varepsilon = 0.5$

Table 3: Comparison between the exact solution and the numerical solution using perturbation method with $\varepsilon = 0.1$ with absolute errors.

x	Exact solutions u(x)	Numerical $u_\varepsilon(x)$ solution by using PM	Absolute error
0.1	0.01001666667	0.01000016667	0.00001650000
0.2	0.04026666667	0.04000266709	0.00026399958
0.3	0.09135000000	0.09001350486	0.00133649514
0.4	0.1642666667	0.1600426940	0.0042239727
0.5	0.2604166667	0.2501042708	0.0103123959
0.6	0.3816000000	0.3602163110	0.0213836890
0.7	0.5300166667	0.4904009510	0.0396157157
0.8	0.7082666667	0.6406844143	0.0675822524
0.9	0.9193500000	0.8110970429	0.1082529571

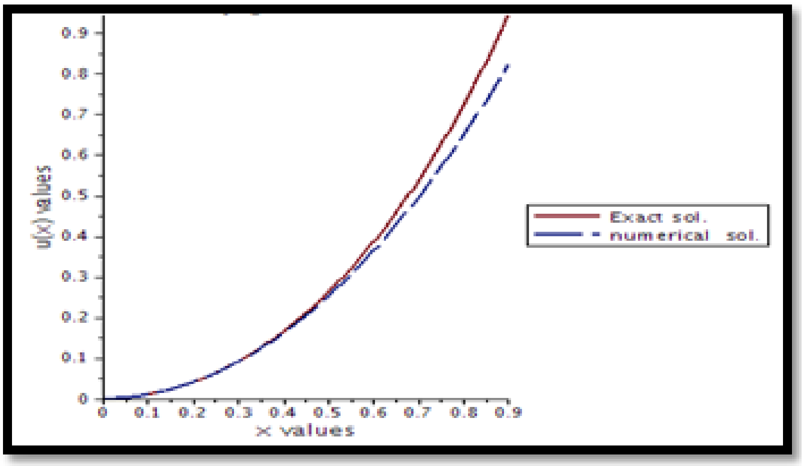

Figure 3: Comparison between graph of exact solution and numerical solution with $\varepsilon = 0.1$

From the last tables and figures, we conclude that the numerical solution using perturbation method it is closed to the exact solution when ε approaches to one. Because if $\varepsilon = 1$ the equation perturbed parameter ε becomes look like the original equation (3.1).

Example 2. Consider the Bratu-type initial value problem (1.1), by choosing $\lambda = -1$ yields:

$$\begin{aligned} &u'' - e^{2u} = 0, \quad 0 < x < 1 \\ &u(0) = u'(0) = 0 \end{aligned} \tag{3.9}$$

In [10], the exact solution of equation (3.9) which is given as follows:

$$u(x) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \frac{17x^8}{2520} + \cdots \tag{3.10}$$

To solve equation (3.9) by perturbation method insert the perturbation parameter yields,

$$\begin{aligned} &u'' - e^{2\varepsilon u} = 0, 0 < x < 1 \\ &u(0) = u'(0) = 0 \end{aligned} \tag{3.11}$$

By substituting (2.2) into (3.11) yields,

$$\begin{aligned} &u_0'' + \varepsilon u_1'' + \varepsilon^2 u_2'' + \varepsilon^3 u_3'' - e^{2\varepsilon(u_0 + \varepsilon u_1 + \ldots)} = 0 \\ &u_0'' + \varepsilon u_1'' + \varepsilon^2 u_2'' + \varepsilon^3 u_3'' - e^{2\varepsilon u_0 + 2\varepsilon^2 u_1} = 0 \\ &u_0'' + \varepsilon u_1'' + \varepsilon^2 u_2'' + \varepsilon^3 u_3'' - \left((1 + 2\varepsilon u_0 + \ldots)\left(1 + 2\varepsilon^2 u_1 + \ldots\right)\right) = 0 \\ &u_0'' + \varepsilon u_1'' + \varepsilon^2 u_2'' + \varepsilon^3 u_3'' - 1 - 2\varepsilon^2 u_1 - 2\varepsilon u_0 - 4\varepsilon^3 u_0 u_1 + \ldots = 0 \end{aligned} \tag{3.12}$$

Require that the terms of the same order are equal one by one:

$$\left(\varepsilon^0\right) u_0'' - 1 = 0 \qquad u_0(0) = u_0'(0) = 0 \tag{3.13}$$

$$\left(\varepsilon^1\right) u_1'' - 2u_0 = 0 \qquad u_1(0) = u_1'(0) = 0 \tag{3.14}$$

$$\left(\varepsilon^2\right) u_2'' - 2u_1 = 0 \qquad u_2(0) = u_2'(0) = 0 \tag{3.15}$$

$$\left(\varepsilon^3\right) u_3'' - 4u_0 u_1 = 0 \qquad u_3(0) = u_3'(0) = 0 \tag{3.16}$$

The solution of equations (3.13),(3.14),(3.15) and (3.16) yields:

$$u_0(x) = \frac{x^2}{2} \quad , u_1(x) = \frac{x^4}{12} \quad , u_2(x) = \frac{x^6}{180}, u_3(x) = \frac{x^8}{336}$$

By substituting the values of u_0, u_1, u_2, u_3 in (2.2), we get the approximate solution for(3.9) as it is shown:-

$$u(x) = \frac{x^2}{2} + \varepsilon \frac{x^4}{12} + \varepsilon^2 \frac{x^6}{180} + \varepsilon^3 \frac{x^8}{336} + \cdots$$

The numerical solution by perturbation method with deferent values of will be showed in tables and figures $4, 5$ and 6.

Table 4: Comparison between the exact solution and the numerical solution using perturbation method with $\varepsilon = 1$ results with absolute errors.

x	Exact solutions u(x)	Numerical $u_\varepsilon(x)$ solution by using PM	Absolute error
0.1	0.005008362302	0.005008338916	2.3386 $\times 10^{-8}$
0.2	0.02013518730	0.02013369588	0.00000149142
0.3	0.04569611786	0.04567922926	0.00001688860
0.4	0.08225198730	0.08215787949	0.00009410781
0.5	0.1306609623	0.1303058117	0.0003551506
0.6	0.0010464518	0.1911050911	0.0010464518
0.7	0.2684164194	0.2658194471	0.0025969723
0.8	0.3617271873	0.3560480828	0.0056791045
0.9	0.4750699179	0.4638035899	0.0112663280

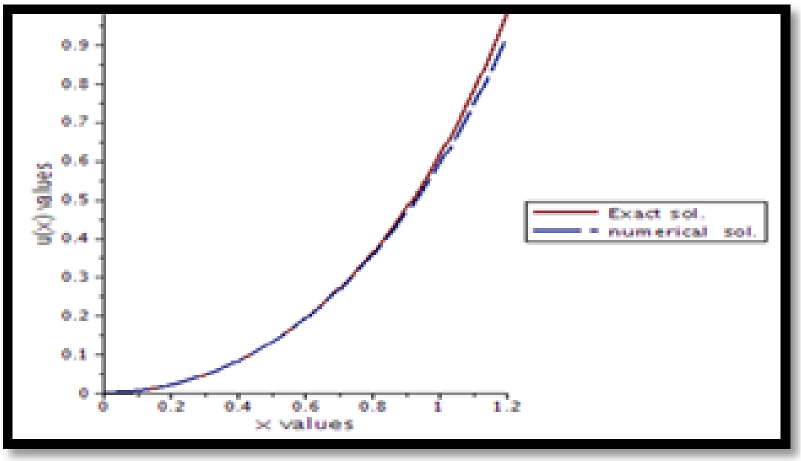

Figure 4: Comparison between graph of exact solution and numerical solution with $\varepsilon = 1$

Table 5: Comparison between the exact solution and the numerical solution using perturbation method with $\varepsilon = 0.5$ with absolute errors.

x	Exact solutions u(x)	Numerical $u_\varepsilon(x)$ solution by using PM	Absolute error
0.1	0.005008362302	0.005004168059	0.000004194243
0.2	0.02013518730	0.02006675643	0.00006843087
0.3	0.04569611786	0.04533853491	0.00035758295
0.4	0.08225198730	0.08107257938	0.00117940792
0.5	0.1306609623	0.1276272022	0.0030337601
0.6	0.0010464518	0.1854705364	0.0066810065
0.7	0.2684164194	0.2551872566	0.0132291628
0.8	0.3617271873	0.3374880548	0.0242391325
0.9	0.4750699179	0.4332226300	0.0418472879

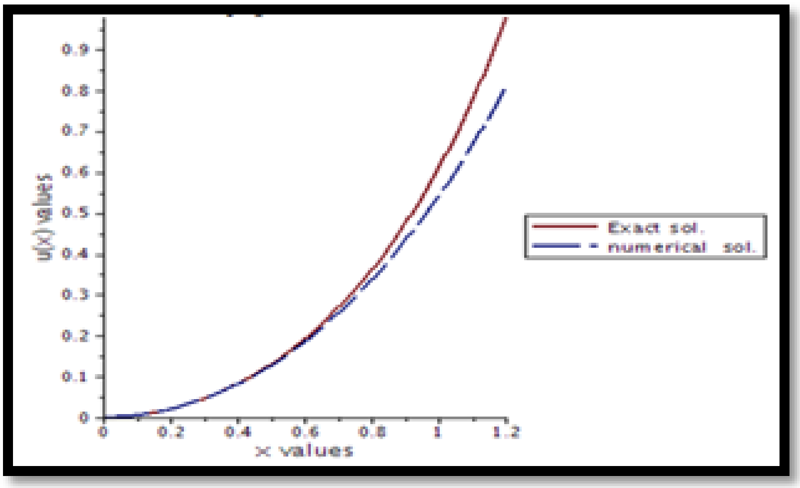

Figure 5: Comparison between graph of exact solution and numerical solution with $\varepsilon = 0.5$

Table 6: Comparison between the exact solution and the numerical solution using perturbation method with $\varepsilon = 0.1$ with absolute errors

x	Exact solutions u(x)	Numerical $u_\varepsilon(x)$ solution by using PM	Absolute error
0.1	0.005008362302	0.005000833389	0.000007528913
0.2	0.02013518730	0.02001333690	0.00012185040
0.3	0.04569611786	0.04506754068	0.00062857718
0.4	0.08225198730	0.08021356268	0.00203842462
0.5	0.1306609623	0.1255217121	0.0051392502
0.6	0.0010464518	0.1810826379	0.0110689050
0.7	0.2684164194	0.2470075269	0.0214088925
0.8	0.3617271873	0.3234283553	0.0382988320
0.9	0.4750699179	0.4104982006	0.0645717173

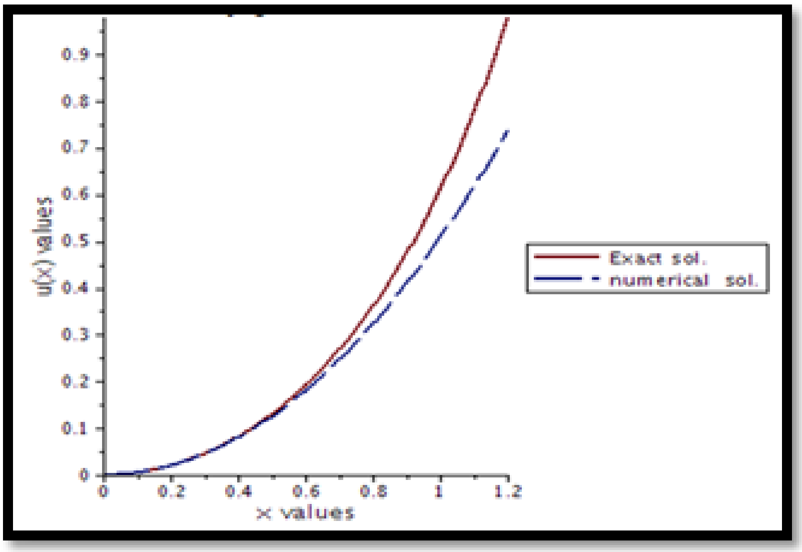

Figure 6: Comparison between graph of exact solution and numerical solution with $\varepsilon = 0.1$

From the last tables and figures, we conclude that the numerical solution using perturbation method it is closed to the exact solution when ε approaches to one. Because if $\varepsilon = 1$ the equation perturbed parameter becomes look like the original equation (3.9).

4. Conclusion

In this paper, the perturbation method has been applied to the nonlinear Bratu differential equation. The approximate solution to this equation has been successfully arrived at, by comparing the exact solution and the numerical solution, we can see that the results are closed and accurate by choosing different values for the perturbated parameter ε . The recommendation in this study is to use this method or any other type of perturbation method and apply it to other types of nonlinear differential equations.

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