Mechanics of Advanced Composite Structures 9 (2022) 89 – 104
This is an Open Access paper licensed under the Creative Commons License CC-BY 4.0 license.
Mechanics of Advanced Composite Structures

Semnan University

journal homepage: https://MACS.journals.semnan.ac.ir

# A New Higher-Order Shear and Normal Deformation Theory for the Free Vibration Analysis of Laminated Shells

**B. M. Shinde[a]*, A. S. Sayyad[b]**

[a]*Department of Civil Engineering, Sanjivani College of Engineering, Savitribai Phule Pune University, Kopargaon-423603, Maharashtra, India*
[b]*Department of Structural Engineering, Sanjivani College of Engineering, Savitribai Phule Pune University, Kopargaon-423603, Maharashtra, India*

**KEYWORDS**

Fifth-order theory;
Shear deformation;
Normal deformation;
Free vibration;
Laminated shells.

**ABSTRACT**

In this paper, the free vibration analysis of laminated composite and sandwich, cylindrical and spherical shells is presented using a new higher-order shear and normal deformation theory. The novelty of the present theory is that it includes the effects of both transverse shear and normal deformations along with higher order expansions of displacement field. A fifth-order polynomial type shape function is used in the in-plane displacements to represent the effect of transverse shear deformation for the first time whereas transverse displacement is a function of x, y, and z coordinates to account for the effect of transverse normal deformation. The equations of motion are derived using Hamilton's principle. Navier's solution technique is employed to obtain the non-dimensional fundamental frequencies. To validate the accuracy of the present theory, the present results are compared with other higher-order theories available in the literature. It is observed that the values of fundamental frequencies obtained using the present theory are in close agreement with those available in the literature.

## 1. Introduction

Laminated composite and sandwich shells are having a wide application in the area of aircraft, spacecraft, undermining, marine constructions, etc. due to their attractive features such as high load carrying capacity, large span-to-depth ratio, high strength-to-weight ratio, high stiffness-to-weight ratio, etc.

In the case of mechanical and structural industries, the structural components get subjected to extreme loads and deformations due to vibration and resonance, which leads to catastrophic failure. Also, in the case of aircraft, to avoid the severe consequences during in-flight conditions the wings need to be designed to eliminate the resonance, in the case of civil structures it needs to be designed considering the wind-induced vibration.

Therefore, static and vibration analysis of laminated composite shells becomes an active area of research among researchers. 169 years ago Kirchoff [1] has developed a classical shell theory (CST) for the analysis of thin shells which neglect the effect of shear deformation. However, this theory is not useful for the analysis of thick shells. Therefore, Mindlin [2] has developed a first-order shear deformation theory (FSDT) which considers the effect of shear deformation for the first time. However, this theory fails to satisfy the zero transverse shear stress condition at top and bottom surfaces of the shell. These drawbacks of the CST and FSDT lead to the development of higher-order shear deformation theories (HSDT). Qatu [3-5], Qatu et al. [6], Mallikarjuna and Kant [7], Thai and Kim [8], Sayyad and Ghugal [9-11] have published several review articles on free vibration analysis of laminated composite beams, plates, and shells.

Reddy [12] has developed a well-known parabolic shear deformation theory (PSDT) for the analysis of laminated composite plates and shells. Bhimaraddi [13] has presented a free

* Corresponding author. Tel.: +91-9096239178
*E-mail address: bhartishinde1987@yahoo.co.in*

DOI: 10.22075/macs.2022.22741.1327
Received 2021-02-22; Received in revised form 2021-12-21; Accepted 2022-01-01.
© 2022 Published by Semnan University Press. All rights reserved.

vibration analysis of doubly curved shells using three dimensional elasticity theory assuming that the ratio of shell thickness to its middle surface radius is negligible. Timaraci and Soldatos [14] studied the dynamic behavior of symmetric cross-ply circular shells using various shear deformation theories. Khare et al. [15] have developed a finite element formulation using a higher-order facet shell element for the free vibration analysis of laminated composite and sandwich cylindrical and spherical shells. A layerwise shear deformation theory is developed by Ferreira [16] for the static analysis of laminated composite plates. Static and free vibration analysis of laminated composite shells is presented by Ferreira et al. [17] by developing a meshless solution of Reddy's higher-order shell theory. Garg et al. [18] presented a closed-form solution for the free vibration analysis of doubly curved laminated composite and sandwich shells. Pradyumna and Bandopadhyay [19] investigated a C0 finite element formulation based on higher-order shear deformation theory for the static and free vibration analysis of laminated composite shells. Matsunga [20] presented free vibration and stability analysis of cross-ply laminated composite shells, considering the effect of transverse shear and normal deformations. Carrera and Brischetto [21] have presented an analysis of laminated composite shells using refined and mixed shear deformation theories. Brischetto et al. [22] studied a free vibration analysis of sandwich plates and shells by introducing a zig-zag function in the displacement field of classical and higher-order theories. Zhao et al. [23] have applied the Ritz method for the static and dynamic analysis of functionally graded cylindrical shells. Noh and Lee [24] have presented the free vibration analysis of laminated composite shells by developing a finite element formulation based on a third-order shear deformation theory. Bending and free vibration analysis of laminated composite plates and shells is presented by Mantari and Soares [25] using higher-order shear deformation theory. Tornabene [26, 27], Tornabene et al. [28, 29] have proposed a GDQ method for the free vibration analysis of laminated composite and functionally graded shells. Qatu and Asadi [30] presented a comprehensive study on free vibration analysis of spherical, cylindrical, and hyperbolic paraboloidal shells using a Ritz method for various boundary conditions. Asadi et al. [31] have presented a 3D solution for static and vibration analysis of thick deep laminated cylindrical shells. The hierarchical trigonometric Ritz formulation is used by Fazzolari and Carrera [32] for the free vibration analysis of doubly curved laminated composite shells. Taj and Chakrabarti [33] have studied the bending analysis of functionally graded ceramic-metal skew shell panels using a C0 finite element. Dai et al. [34] have obtained an exact series solution for the free vibration analysis of cylindrical shells for various boundary conditions. Wang et al. [35] have predicted the free vibration response of laminated composite circular panels and shells of revolution using a Fourier-Ritz method for various boundary conditions. Rawat et al. [36] have developed a finite element model for the free vibration analysis of thin circular cylindrical shells. Pandey and Pradyumna [37] have presented a thermally induced vibration analysis of functionally graded sandwich plates and shell panels. Biswal et al. [38] studied free vibration and stability analysis of doubly curved laminated shell panels based on Sander's approximation. Fares et al. [39] have presented the bending and free vibration analysis of functionally graded doubly curved shells using a layerwise theory. Monge et al. [40] have carried out an asymptotic evaluation of the best theories for the free vibration analysis of laminated composite and sandwich shells. Cong et al. [41] have extended a third order shear deformation theory of Reddy by developing a new approach to investigate the nonlinear dynamic response of doubly curved sandwich shells. A semi-analytical method is used by Li et al. [42] and Pang et al. [43] to study the free vibration analysis for laminated composite doubly curved shells of revolution. Kiani et al. [44] investigated the free vibration characteristics of functionally graded carbon nanotube skew cylindrical shells based on Chebyshev-Ritz formulation. Sayyad and Ghugal [45] developed a generalized shell theory to investigate the static and dynamic response of laminated composite and sandwich spherical shells. Draiche et al. [46], Allam et al. [47], Zine et al. [48] presented a higher order shear deformation theory for the bending and free vibration analysis of laminated composite, sandwich plates, and shells. Arefi [49-53], Arefi and Rabczuk [54], Arefi and Elyas [55], Arefi and Amabili [56] have presented an electro-elastic and free vibration analysis of piezoelectric doubly curved nanoshells. Arefi and Zenkour [57-58] have presented the bending and free vibration analysis of functionally graded nanobeams whereas Arefi et al. [59] presented the bending response of FG composite doubly curved nanoshells considering the thickness stretching effects.

Draiche et al. [60] presented a static analysis of laminated reinforced plates using first order shear deformation theory. Belbachir et al. [61-62], Abualnour et al. [63], Sahla et al. [64] have applied a four variable refined theory for the static and free vibration analysis of laminated composite plates and shells under mechanical and thermal load.

### 1.1. The Present Contribution

Carrera [65] reported in his research that for the accurate structural analysis of composite laminates, higher-order theories must be expanded up to minimum fifth-order polynomial. It is also recommended by Carrera that it is important to consider the effect of transverse normal deformation for the analysis of composite laminates. However, limited literature is available on refined theories considering the effects of transverse normal strain. Also, refined theories representing higher order (minimum fifth-order) expansion of displacement field are limited. Based on these observations Sayyad and his coauthors Sayyad and Naik, [66], Naik and Sayyad, [67-69], Ghumare and Sayyad [70-72], Shinde and Sayyad [73] have given due consideration to these recommendations of Carrera and developed a fifth-order shear and normal deformation theory for the analysis of laminated composite, sandwich, and functionally graded beams, plates, and shells. In the present work, this theory is extended for the free vibration analysis of laminated shells. The theory considers the effects of transverse shear and normal deformations which is neglected by classical theories as well as many other higher-order shell theories including Reddy's theory. A fifth-order polynomial type shear strain function is used in the in-plane displacements to consider the effect of transverse shear deformation, whereas a fourth-order function is considered in the transverse displacement to account for the effect of normal deformation i.e. transverse normal stress. The equations of motion are derived using Hamilton's principle. Navier's solution technique is employed to obtain the non-dimensional fundamental frequencies. The numerical results obtained for the natural frequencies using the present theory are compared with other higher-order theories presented by Asadi et al. [31], Bhimaraddi [13], Sayyad, and Ghugal [45], etc. In overall, the numerical results predicted by the present theory are in excellent agreement with the 3D elasticity solution.

## 2. Mathematical Formulation of the Present Theory

A simply supported laminated composite shell on rectangular planform with a width $a$ in the $x$-direction, breadth $b$ in the $y$-direction, and thickness $h$ in the $z$-direction as shown in Fig. 1 is considered. The shell has $N$ number of orthotropic layers made up of fibrous composite materials. For spherical shell $R_1=R_2=R$ and for cylindrical shell $R_1=R$, $R_2=\infty$.

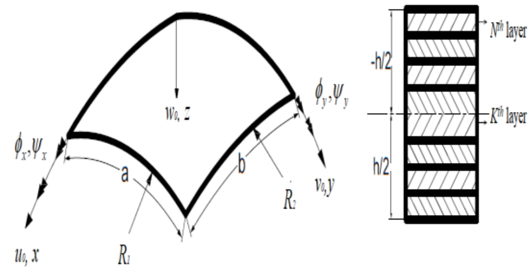


**Fig. 1.** Geometry and coordinate system of a spherical shell.

The present theory is built upon classical shell theory. Extension and bending components associated with the present theory are analogous to the classical shell theory. Fifth-order polynomial type shape functions are used in the in-plane displacements to account for the effect of transverse shear deformation. Whereas, fourth-order shape function in terms of thickness coordinates is used in the transverse displacement to account for the effect of transverse normal deformations. The theory is presented by Sayyad and Naik, [65], Naik and Sayyad, [67-69], Ghumare and Sayyad [70-72] and, Shinde and Sayyad [73]. The displacement field of the present theory is as follows.

$$
\begin{aligned}
u(x,y,z,t) &= \left(1+\frac{z}{R_1}\right)u_0 - z\frac{\partial w_0}{\partial x} + \varphi_1(z)\phi_x + \varphi_2(z)\psi_x \\
v(x,y,z,t) &= \left(1+\frac{z}{R_2}\right)v_0 - z\frac{\partial w_0}{\partial y} + \varphi_1(z)\phi_y + \varphi_2(z)\psi_y \qquad (1) \\
w(x,y,z,t) &= w_0 + \varphi_1'(z)\phi_z + \varphi_2'(z)\psi_z
\end{aligned}
$$

where

$$
\begin{aligned}
\varphi_1(z) &= \left(z-\frac{4z^3}{3h^2}\right),\ \varphi_2(z) = \left(z-\frac{16z^5}{5h^4}\right), \\
\varphi_1'(z) &= \left(1-\frac{4z^2}{h^2}\right),\ \varphi_2'(z) = \left(1-\frac{16z^4}{h^4}\right) \qquad (2)
\end{aligned}
$$

where $u$, $v$, and $w$ are the displacements in the $x$-, $y$- and $z$- directions, respectively; $u_0$, $v_0$, and $w_0$ are the unknown displacements of the mid-plane of the shell in $x$-, $y$- and $z$- directions respectively; $(\phi_x,\psi_x)$ are the shear slopes about the y-axis, $(\phi_y,\psi_y)$ are the shear slopes about the x-axis, $(\phi_z,\psi_z)$ are the shear slopes about the z-axis. All these shear slopes are unknowns that need to be determined. (ʹ) represents the derivative of the function with respect to the $z$ coordinate. The present theory has nine unknowns. The following general strain-displacement relationships are used to determine nonzero strain components, Bhimaraddi [13].

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} + \frac{w}{R_1} \\ \frac{\partial v}{\partial y} + \frac{w}{R_2} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} - \frac{u_0}{R_1} \\ \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} - \frac{v_0}{R_2} \end{Bmatrix} \tag{3}$$

Substituting displacement expressions from the displacement field, stated in Eq. (1) one can obtain the following expressions for nonzero strains.

$$\begin{aligned}
\varepsilon_x &= \left(\frac{\partial u_0}{\partial x} + \frac{w_0}{R_1}\right) - z\,\frac{\partial^2 w_0}{\partial x^2} + \varphi_1(z)\frac{\partial \phi_x}{\partial x} \\
&\quad + \varphi_2(z)\frac{\partial \psi_x}{\partial x} + \frac{\varphi_1'(z)}{R_1}\phi_z + \frac{\varphi_2'(z)}{R_1}\psi_z \\
\varepsilon_y &= \left(\frac{\partial v_0}{\partial y} + \frac{w_0}{R_2}\right) - z\,\frac{\partial^2 w_0}{\partial y^2} + \varphi_1(z)\frac{\partial \phi_y}{\partial y} \\
&\quad + \varphi_2(z)\frac{\partial \psi_y}{\partial y} + \frac{\varphi_1'(z)}{R_2}\phi_z + \frac{\varphi_2'(z)}{R_2}\psi_z \\
\varepsilon_z &= \varphi_1''(z)\phi_z + \varphi_2''(z)\psi_z \\
\gamma_{xy} &= \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z\,\frac{\partial^2 w_0}{\partial x\,\partial y} + \varphi_1(z)\frac{\partial \phi_x}{\partial y} \\
&\quad + \varphi_1(z)\frac{\partial \phi_y}{\partial x} + \varphi_2(z)\frac{\partial \psi_x}{\partial y} + \varphi_2(z)\frac{\partial \psi_y}{\partial x} \\
\gamma_{xz} &= \varphi_1'(z)\phi_x + \varphi_2'(z)\psi_x + \varphi_1'(z)\frac{\partial \phi_z}{\partial x} \\
&\quad + \varphi_2'(z)\frac{\partial \psi_z}{\partial x} \\
\gamma_{yz} &= \varphi_1'(z)\phi_y + \varphi_2'(z)\psi_y + \varphi_1'(z)\frac{\partial \phi_z}{\partial y} \\
&\quad + \varphi_2'(z)\frac{\partial \psi_z}{\partial y}
\end{aligned} \tag{4}$$

The three-dimensional Hooke's law is used to derive expressions for stresses in the shell domain.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix}^k = \left[Q_{ij}\right]^k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}^k \tag{5}$$

where,

$$\left[Q_{ij}\right]^k = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_{44} \end{bmatrix} \tag{6}$$

$$\begin{aligned}
&Q_{11} = \frac{E_1(1-\mu_{23}\mu_{32})}{\Delta}, Q_{12} = \frac{E_1(\mu_{21}+\mu_{23}\mu_{31})}{\Delta}, \\
&Q_{13} = \frac{E_1(\mu_{31}+\mu_{21}\mu_{32})}{\Delta}, Q_{22} = \frac{E_2(1-\mu_{13}\mu_{31})}{\Delta}, \\
&Q_{23} = \frac{E_2(\mu_{32}+\mu_{21}\mu_{31})}{\Delta}, Q_{33} = \frac{E_3(1-\mu_{12}\mu_{21})}{\Delta} \\
&Q_{66} = G_{12},\quad Q_{55} = G_{13},\quad Q_{44} = G_{23},
\end{aligned} \tag{7}$$

where

$$\Delta = 1 - \mu_{21}\mu_{12} - \mu_{23}\mu_{32} - \mu_{31}\mu_{13} - 2\mu_{12}\mu_{23}\mu_{31}$$

$E_i$'s ($i$=1, 2, 3) are the modulus of elasticity; $G_{ij}$'s are the modulus of rigidity, and $\mu_{ij}$'s are the Poisson's ratio. The equations of motion for the free vibration analysis of the shell are obtained using Hamilton's principle.

$$\int_{t_1}^{t_2} \left[\delta U - \delta V + \delta K\right] dt = 0 \tag{8}$$

where, $\delta K$ represents the kinetic energy due to inertia forces, $\delta U$ represents the strain energy due to stresses, and $\delta V$ represents the potential energy due to external load. $\delta$ is the variational operator. $t_1$ and $t_2$ are the initial and final times respectively. Substituting values of these energies in Eq. (8), one can rewrite the Eq. (8) as

$$\begin{aligned}
&\int_{dv} \sigma_{ij}\,\delta\varepsilon_{ij}\,dv - \int_{dA} q(x,y)\,\delta w\,dA + \\
&\rho\int_{dv}\left(\frac{\partial^2 u}{\partial t^2}\delta u + \frac{\partial^2 v}{\partial t^2}\delta v + \frac{\partial^2 w}{\partial t^2}\delta w\right) dv = 0
\end{aligned} \tag{9}$$

where $\rho$ represents the mass density. Substituting the expressions of stresses and strains from Eqs. (3)-(7) into the Eq. (9), integrating by parts, collecting the coefficients of $\left(\delta u_0, \delta v_0, \delta w_0, \delta\phi_x, \delta\phi_y, \delta\phi_z, \delta\psi_x, \delta\psi_y, \delta\psi_z\right)$ and setting them equal to zero, one can derive the following equations of motion.

$$\delta u_0 : \frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}$$
$$=\left(I_1+2\frac{I_2}{R_1}+\frac{I_3}{R_1^2}\right)\frac{\partial^2 u}{\partial t^2}-\left(I_2+\frac{I_3}{R_1}\right)\frac{\partial^3 w}{\partial x\,\partial t^2}$$
$$+\left(I_4+\frac{I_8}{R_1}\right)\frac{\partial^2 \phi_x}{\partial t^2}+\left(I_5+\frac{I_9}{R_1}\right)\frac{\partial^2 \psi_x}{\partial t^2}$$
$$\delta v_0 : \frac{\partial N_y}{\partial y}+\frac{\partial N_{xy}}{\partial x}$$
$$=\left(I_1+2\frac{I_2}{R_2}+\frac{I_3}{R_2^2}\right)\frac{\partial^2 v}{\partial t^2}-\left(I_2+\frac{I_3}{R_2}\right)\frac{\partial^3 w}{\partial y\,\partial t^2}$$
$$+\left(I_4+\frac{I_8}{R_2}\right)\frac{\partial^2 \phi_y}{\partial t^2}+\left(I_5+\frac{I_9}{R_2}\right)\frac{\partial^2 \psi_y}{\partial t^2}$$
$$\delta w_0 : \frac{\partial^2 M_x^b}{\partial x^2}+\frac{\partial^2 M_y^b}{\partial y^2}+2\frac{\partial^2 M_{xy}^b}{\partial x\,\partial y}-\frac{N_x}{R_1}-\frac{N_y}{R_2}+q$$
$$=\left(I_2+\frac{I_3}{R_1}\right)\frac{\partial^3 u}{\partial x\,\partial t^2}-I_3\frac{\partial^4 w}{\partial x^2\partial t^2}+I_8\frac{\partial^3 \phi_x}{\partial x\,\partial t^2}$$
$$+I_9\frac{\partial^3 \psi_x}{\partial x\,\partial t^2}+\left(I_2+\frac{I_3}{R_2}\right)\frac{\partial^3 v}{\partial y\,\partial t^2}-I_3\frac{\partial^4 w}{\partial y^2\partial t^2}+$$
$$I_8\frac{\partial^3 \phi_y}{\partial y\,\partial t^2}+I_9\frac{\partial^3 \psi_y}{\partial y\,\partial t^2}+I_1\frac{\partial^2 w}{\partial t^2}+I_{10}\frac{\partial^2 \phi_z}{\partial t^2}+I_{11}\frac{\partial^2 \psi_z}{\partial t^2}$$
$$\delta\phi_x : \frac{\partial M_x^{S1}}{\partial x}+\frac{\partial M_{xy}^{S1}}{\partial y}-Q_x^{S1}= \qquad (10)$$
$$\left(I_4+\frac{I_8}{R_1}\right)\frac{\partial^2 u}{\partial t^2}-I_8\frac{\partial^3 w}{\partial x\,\partial t^2}+I_6\frac{\partial^2 \phi_x}{\partial t^2}+I_{15}\frac{\partial^2 \psi_x}{\partial t^2}$$
$$\delta\psi_x : \frac{\partial M_x^{S2}}{\partial x}+\frac{\partial M_{xy}^{S2}}{\partial y}-Q_x^{S2}=$$
$$\left(I_5+\frac{I_9}{R_1}\right)\frac{\partial^2 u}{\partial t^2}-I_9\frac{\partial^3 w}{\partial x\,\partial t^2}+I_{15}\frac{\partial^2 \phi_x}{\partial t^2}+I_7\frac{\partial^2 \psi_x}{\partial t^2}$$
$$\delta\phi_y : \frac{\partial M_y^{S1}}{\partial y}+\frac{\partial M_{xy}^{S1}}{\partial x}-Q_y^{S1}=$$
$$\left(I_4+\frac{I_8}{R_2}\right)\frac{\partial^2 v}{\partial t^2}-I_8\frac{\partial^3 w}{\partial y\,\partial t^2}+I_6\frac{\partial^2 \phi_y}{\partial t^2}+I_{15}\frac{\partial^2 \psi_y}{\partial t^2}$$
$$\delta\psi_y : \frac{\partial M_y^{S2}}{\partial y}+\frac{\partial M_{xy}^{S2}}{\partial x}-Q_y^{S2}=$$
$$\left(I_5+\frac{I_9}{R_2}\right)\frac{\partial^2 v}{\partial t^2}-I_9\frac{\partial^3 w}{\partial y\,\partial t^2}+I_{15}\frac{\partial^2 \phi_y}{\partial t^2}+I_7\frac{\partial^2 \psi_y}{\partial t^2}$$
$$\delta\phi_z : \frac{\partial Q_x^{S1}}{\partial x}+\frac{\partial Q_y^{S1}}{\partial y}-\frac{S_{xs1}}{R_1}-\frac{S_{ys1}}{R_2}-S^{S1}=$$
$$I_{10}\frac{\partial^2 w}{\partial t^2}+I_{13}\frac{\partial^2 \phi_z}{\partial t^2}+I_{12}\frac{\partial^2 \psi_z}{\partial t^2}$$
$$\delta\psi_z : \frac{\partial Q_x^{S2}}{\partial x}+\frac{\partial Q_y^{S2}}{\partial y}-\frac{S_{xs2}}{R_1}-\frac{S_{ys2}}{R_2}-S^{S2}=$$
$$I_{11}\frac{\partial^2 w}{\partial t^2}+I_{12}\frac{\partial^2 \phi_z}{\partial t^2}+I_{14}\frac{\partial^2 \psi_z}{\partial t^2}$$

Boundary conditions of the present theory are expressed in the following form

Along the edges $x$ =0 and $x$ =$a$

Either $u_0=0$ or $N_x$ is prescribed

Either $v_0=0$ or $N_{xy}$ is prescribed

Either $w_0=0$ or $\frac{dM_x^b}{dx}+2\frac{dM_{xy}^b}{dy}$ is prescribed

Either $\frac{\partial w_0}{\partial x}=0$ or $M_{xy}^b$ is prescribed

Either $\phi_x=0$ or $M_x^{S1}$ is prescribed

Either $\psi_x=0$ or $M_x^{S2}$ is prescribed

Either $\phi_y=0$ or $M_{xy}^{S1}$ is prescribed

Either $\psi_y=0$ or $M_{xy}^{S2}$ is prescribed

Either $\phi_z=0$ or $Q_x^{S1}$ is prescribed

Either $\psi_z=0$ or $Q_{xz}^{S2}$ is prescribed

Along with the edges $y$ =0 and $y$ =$b$,

Either $u_0=0$ or $N_{xy}$ is prescribed

Either $v_0=0$ or $N_y$ is prescribed

Either $w_0=0$ or $\frac{dM_y^b}{dy}+2\frac{dM_{xy}^b}{dx}$ is prescribed

Either $\frac{\partial w_0}{\partial y}=0$ or $M_{xy}^b$ is prescribed

Either $\phi_x=0$ or $M_{xy}^{S1}$ is prescribed

Either $\psi_x=0$ or $M_{xy}^{S2}$ is prescribed

Either $\phi_y=0$ or $M_y^{S1}$ is prescribed

Either $\psi_y=0$ or $M_y^{S2}$ is prescribed

Either $\phi_z=0$ or $Q_y^{S1}$ is prescribed

Either $\psi_z=0$ or $Q_{yz}^{S2}$ is prescribed

where force and moment resultants can be derived from the following relations

$$\left(N_x,N_y,N_{xy}\right)=\int_{-h/2}^{h/2}\left[\sigma_x,\sigma_y,\tau_{xy}\right]dz$$
$$\left(M_x^b,\,M_y^b,M_{xy}^b\right)=\int_{-h/2}^{h/2}\left[z\sigma_x,\,z\sigma_y,z\tau_{xy}\right]dz$$
$$\left(M_x^{S_1},M_y^{S_1},M_{xy}^{S_1}\right)=\int_{-h/2}^{h/2}\left\{\left[\varphi_1(z)\left(\sigma_x,\sigma_y,\tau_{xy}\right)\right]\right\}dz$$
$$\left(M_x^{S_2},M_y^{S_2}M_{xy}^{S_2}\right)=\int_{-h/2}^{h/2}\left\{\left[\varphi_2(z)\left(\sigma_x,\sigma_y,\tau_{xy}\right)\right]\right\}dz$$
$$\left(Q_x^{S_1},Q_y^{S_1}\right)=\int_{-h/2}^{h/2}\left\{\left[\varphi_1'(z)\left(\tau_{xz},\tau_{yz}\right)\right]\right\}dz \qquad (11)$$
$$\left(Q_x^{S_2},Q_y^{S_2}\right)=\int_{-h/2}^{h/2}\left\{\left[\varphi_2'(z)\left(\tau_{xz},\tau_{yz}\right)\right]\right\}dz$$
$$\left(S^{S_1},S^{S_2}\right)=\int_{-h/2}^{h/2}\left\{\sigma_z\left[\varphi_1''(z),\varphi_2''(z)\right]\right\}dz$$
$$\left(S_{xS1},S_{xS2}\right)=\int_{-h/2}^{h/2}\left\{\sigma_x\left[\varphi_1'(z),\varphi_2'(z)\right]\right\}dz$$
$$\left(S_{yS1},S_{yS2}\right)=\int_{-h/2}^{h/2}\left\{\sigma_y\left[\varphi_1'(z),\varphi_2'(z)\right]\right\}dz$$

The inertia constants are derived as,

$$\left(I_1,I_2,I_3,I_4,I_5\right)=\rho\int_{-h/2}^{h/2}\left[1,z,z^2,\varphi_1(z),\varphi_2(z)\right]dz$$
$$\left(I_6,I_8,I_{15}\right)=\rho\int_{-h/2}^{h/2}\varphi_1(z)\left[\varphi_1(z),z,\varphi_2(z)\right]dz$$
$$\left(I_7,I_9\right)=\rho\int_{-h/2}^{h/2}\varphi_2(z)\left[\varphi_2(z),z\right]dz \qquad (12)$$
$$\left(I_{10},I_{12},I_{13}\right)=\rho\int_{-h/2}^{h/2}\varphi_1^{'}(z)\left[1,\varphi_2^{'}(z),\varphi_1^{'}(z)\right]dz$$
$$\left(I_{11},I_{14}\right)=\rho\int_{-h/2}^{h/2}\varphi_2^{'}(z)\left[1,\varphi_2^{'}(z)\right]dz$$

## 3. Analytical Solutions

The Navier's solution technique is used to obtain the analytical solutions for the free vibration analysis of simply supported cross-ply laminated composite shells. The boundary conditions at the simply supported edges of the shell are as follows.

Along the edges $x$ =0 and $x$ =$a$

$$v_0=w_0=\phi_y=\psi_y=\phi_z=\psi_z=0$$
$$M_x^b=M_x^{S1}=M_x^{S2}=N_x=0 \qquad (13)$$

Along the edges $y$ =0 and $y$ =$b$

$$u_0=w_0=\phi_x=\psi_x=\phi_z=\psi_z=0$$
$$M_y^b=M_y^{S1}=M_y^{S2}=N_y=0 \qquad (14)$$

The following trigonometric form of unknown variables is assumed which satisfies the simply supported boundary conditions exactly.

$$\begin{bmatrix}u_0\\\phi_x\\\psi_x\\v_0\\\phi_y\\\psi_y\\w_0\\\phi_z\\\psi_z\end{bmatrix}=\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\begin{bmatrix}u_1\cos\alpha x\sin\beta y e^{i\omega t}\\\phi_{x1}\cos\alpha x\sin\beta y e^{i\omega t}\\\psi_{x1}\cos\alpha x\sin\beta y e^{i\omega t}\\v_1\sin\alpha x\cos\beta y e^{i\omega t}\\\phi_{y1}\sin\alpha x\cos\beta y e^{i\omega t}\\\psi_{y1}\sin\alpha x\cos\beta y e^{i\omega t}\\w_1\sin\alpha x\sin\beta y e^{i\omega t}\\\phi_{z1}\sin\alpha x\sin\beta y e^{i\omega t}\\\psi_{z1}\sin\alpha x\sin\beta y e^{i\omega t}\end{bmatrix} \qquad (15)$$

where $\alpha=m\pi/a,\ \beta=n\pi/b$ ; $i=\sqrt{-1}$ ; $\omega$ is the fundamental frequency and $u_1,\phi_{x1},\psi_{x1},v_1,\phi_{y1},\psi_{y2},w_1,\phi_{z1},\psi_{z1}$ are the unknown parameters to be determined. Substitution of Eq. (15) into Eq. (10) by setting $q$ = 0 leads to the following eigenvalue problem.

$$\left\{[K]-\omega^2[M]\right\}\{\Delta\}=\{0\} \qquad (16)$$

where [K] is stiffness matrix, [$M$] is the mass matrix, and $\{\Delta\}$ is the vector of amplitudes. Elements of these matrices are mentioned in Appendix.

## 4. Numerical Results and Discussion

In the present study, natural frequencies for a homogeneous two-layer ($0^0/90^0$) and three-layer ($0^0/90^0/0^0$) laminated composite cylindrical and spherical shells are obtained using MATLAB 2015a. In laminated shells, all layers are considered of equal thickness. The numerical results are obtained for different values of the $R/a$ ratio and aspect ($a/h$) ratio. The following material properties for the laminated composite shells are used, Bhimaraddi [13]

$$\frac{E_1}{E_2}=25,\ \frac{E_3}{E_2}=1,\ \frac{G_{12}}{E_2}=\frac{G_{13}}{E_2}=0.5,\ \frac{G_{23}}{E_2}=0.2, \qquad (17)$$
$$\mu_{12}=\mu_{13}=\mu_{23}=0.25,\ \rho=\text{constant}$$

The natural frequency is presented in the following non-dimensional form unless and until specified,

$$\bar{\omega}=\omega\left(a^2/h\right)\sqrt{\rho/E_2} \qquad (18)$$

The comparison of the first five natural frequencies of two-layer ($0^0/90^0$) and three-layer ($0^0/90^0/0^0$) laminated composite cylindrical shells is shown in Tables 1-2 respectively. Natural frequencies are presented for $a/h$=10 and 20 at $R/a$=2, 1, 0.5. Material properties of laminated shells are mentioned in Eq. (17). The comparison shows that the present results are in good agreement with those results presented by Asadi et al. [31] by different models. It is observed that as the a/h ratio and R/a ratio increases the natural frequency increases. It is found that the natural frequencies are decreasing with respect to the increase in radius of curvature (R/a).

Also, from Tables 1 and 2 it is clearly observed that the natural frequency is less in deep shells and more in shallow shells. The variation of fundamental frequency with respect to the aspect ratio (a/h) is also shown in Tables 1 and 2, which predicts that the frequency is found more in thick shells.

Table 3 shows a comparison of natural frequencies for varying modes of vibration of a two-layer $0^0/90^0$ laminated cylindrical shells at $a/h$=10 and $R/a$=1. The numerical results are obtained for $m$=$n$=1, 2, 3, 4, 5, 6 to change modes of vibration. The comparison shows that the present theory predicts natural frequencies in good agreement with those presented by Bhimaraddi [13] using various theories.

Table 4 shows the effect of *R/a* and *h/a* ratios on the natural frequencies of orthotropic and laminated composite cylindrical shells. Variations of natural frequencies in laminated composite cylindrical shells are presented with the help of Fig. 2.

Table 5 through 7 and Figures.3 through 6 deal with the natural frequencies of orthotropic and laminated composite spherical shells using the present theory. The present results are obtained for different modes of vibrations, *a/h* ratios, and *R/a* ratios. The comparison of all tables shows that the present results are compared with those presented by Bhimaraddi [13] and, Sayyad and Ghugal [45] using various theories. The present results are found in good agreement with previously published results for parameters.

From Tables 5 through 7 it is seen that the fundamental frequency of laminated shells decreases with an increase in the radii of curvature (R/a ratio). These results are also plotted in Figures 3 through 6.

Figures 5 and 6 show the variation of fundamental frequency with respect to modes of vibration (m, n) in a single layer ($0^0$) and a double-layer ($0^0/90^0$) laminated shell.

From Figures 5 and 6 it is observed that as the mode (m, n) increases the value of frequency also increases.

Table 8 shows the fundamental frequencies of three layered ($0^0$/core/$0^0$) sandwich spherical shells. The thickness of each face sheet is 0.1*h* and the middle core is 0.8*h*, where *h* is the total thickness of the shell. Following are the material properties used for the middle core of sandwich shell, Sayyad and Ghugal [46]:

$$\frac{Q_{12}}{Q_{11}} = 0.23319, \quad \frac{Q_{13}}{Q_{11}} = 0.010776,$$

$$\frac{Q_{22}}{Q_{11}} = 0.543104, \quad \frac{Q_{23}}{Q_{11}} = 0.098276,$$

$$\frac{Q_{33}}{Q_{11}} = 0.530172, \quad \frac{Q_{44}}{Q_{11}} = 0.266810,$$

$$\frac{Q_{55}}{Q_{11}} = 0.159914, \quad \frac{Q_{22}}{Q_{11}} = 0.262931$$

The elastic properties of the face sheets are assumed as '*C*' times the elastic properties of the core and the value of *C* are taken as 1, 2, 5, 10, and 15. From Table 8 it is observed that the result obtained by using the present theory are in good agreement with other theories available in the literature. The fundamental frequency of sandwich shells decreases with an increase in the radii of curvature (R/a).

Also, the results quoted in Table 8 show that the values of fundamental frequency increase with an increase in softness of the core.

**Table 1.** First five non-dimensional natural frequencies in two-layer ($0^0/90^0$) laminated composite cylindrical shell for varying *a/R* and a/h ratio ($R_1$=R and $R_2$=∞).

| *a/h* | *R/a* | Theory | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ | $\omega_5$ |
|---|---|---|---|---|---|---|---|
| | | Present | 11.579 | 25.514 | 28.015 | 36.485 | 50.747 |
| | 2.0 | FSDTQ [31] | 11.530 | 25.357 | 27.913 | 36.324 | 50.210 |
| | | 3D-FEM [31] | 11.537 | 25.378 | 27.951 | 36.434 | 50.253 |
| | | Present | 15.967 | 26.053 | 34.840 | 38.115 | 50.925 |
| 20 | 1.0 | FSDTQ [31] | 15.859 | 25.648 | 34.867 | 37.831 | 50.263 |
| | | 3D-FEM [31] | 15.861 | 25.658 | 34.890 | 37.942 | 50.297 |
| | | Present | 24.855 | 27.686 | 43.444 | 50.689 | 51.091 |
| | 0.5 | FSDTQ [31] | 24.809 | 26.193 | 42.664 | 49.382 | 51.170 |
| | | 3D-FEM [31] | 24.805 | 26.162 | 42.743 | 49.359 | 51.167 |
| | | Present | 9.5271 | 21.995 | 22.464 | 30.415 | 39.651 |
| | 2.0 | FSDTQ [31] | 9.4577 | 21.676 | 22.150 | 29.959 | 38.608 |
| | | 3D-FEM [31] | 9.4855 | 21.743 | 22.246 | 30.193 | 38.745 |
| | | Present | 10.859 | 22.190 | 24.319 | 30.899 | 39.820 |
| 10 | 1.0 | FSDTQ [31] | 10.666 | 21.705 | 24.090 | 30.368 | 38.722 |
| | | 3D-FEM [31] | 10.686 | 21.767 | 24.191 | 30.614 | 38.896 |
| | | Present | 14.099 | 22.496 | 29.446 | 32.266 | 39.788 |
| | 0.5 | FSDTQ [31] | 13.771 | 21.037 | 29.574 | 31.200 | 38.073 |
| | | 3D-FEM [31] | 13.772 | 21.040 | 29.639 | 31.411 | 38.266 |

**Table 2.** First five non-dimensional natural frequencies in three-layer ($0^0/90^0/0^0$) laminated composite cylindrical shell for varying *a/R* and *a/h* ratio ($R_1$=*R* and $R_2$=∞).

| *a/h* | *R/a* | Theory | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ | $\omega_5$ |
|---|---|---|---|---|---|---|---|
| 20 | 2.0 | Present | 15.353 | 24.850 | 41.093 | 42.808 | 46.932 |
| | | FSDTQ [31] | 15.551 | 21.646 | 37.022 | 46.309 | 48.938 |
| | | 3D-FEM [31] | 15.245 | 21.370 | 36.803 | 43.529 | 46.148 |
| | 1.0 | Present | 18.596 | 32.641 | 42.761 | 47.928 | 50.801 |
| | | FSDTQ [31] | 18.710 | 21.974 | 36.794 | 49.770 | 49.852 |
| | | 3D-FEM [31] | 18.471 | 21.703 | 36.567 | 47.074 | 47.416 |
| | 0.5 | Present | 25.896 | 42.591 | 49.308 | 51.416 | 66.365 |
| | | FSDTQ [31] | 23.178 | 25.978 | 35.923 | 52.746 | 57.077 |
| | | 3D-FEM [31] | 22.924 | 25.840 | 35.668 | 50.360 | 56.448 |
| 10 | 2.0 | Present | 11.982 | 19.324 | 28.921 | 31.916 | 32.717 |
| | | FSDTQ [31] | 12.443 | 18.677 | 30.839 | 31.323 | 34.456 |
| | | 3D-FEM [31] | 11.769 | 18.159 | 28.600 | 30.471 | 31.928 |
| | 1.0 | Present | 12.854 | 21.665 | 28.786 | 32.931 | 34.688 |
| | | FSDTQ [31] | 13.187 | 18.524 | 30.564 | 32.232 | 34.523 |
| | | 3D-FEM [31] | 12.590 | 18.005 | 29.732 | 30.189 | 32.037 |
| | 0.5 | Present | 15.134 | 27.579 | 28.281 | 33.701 | 42.195 |
| | | FSDTQ [31] | 15.250 | 17.989 | 29.491 | 34.795 | 34.913 |
| | | 3D-FEM [31] | 14.840 | 17.468 | 29.094 | 32.464 | 33.046 |

**Table 3.** Non-dimensional natural frequencies in two-layer ($0^0/90^0$) laminated composite cylindrical shell for varying modes of vibration ($R_1$=*R*, $R_2$=∞, *R/a*=1 and *a/h*=10). $\left(\bar{\omega}=\omega\sqrt{\rho/E_2}\right)$

| *n* | Source | *m* =1 | *m* =2 | *m* =3 | *m* =4 | *m* =5 | *m* =6 |
|---|---|---|---|---|---|---|---|
| 1 | Present | 1.0859 | 2.2190 | 3.9820 | 5.8774 | 7.7433 | 9.4882 |
| | 3D-Elasticity [13] | 1.0408 | 2.4127 | 4.1157 | 5.9337 | 7.7818 | 9.6281 |
| 2 | Present | 2.4319 | 3.0899 | 4.554 | 6.3011 | 8.0866 | 9.7855 |
| | 3D-Elasticity [13] | 2.0956 | 3.0069 | 4.4760 | 6.1778 | 7.9611 | 9.7672 |
| 3 | Present | 4.1705 | 4.5649 | 5.6724 | 7.1589 | 8.7769 | 10.367 |
| | 3D-Elasticity [13] | 3.7949 | 4.4010 | 5.5338 | 6.9958 | 8.6193 | 10.316 |
| 4 | Present | 6.0501 | 6.3079 | 7.1502 | 8.3748 | 9.7868 | 11.225 |
| | 3D-Elasticity [13] | 5.6331 | 6.0816 | 6.9643 | 8.1881 | 9.6232 | 11.177 |
| 5 | Present | 7.9595 | 8.1370 | 8.8001 | 9.8109 | 11.025 | 12.300 |
| | 3D-Elasticity [13] | 7.4876 | 7.8550 | 8.5704 | 9.6035 | 10.864 | 12.274 |
| 6 | Present | 9.8524 | 9.9762 | 10.514 | 11.360 | 12.402 | 13.520 |
| | 3D-Elasticity [13] | 8.6842 | 9.4979 | 10.254 | 11.143 | 12.258 | 13.536 |

**Table 4.** Non-dimensional natural frequencies in laminated composite cylindrical shell ($R_1$=*R*, $R_2$=∞) for different *R/a* and *a/h* ratios. $\left(\bar{\omega}=\omega\sqrt{\rho/E_2}\right)$

| *R/a* | Source | Orthotropic | | | $0^0/90^0$ | | |
|---|---|---|---|---|---|---|---|
| | | *h/a*=0.05 | *h/a* =0.1 | *h/a*=0.15 | *h/a* =0.05 | *h/a* =0.1 | *h/a* =0.15 |
| 1 | Present | 0.8727 | 1.2919 | 1.5740 | 0.7983 | 1.0859 | 1.3714 |
| | 3D-Elasticity [13] | 0.8917 | 1.3241 | 1.6169 | 0.7868 | 1.0408 | 1.2909 |
| 2 | Present | 0.7602 | 1.2495 | 1.5609 | 0.5789 | 0.9527 | 1.2815 |
| | 3D-Elasticity [13] | 0.7663 | 1.2674 | 1.5924 | 0.5725 | 0.9362 | 1.2537 |
| 3 | Present | 0.7354 | 1.2407 | 1.5582 | 0.5243 | 0.9231 | 1.2609 |
| | 3D-Elasticity [13] | 0.7396 | 1.2562 | 1.5878 | 0.5207 | 0.9144 | 1.2450 |
| 4 | Present | 0.7263 | 1.2376 | 1.5573 | 0.5034 | 0.9120 | 1.2529 |
| | 3D-Elasticity [13] | 0.7304 | 1.2522 | 1.5452 | 0.5011 | 0.9061 | 1.2409 |
| 5 | Present | 0.7220 | 1.2361 | 1.5568 | 0.4933 | 0.9067 | 1.2487 |
| | 3D-Elasticity [13] | 0.7255 | 1.2503 | 1.5842 | 0.4916 | 0.9020 | 1.2384 |
| 10 | Present | 0.7163 | 1.2342 | 1.5563 | 0.4793 | 0.8989 | 1.2423 |
| | 3D-Elasticity [13] | 0.7194 | 1.2473 | 1.5825 | 0.4785 | 0.8956 | 1.2337 |
| 20 | Present | 0.7149 | 1.2337 | 1.5562 | 0.4757 | 0.8965 | 1.2399 |
| | 3D-Elasticity [13] | 0.7179 | 1.2463 | 1.5821 | 0.4750 | 0.8934 | 1.2314 |
| ∞ | Present | 0.7144 | 1.2336 | 1.5562 | 0.4743 | 0.8952 | 1.2381 |
| | 3D-Elasticity [13] | 0.7173 | 1.2461 | 1.5812 | 0.4736 | 0.8917 | 1.2290 |

**Table 5.** Non-dimensional natural frequencies in two-layer ($0^0/90^0$) laminated composite spherical shell ($R_1= R_2=R$) for varying modes of vibration (*R/a*=1 and *a/h*=10). $\left(\bar{\omega}=\omega\sqrt{\rho/E_2}\right)$

| *n* | Source | *m*=1 | *m* =2 | *m* =3 | *m* =4 | *m* =5 | *m* =6 |
|---|---|---|---|---|---|---|---|
| 1 | Present | 1.4818 | 2.6042 | 4.3235 | 6.2130 | 8.0999 | 9.8893 |
| | 3D-Elasticity [13] | 1.3997 | 2.4387 | 4.0531 | 5.8455 | 7.6895 | 8.7973 |
| 2 | Present | 2.5333 | 3.2067 | 4.6702 | 6.4332 | 8.2533 | 10.006 |
| | 3D-Elasticity [13] | 2.4420 | 3.0452 | 4.4168 | 5.7938 | 8.0320 | 9.6841 |
| 3 | Present | 4.1621 | 4.5650 | 5.6853 | 7.2038 | 8.8718 | 10.534 |
| | 3D-Elasticity [13] | 4.0841 | 4.4327 | 5.4741 | 6.9051 | 8.5234 | 10.226 |
| 4 | Present | 5.9659 | 6.2301 | 7.0901 | 8.3601 | 9.8434 | 11.383 |
| | 3D-Elasticity [13] | 5.7102 | 6.1128 | 6.9091 | 8.0983 | 9.5244 | 11.083 |
| 5 | Present | 7.7957 | 7.9834 | 8.6702 | 9.7402 | 11.051 | 12.465 |
| | 3D-Elasticity [13] | 7.4002 | 7.8904 | 8.5227 | 9.5188 | 10.766 | 12.177 |
| 6 | Present | 9.5939 | 9.7358 | 10.306 | 11.226 | 12.393 | 13.692 |
| | 3D-Elasticity [13] | 9.0324 | 9.6948 | 10.214 | 11.065 | 12.164 | 13.440 |

**Table 6.** Non-dimensional natural frequencies in two-layer ($0^0/90^0$) laminated composite spherical shell ($R_1= R_2=R$) for different *R/a* and *a/h* ratios.

| *R/a* | Theory | *a/h* | | | | |
|---|---|---|---|---|---|---|
| | | 5 | 10 | 20 | 50 | 100 |
| 5 | Present | 7.6370 | 9.3431 | 10.932 | 16.7371 | 29.0279 |
| | PSDT [45] | 7.6781 | 9.3424 | 10.923 | 16.7059 | 29.0271 |
| | ESDT [45] | 7.7826 | 9.3759 | 10.931 | 16.7068 | 29.0272 |
| 10 | Present | 7.5733 | 9.0750 | 9.8931 | 11.8618 | 16.8058 |
| | PSDT [45] | 7.6122 | 9.0738 | 9.8893 | 11.8560 | 16.8218 |
| | FSDT [45] | 7.6482 | 9.0991 | 9.8978 | 11.8575 | 16.8222 |
| 50 | Present | 7.5527 | 8.9870 | 9.5339 | 9.7865 | 10.0960 |
| | PSDT [45] | 7.5908 | 8.9856 | 9.5323 | 9.7943 | 10.1312 |
| | ESDT [45] | 7.6974 | 9.0208 | 9.5420 | 9.7959 | 10.1316 |
| 100 | Present | 7.5520 | 8.9842 | 9.5225 | 9.7144 | 9.8121 |
| | PSDT [45] | 7.5902 | 8.9828 | 9.5209 | 9.7227 | 9.8487 |
| | ESDT [45] | 7.6967 | 9.0180 | 9.5307 | 9.7243 | 9.8491 |
| Plate | Present | 7.5753 | 9.0123 | 9.5498 | 9.7220 | 9.7476 |
| | PSDT [45] | 7.5899 | 9.9819 | 9.5171 | 9.6988 | 9.7527 |
| | ESDT [45] | 7.6965 | 9.0171 | 9.5269 | 9.7004 | 9.7531 |

**Table 7.** Non-dimensional natural frequencies in three-layer ($0^0/90^0/0^0$) laminated composite spherical shell ($R_1= R_2=R$) for different *R/a* and *a/h* ratios.

| *R/a* | Theory | *a/h* | | | | |
|---|---|---|---|---|---|---|
| | | 5 | 10 | 20 | 50 | 100 |
| 5 | Present | 8.3515 | 12.0792 | 15.1567 | 20.4682 | 31.4974 |
| | PSDT [45] | 8.3200 | 12.0613 | 15.0499 | 20.2525 | 31.2192 |
| | ESDT [45] | 8.3425 | 12.0412 | 15.0365 | 20.2601 | 31.2189 |
| 10 | Present | 8.2908 | 11.8770 | 14.4306 | 16.6907 | 20.6521 |
| | PSDT [45] | 8.2593 | 11.8633 | 14.3366 | 16.5276 | 20.4844 |
| | ESDT [45] | 8.2820 | 11.8428 | 14.3225 | 16.5247 | 20.4837 |
| 50 | Present | 8.2711 | 11.8111 | 14.1887 | 15.2760 | 15.6296 |
| | PSDT [45] | 8.2396 | 11.7988 | 14.0991 | 15.1334 | 15.5166 |
| | ESDT [45] | 8.2625 | 11.7781 | 14.0847 | 15.1302 | 15.5158 |
| 100 | Present | 8.2705 | 11.8090 | 14.1811 | 15.2296 | 15.4460 |
| | PSDT [45] | 8.2390 | 11.7968 | 14.0916 | 15.0876 | 15.3352 |
| | ESDT [45] | 8.2619 | 11.7760 | 14.0772 | 15.0845 | 15.3343 |
| Plate | Present | 8.2878 | 11.8281 | 14.1997 | 15.2359 | 15.4063 |
| | PSDT [45] | 8.2388 | 11.7961 | 14.0891 | 15.0724 | 15.2742 |
| | ESDT [45] | 8.2617 | 11.7754 | 14.0747 | 15.0692 | 15.2734 |

**Table 8.** Non-dimensional natural frequencies in three-layer ($0^0$/core/$0^0$) sandwich laminated composite spherical shell ($R_1$= $R_2$=$R$) for different R/a ratios.

| *R/a* | Model | *C* | | | | |
|---|---|---|---|---|---|---|
| | | 1 | 2 | 5 | 10 | 15 |
| 5 | Present | 5.0011 | 5.9644 | 8.0349 | 10.2930 | 11.868 |
| | PSDT [45] | 5.0209 | 5.9690 | 8.0090 | 10.023 | 10.249 |
| | ESDT [45] | 5.0205 | 5.9683 | 8.0076 | 10.0216 | 10.246 |
| 10 | Present | 5.0480 | 5.9841 | 8.0227 | 10.041 | 10.269 |
| | PSDT [45] | 4.8082 | 5.7718 | 7.8223 | 10.035 | 11.563 |
| | ESDT [45] | 4.8274 | 5.6883 | 7.6248 | 9.9028 | 11.332 |
| 20 | Present | 4.8280 | 5.6882 | 7.6236 | 9.9008 | 11.329 |
| | PSDT [45] | 4.8556 | 5.7042 | 7.6392 | 9.9209 | 11.353 |
| | ESDT [45] | 4.7585 | 5.7225 | 7.7681 | 9.9693 | 11.485 |
| 50 | Present | 4.7771 | 5.6150 | 7.5237 | 9.8689 | 11.645 |
| | PSDT [45] | 4.7783 | 5.6156 | 7.5241 | 9.8692 | 11.645 |
| | ESDT [45] | 4.8061 | 5.6318 | 7.5399 | 9.8895 | 11.669 |
| 100 | Present | 4.7635 | 5.5953 | 7.4971 | 9.8623 | 11.741 |
| | PSDT [45] | 4.7630 | 5.5944 | 7.4956 | 9.8600 | 11.738 |
| | ESDT [45] | 4.7642 | 5.5951 | 7.4959 | 9.8602 | 11.738 |
| Plate | Present | 4.7425 | 5.7066 | 7.7507 | 9.9482 | 11.460 |
| | PSDT [45] | 4.7615 | 5.5923 | 7.4931 | 9.8610 | 11.754 |
| | ESDT [45] | 4.7610 | 5.5915 | 7.4916 | 9.8587 | 11.751 |

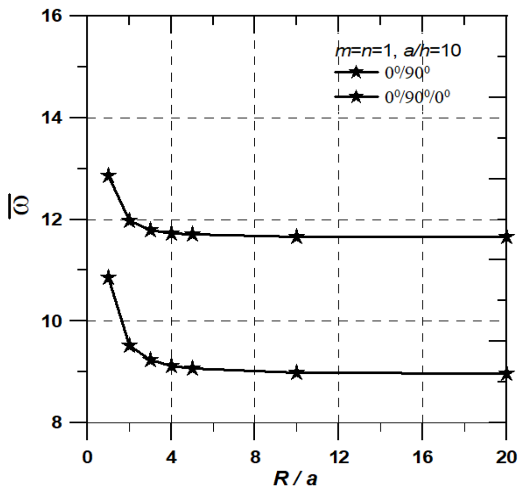


**Fig. 2.** Variations of natural frequencies with respect to R/a ratio in laminated composite cylindrical shell.

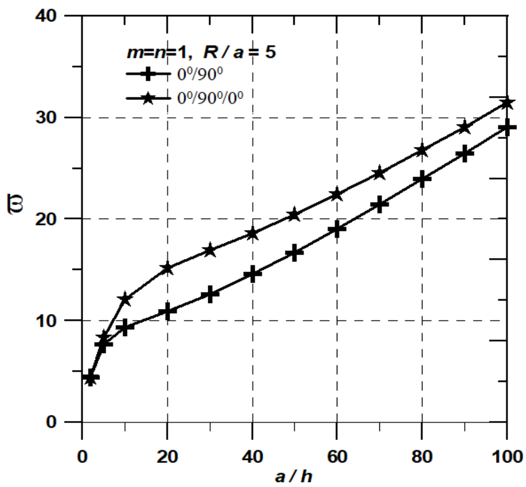


**Fig. 4.** Variations of natural frequencies with respect to a/h ratio in laminated composite spherical shell.

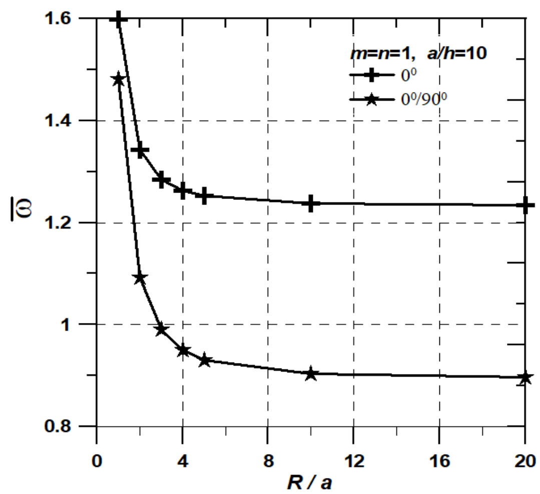


**Fig. 3.** Variations of natural frequencies with respect to R/a ratio in laminated composite spherical shell

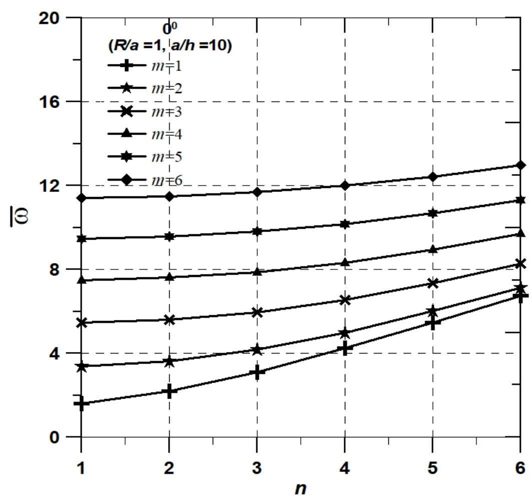


**Fig. 5.** Variations of natural frequencies with respect to modes of vibration in orthotropic spherical shell.

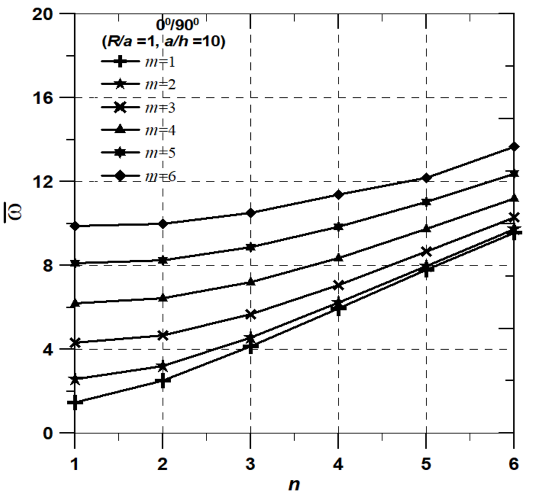


**Fig. 6.** Variations of natural frequencies $(\bar{\omega})$ with respect to modes of vibration in two layered ($0^0/90^0$) laminated composite spherical shell.

## 5. Conclusion

In the present study, a new fifth-order shear and normal deformation theory is developed and applied for the free vibration analysis of laminated composite and sandwich shells. The present theory includes the effects of both transverse shear and normal deformations. A polynomial type transverse shear strain shape function is used in the displacement field to account for these effects. The fundamental frequency analysis is performed for different types of shell problems, to prove the efficacy and validity of the present theory. Based on the numerical results and discussion, the following conclusions are drawn.

1. The present results are compared with previously published results and found in good agreement with those.
2. Natural frequencies of laminated shells decrease with an increase in the radii of curvature which shows that the deep shells predict higher frequencies whereas shallow shells predict lower frequencies.
3. It is concluded that the natural frequency increases with an increase in a/h ratio which ultimately shows that the thin shell predicts higher frequency whereas the thick shell predicts lower frequency.
4. In the case of sandwich shells, natural frequency increases with the increase in softness of the core material.

Based on the literature review, illustrated examples, numerical results, and discussion, the authors recommend the use of the present theory for many other problems of composite shells.

## Appendix

The elements of stiffness matrix [K] in Eq. (16) are: [A.1]

$$K_{11}=-A_{11}\alpha^2-A_{66}\beta^2, \quad K_{12}=K_{21}=-A_{12}\alpha\beta-A_{66}\alpha\beta,$$

$$K_{13}=K_{31}=\frac{A_{11}}{R_1}\alpha+\frac{A_{12}}{R_R}\beta+B_{11}\alpha^3+B_{12}\alpha\beta^2+2B_{66}\alpha\beta^2,$$

$$K_{14}=K_{41}=-A_{S1_{11}}\alpha^2-A_{S2_{66}}\beta^2,$$

$$K_{15}=K_{51}=-A_{S2_{11}}\alpha^2-A_{S2_{66}}\beta^2,$$

$$K_{16}=K_{61}=-A_{S1_{12}}\alpha\beta-A_{S1_{66}}\alpha\beta,$$

$$K_{17}=K_{71}=-A_{S2_{12}}\alpha\beta-A_{S2_{66}}\alpha\beta,$$

$$K_{18}=K_{81}=\left(E_{13}+\frac{Q_{1_{11}}}{R_1}+\frac{Q_{1_{12}}}{R_2}\right)\alpha,$$

$$K_{19}=K_{91}=\left(F_{13}+\frac{Q_{2_{11}}}{R_1}+\frac{Q_{2_{12}}}{R_2}\right)\alpha,$$

$$K_{22}=-A_{22}\beta^2-A_{66}\alpha^2,$$

$$K_{23}=K_{32}=\left(\frac{A_{12}}{R_1}+\frac{A_{22}}{R_2}\right)\beta+B_{12}\alpha^2\beta+B_{22}\beta^3+2B_{66}\alpha^2\beta,$$

$$K_{24}=K_{42}=-A_{S1_{12}}\alpha\beta-A_{S1_{66}}\alpha\beta,$$

$$K_{25}=K_{52}=-A_{S2_{12}}\alpha\beta-A_{S2_{66}}\alpha\beta,$$

$$K_{26}=K_{62}=-A_{S1_{22}}\beta^2-A_{S1_{66}}\alpha^2,$$

$$K_{27}=K_{72}=-A_{S2_{22}}\beta^2-A_{S2_{66}}\alpha^2,$$

$$K_{28}=K_{82}=\left(E_{23}+\frac{Q_{1_{12}}}{R_1}+\frac{Q_{1_{22}}}{R_2}\right)\beta,$$

$$K_{29}=K_{92}=\left(F_{23}+\frac{Q_{2_{12}}}{R_1}+\frac{Q_{2_{22}}}{R_2}\right)\beta$$

$$K_{33}=-\left(D_{11}\alpha^4+D_{22}\beta^4\right)-2\alpha^2\beta^2\left(D_{12}+2D_{66}\right)$$
$$-2\alpha^2\left(\frac{B_{11}}{R_1}+\frac{B_{12}}{R_2}\right)-2\beta^2\left(\frac{B_{12}}{R_1}+\frac{B_{22}}{R_2}\right)-\left(\frac{A_{11}}{R_1^2}+\frac{A_{22}}{R_2^2}+\frac{2A_{12}}{R_1R_2}\right),$$

$$K_{34}=K_{43}=B_{S1_{11}}\alpha^3+B_{S1_{12}}\alpha\beta^2+2B_{S1_{66}}\alpha\beta^2+\frac{A_{S1_{11}}}{R_1}\alpha+\frac{A_{S1_{12}}}{R_2}\alpha,$$

$$K_{35}=K_{53}=B_{S2_{11}}\alpha^3+B_{S2_{12}}\alpha\beta^2+2B_{S2_{66}}\alpha\beta^2+\frac{A_{S2_{11}}}{R_1}\alpha+\frac{A_{S2_{12}}}{R_2}\alpha,$$

$$K_{36}=K_{63}=B_{S1_{12}}\alpha^2\beta+B_{S1_{22}}\beta^3+2B_{S1_{66}}\alpha^2\beta+\frac{A_{S1_{12}}}{R_1}\beta+\frac{A_{S1_{22}}}{R_2}\beta,$$

$$K_{37}=K_{73}=B_{S2_{12}}\alpha^2\beta+B_{S2_{22}}\beta^3+2B_{S2_{66}}\alpha^2\beta+\frac{A_{S2_{12}}}{R_1}\beta+\frac{A_{S2_{22}}}{R_2}\beta,$$

$$K_{38}=K_{83}=-J_{13}\alpha^2-J_{23}\beta^2-\frac{E_{13}}{R_1}-\frac{E_{23}}{R_2}-\left(\frac{Q_{3_{11}}}{R_1}+\frac{Q_{3_{12}}}{R_2}\right)\alpha^2$$
$$-\left(\frac{Q_{3_{12}}}{R_1}+\frac{Q_{3_{22}}}{R_2}\right)\beta^2-\left(\frac{Q_{1_{11}}}{R_1^2}+\frac{Q_{1_{12}}}{R_1R_2}\right)-\left(\frac{Q_{1_{12}}}{R_1R_2}+\frac{Q_{1_{22}}}{R_2^2}\right),$$

$$K_{39}=K_{93}=-O_{13}\alpha^2-O_{23}\beta^2-\frac{F_{13}}{R_1}-\frac{F_{23}}{R_2}-\left(\frac{Q_{4_{11}}}{R_1}+\frac{Q_{4_{12}}}{R_2}\right)\alpha^2$$
$$-\left(\frac{Q_{4_{12}}}{R_1}+\frac{Q_{4_{22}}}{R_2}\right)\beta^2-\left(\frac{Q_{2_{11}}}{R_1^2}+\frac{Q_{2_{12}}}{R_1R_2}\right)-\left(\frac{Q_{2_{12}}}{R_1R_2}+\frac{Q_{2_{22}}}{R_2^2}\right),$$

$$K_{44} = -A_{SS1_{11}}\alpha^2 - A_{SS1_{66}}\beta^2 - G_{44},$$
$$K_{45} = K_{54} = -C_{11}\alpha^2 - C_{66}\beta^2 - I_{44},$$
$$K_{46} = K_{64} = -A_{SS1_{12}}\alpha\beta - A_{SS1_{66}}\alpha\beta,$$
$$K_{47} = K_{74} = -C_{12}\alpha\beta - C_{66}\alpha\beta,$$
$$K_{48} = K_{84} = \left(L_{1_{13}}\alpha - G_{55}\alpha + \frac{Q_{511}}{R_1} + \frac{Q_{512}}{R_2}\right),$$
$$K_{49} = K_{94} = \left(L_{2_{13}}\alpha - I_{55}\alpha + \frac{Q_{6_{11}}}{R_1} + \frac{Q_{6_{12}}}{R_2}\right),$$
$$K_{55} = -A_{SS2_{11}}\alpha^2 - A_{SS2_{66}}\beta^2 - H_{44},$$
$$K_{56} = K_{65} = -C_{12}\alpha\beta - C_{66}\alpha\beta,$$
$$K_{57} = K_{75} = -A_{S2_{12}}\alpha\beta - A_{SS2_{66}}\alpha\beta,$$
$$K_{58} = K_{85} = -I_{55}\alpha + M_{1_{13}}\alpha + \frac{Q_{7_{11}}}{R_1} + \frac{Q_{7_{12}}}{R_2},$$
$$K_{59} = K_{95} = -H_{55}\alpha + M_{2_{13}}\alpha + \frac{Q_{8_{11}}}{R_1} + \frac{Q_{8_{12}}}{R_2},$$
$$K_{66} = -A_{SS1_{22}}\beta^2 - A_{SS1_{66}}\alpha^2 - G_{55},$$
$$K_{67} = K_{76} = -C_{22}\beta^2 - C_{66}\alpha^2 - I_{55},$$
$$K_{68} = K_{86} = L_{1_{23}}\beta - G_{55}\beta + \frac{Q_{512}}{R_1} + \frac{Q_{522}}{R_2},$$
$$K_{69} = K_{96} = L_{2_{23}}\beta - I_{44}\beta + \frac{Q_{6_{12}}}{R_1} + \frac{Q_{6_{22}}}{R_2},$$
$$K_{77} = -A_{SS2_{22}}\beta^2 - A_{SS2_{66}}\alpha^2 - H_{55},$$
$$K_{78} = K_{87} = M_{1_{23}}\beta - I_{44}\beta + \frac{Q_{7_{12}}}{R_1} + \frac{Q_{7_{22}}}{R_2},$$
$$K_{79} = K_{97} = M_{2_{23}}\beta - H_{44}\beta + \frac{Q_{8_{12}}}{R_1} + \frac{Q_{8_{22}}}{R_2},$$
$$K_{88} = -G_{55}\alpha^2 - G_{44}\beta^2 - N_{1_{33}} - 2\frac{Q_{16_{13}}}{R_1} - 2\frac{Q_{16_{23}}}{R_2} - \frac{Q_{13_{11}}}{R_1^2} - 2\frac{Q_{13_{12}}}{R_1R_2} - \frac{Q_{13_{22}}}{R_2^2},$$
$$K_{89} = K_{98} = -I_{55}\alpha^2 - I_{44}\beta^2 - N_{3_{33}} - \frac{Q_{17_{13}}}{R_1} - \frac{Q_{17_{23}}}{R_2} - \frac{Q_{18_{13}}}{R_1} - \frac{Q_{18_{23}}}{R_2} - \frac{Q_{14_{11}}}{R_1^2} - 2\frac{Q_{14_{12}}}{R_1R_2} - \frac{Q_{14_{22}}}{R_2^2},$$
$$K_{99} = -H_{55}\alpha^2 - H_{44}\beta^2 - N_{2_{33}} - 2\frac{Q_{19_{13}}}{R_1} - 2\frac{Q_{19_{23}}}{R_2} - \frac{Q_{15_{11}}}{R_1^2} - 2\frac{Q_{15_{12}}}{R_1R_2} - \frac{Q_{15_{22}}}{R_2^2}$$

The elements of mass matrix [M] in Eq. (16) are: [A.2]

$$M_{11} = -\left(I_1 + 2\frac{I_2}{R_1} + \frac{I_3}{R_1^2}\right),$$
$$M_{13} = -\alpha\left(I_2 + \frac{I_3}{R_1}\right),$$
$$M_{14} = -\left(I_4 + \frac{I_8}{R_1}\right),$$
$$M_{15} = -\left(I_5 + \frac{I_9}{R_1}\right),$$
$$M_{22} = -\left(I_1 + 2\frac{I_2}{R_2} + \frac{I_3}{R_2^2}\right),$$
$$M_{23} = -\beta\left(I_2 + \frac{I_3}{R_2}\right),$$
$$M_{26} = -\left(I_4 + \frac{I_8}{R_2}\right),$$
$$M_{27} = -\left(I_5 + \frac{I_9}{R_2}\right),$$
$$M_{33} = -\left(I_3\alpha^2 + I_3\beta^2 + I_1\right),$$
$$M_{34} = I_8\alpha,\quad M_{35} = I_9\alpha,$$
$$M_{36} = I_8\beta,\quad M_{37} = I_9\beta,$$
$$M_{38} = -I_{10},\quad M_{35} = -I_{11},$$
$$M_{44} = -I_6,\quad M_{45} = -I_{15},$$
$$M_{55} = -I_7,\quad M_{66} = -I_6,$$
$$M_{67} = -I_{15}, M_{77} = -I_7, M_{88} = -I_{13},$$
$$M_{89} = -I_{12}, M_{99} = -I_{14}$$
$$M_{12} = M_{16} = M_{17} = M_{18} = 0$$
$$M_{19} = M_{24} = M_{25} = M_{28} = M_{29} = 0$$
$$M_{46} = M_{47} = M_{48} = M_{49} = 0$$
$$M_{56} = M_{57} = M_{58} = M_{59} = M_{68} = 0$$
$$M_{69} = M_{78} = M_{79} = 0$$

$$\Delta = \left\{u_1, \phi_{x1}, \psi_{x1}, v_1, \phi_{y1}, \psi_{y1}, w_1, \phi_{z1}, \psi_{z1}\right\}^T$$

where [A.3] is,

$$\left(A_{ij}, B_{ij}, D_{ij}\right) = C_{ij}\int_{-h/2}^{h/2}\left[1.0, z, z^2\right]dz$$

$$\left(A_{S1_{ij}}, A_{S2_{ij}}\right) = C_{ij}\int_{-h/2}^{h/2}\left[\varphi_1(z), \varphi_2(z)\right]dz$$

$$\left(B_{S1_{ij}},B_{S2_{ij}}\right)=C_{ij}\int_{-h/2}^{h/2}\left[z\,\varphi_1(z),\,z\,\varphi_2(z)\right]dz$$

$$\left(Q_{1_{ij}},Q_{2_{ij}}\right)=C_{ij}\int_{-h/2}^{h/2}\left[\varphi_1^{'}(z),\varphi_2^{'}(z),\right]dz$$

$$\left(Q_{3_{ij}},Q_{4_{ij}}\right)=C_{ij}\int_{-h/2}^{h/2}\left[z\,\varphi_1^{'}(z),\,z\,\varphi_2^{'}(z)\right]dz$$

$$\left(A_{SS1_{ij}},A_{SS2_{ij}}\right)=C_{ij}\int_{-h/2}^{h/2}\left\{\left[\varphi_1(z)\right]^2,\left[\varphi_2(z)\right]^2\right\}dz$$

$$\left(C_{ij}\right)=C_{ij}\int_{-h/2}^{h/2}\left\{\left[\varphi_1(z)\varphi_2(z)\right]\right\}dz$$

$$\left(Q_{5_{ij}},Q_{6_{ij}}\right)=C_{ij}\int_{-h/2}^{h/2}\left\{\varphi_1(z)\left[\varphi_1^{'}(z),\varphi_2^{'}(z)\right]\right\}dz$$

$$\left(Q_{7_{ij}},Q_{8_{ij}}\right)=C_{ij}\int_{-h/2}^{h/2}\left\{\varphi_2(z)\left[\varphi_1^{'}(z),\varphi_2^{'}(z)\right]\right\}dz$$

$$\left(Q_{13_{ij}},Q_{15_{ij}}\right)=C_{ij}\int_{-h/2}^{h/2}\left\{\left[\varphi_1^{'}(z)\right]^2,\left[\varphi_2^{'}(z)\right]^2\right\}dz$$

$$\left(Q_{14_{ij}}\right)=C_{ij}\int_{-h/2}^{h/2}\left\{\left[\varphi_1^{'}(z)\varphi_2^{'}(z)\right]\right\}dz$$

$$\left(G_{ij},H_{ij}\right)=C_{ij}\int_{-h/2}^{h/2}\left\{\left[\varphi_1^{'}(z)\right]^2,\left[\varphi_2^{'}(z)\right]^2\right\}dz$$

$$\left(I_{ij}\right)=C_{ij}\int_{-h/2}^{h/2}\left\{\left[\varphi_1^{'}(z)\varphi_2^{'}(z)\right]\right\}dz$$

$$\left(E_{ij},F_{ij}\right)=C_{ij}\int_{-h/2}^{h/2}\left\{\left[\varphi_1^{''}(z)\right],\left[\varphi_2^{''}(z)\right]\right\}dz$$

$$\left(N_{1_{ij}},N_{2_{ij}}\right)=C_{ij}\int_{-h/2}^{h/2}\left\{\left[\varphi_1^{''}(z)\right]^2,\left[\varphi_2^{''}(z)\right]^2\right\}dz$$

$$\left(N_{3_{ij}}\right)=C_{ij}\int_{-h/2}^{h/2}\left\{,\left[\varphi_1^{''}(z)\varphi_2^{'}(z)\right]\right\}dz$$

$$\left(Q_{17_{ij}}\right)=C_{ij}\int_{-h/2}^{h/2}\left\{\varphi_1^{'}(z)\varphi_1^{''}(z)\right\}dz$$

$$\left(Q_{17_{ij}}\right)=C_{ij}\int_{-h/2}^{h/2}\left\{\varphi_1^{'}(z)\varphi_2^{''}(z)\right\}dz$$

$$\left(Q_{18_{ij}}\right)=C_{ij}\int_{-h/2}^{h/2}\left\{\varphi_2(z)\varphi_1^{''}(z)\right\}dz$$

$$\left(Q_{19_{ij}}\right)=C_{ij}\int_{-h/2}^{h/2}\left\{\varphi_2(z)\varphi_2^{''}(z)\right\}dz$$

$$\left(J_{ij}\right)=C_{ij}\int_{-h/2}^{h/2}z\,\varphi_1^{''}(z)\,dz$$

$$\left(L_{1_{ij}},L_{2_{ij}}\right)=C_{ij}\int_{-h/2}^{h/2}\varphi_1^{''}(z)\left[\varphi_1(z),\varphi_2(z)\right]dz$$

$$\left(O_{ij}\right)=C_{ij}\int_{-h/2}^{h/2}z\,\varphi_2^{''}(z)\,dz$$

$$\left(M_{1_{ij}},M_{2_{ij}}\right)=C_{ij}\int_{-h/2}^{h/2}\varphi_2^{''}(z)\left[\varphi_1(z),\varphi_2(z)\right]dz$$

# References


[1] Kirchhoff, G. R., 1850. Uber das Gleichgewicht und die Bewegung einer Elastischen Scheibe. *Journal of Reine Angew. Math. (Crelle*), 40, pp. 51-88.

[2] Mindlin, R.D., 1951. Influence of rotatory inertia and shear on flexural motions of isotropic elastic plates. *ASME Journal of Applied Mechanics, 18*, pp. 31-38.

[3] Qatu, M.S., 1992. Review of shallow shell vibration research, *Shocks and Vibrations*, *24(9)*, pp. 3-15.

[4] Qatu, M.S., 2002a. Recent research advances in the dynamic behavior of shells: 1989–2000, Part 1: laminated composite shells. *Applied and Mechanics Revision*, *55(4)*, pp. 325–350.

[5] Qatu, M.S., 2002b. Recent research advances in the dynamic behavior of shells: 1989–2000, Part 2: Homogeneous shells. *Applied and Mechanics Revision, 55(5)*, pp. 415–434.

[6] Qatu, M.S., Sullivan, R. W. and Wang, W., 2010. Recent Research Advances in the Dynamic Behavior of Composite Shells: 2000-2009*, Composite Structures*, *93(1)*, pp. 14-31.

[7] Mallikarjuna and Kant, T., 1993. A critical review and some results of recently developed refined theories of fiber-reinforced laminated composites and sandwiches. *Composite Structures*, *23*, pp. 293-312.

[8] Thai, H.T. and Kim, S.E., 2015. A review of theories for the modeling and analysis of functionally graded plates and shells. *Composite Structures, 128*, pp. 70-86.

[9] Sayyad, A.S. and Ghugal, Y.M., 2015. On the free vibration analysis of laminated composite and sandwich plates: A review of recent literature with some numerical results. *Composite Structures, 129*, pp. 177-201.

[10] Sayyad, A.S. and Ghugal, Y.M., 2017. Bending, buckling and free vibration of laminated composite and sandwich beams: A critical review of literature. *Composite Structures*, *171*, pp. 486-504.

[11] Sayyad, A.S. and Ghugal, Y.M., 2018. Modeling and analysis of functionally graded sandwich beams: A review. *Mechanics of Advanced Materials and Structures*, *26(21)*, pp. 1776-1795.

[12] Reddy, J.N., 1984. A Simple Higher-Order Theory for Laminated Composite Plates. *Journal of Applied Mechanics*, *51 (4)*, pp. 745-752.

[13] Bhimaraddi, A., 1991. Free vibration analysis of doubly curved shallow shells on rectangular plan form using three-dimensional elasticity theory. *International*

*Journal of Solids and Structures*, *27(7)*, pp. 897-913.

[14] Timarci, T. and Soldatos, K.P., 1995. Comparative dynamic studies for symmetric cross-ply circular cylindrical shells on the basis of a unified shear deformable shell theory. *Journal of Sound and Vibration*, *187(4)*, pp. 609-624.

[15] Khare R.K, Kant T, Garg A. K., 2004. Free vibration of composite and sandwich laminates with a higher-order facet shell element. *Composite Structures*, *65*, pp. 405-418.

[16] Ferreria, A.J.M., 2005. Analysis of composite plates using a layerwise theory and multiquadrucs discretization. *Mechanics of Advanced Materials and Structures*, *12*, pp. 99-112.

[17] Ferreria, A.J.M., Roque, C.M.C. and Jorge, R.M.N., 2006. Static and free vibration analysis of composite shells by radial basis functions. *Engineering Analysis with Boundary Elements*, *30*, pp. 719-733.

[18] Garg, A.K., Khare, R.K. and Kant, T., 2006. Higher-order closed-form solutions for free vibration of laminated composite and sandwich shells. *Journal of Sandwich Structures and Materials*, *8*, pp. 205-235.

[19] Pradyumna, S., and Bandyopadhyay, J.N., 2007. Static and free vibration analyses of laminated shells using a higher-order theory. *Journal of Reinforced Plastic Composites*, *27(2)*, pp. 167–186.

[20] Matsunaga, H., 2007. Vibration and stability of cross-ply laminated composite shallow shells subjected to in-plane stresses. *Composite Structures*, *78*, pp. 377-391.

[21] Carrera, E. and Brischetto, S., 2008. Analysis of thickness locking in classical, refined and mixed theories for layered shells. *Composite Structures*, *85 (1)*, pp. 83–90.

[22] Brischetto, S. Carrera, E. and Demasai, L., 2009. Free vibration of sandwich plates and shells by using zig-zag function. *Shocks and Vibration*, *16*, pp. 495-503.

[23] Zhao, X., Lee, Y.Y., and Liew, K.M., 2009. Thermoelastic and vibration analysis of functionally graded cylindrical shells. *International Journal of Mechanical Sciences*, *51*, pp. 694-707.

[24] Noh, M.H. and Lee, S.Y., 2012. Vibration of composite shells containing embedded delaminations based on the third-order shear deformation theory. *KSCE Journal of Civil Engineering*, *16(7)*, pp. 1193-1201.

[25] Mantari, J.L. and Soares, C.G., 2012. Analysis of isotropic and multilayered plates and shells by using a generalized higher-order shear deformation theory. *Composite Structures*, *94*, pp. 2640-2656.

[26] Tornabene, F., 2009. Free vibration analysis of functionally graded conical, cylindrical shell and annular plate structures with a four-parameter power law distribution. *Computational Methods Applied Mechanical Engineering*, *198*, pp. 2911-2935.

[27] Tornabene, F., 2011. 2-D GDQ solution for free vibrations of anisotropic doubly-curved shells and panels of revolution. *Composite Structures*, *93*, pp. 1854-1876.

[28] Tornabene, F., Viola, E. and Fantuzzi, N., 2013. General higher-order equivalent single layer theory for the free vibrations of doubly-curved laminated composite shells and panels. *Composite Structures*, *104*, pp. 94-117.

[29] Tornabene, F., Liverani, A. and Caligiana, G., 2012. General anisotropic doubly-curved shell theory: A differential quadrature solution for free vibrations of shell and panels of revolution with a free-form meridian. *Journal of Sound and Vibration*, *331*, pp. 4848-4869.

[30] Qatu, M.S. and Asadi, E., 2012. Vibration of doubly curved shallow shells with arbitrary boundaries. *Applied Accoustics*, *73*, pp. 21-27.

[31] Asadi, E., Wang. and Qatu, M., 2012. Static and vibration analyses of thick deep laminated cylindrical shells using 3D various shear deformation theories. *Composite Structures*, *94*, pp. 494-500.

[32] Fazzolari, F.A. and Carrera, E., 2013. Advances in the Ritz formulation for free vibration response of doubly-curved anisotropic laminated composite shallow and deep shells. *Composite Structures*, 101, pp. 111-128.

[33] Taj, G. and Chakrabarti, A., 2013. An efficient C0 finite element approach for bending analysis of functionally graded ceramic-metal skew shell panels. *Journal of Solid Mechanics*, *5(1)*, pp. 47-62.

[34] Dai, L. Yang, T. Du, J. Li, W.L. and Brennan, M.J., 2013. An exact series solution for the vibration analysis of cylindrical shells with arbitrary boundary conditions. *Applied Acoustics*, *74*, pp. 440-449.

[35] Wang, Q., Shi, D, Pang, F. and Liang Q, 2016. Vibrations of composite laminated circular panels and shells of revolution with general elastic boundary conditions via fourier-ritz method. *Curved and Layered Structures*, *3(1)*, pp. 105-136.

[36] Rawat, A., Matsagar, V. and Nagpal, A.K., 2016. Finite element analysis of thin circular cylindrical shells. *Proc. of Indian National Science Academy*, *82(2)*, pp. 349-355.

[37] Pandey, S. and Pradyumna, S., 2017. A finite element formulation for thermally induced vibrations of functionally graded material

sandwich plates and shell panels. *Composite Structures*, *160*, pp. 877-886.

[38] Biswal, D.K., Joseph, S.V. and Mohanty, S.C., 2018. Free vibration and buckling study of doubly curved Sander's approximation. *Proc. of IMechE Part C: Journal of Mechanical Engineering Science*, *232(20)*, pp. 3612-3628.

[39] Fares,M.E., Elmarghany, M.K., Atta, D. and Salem, M.G., 2018. Bending and free vibration of multilayered functionally graded curved shells by an improved layerwise theory. *Composites Part B*, 154, pp. 272-284.

[40] Monge, J.C., Mantari, J.L., Charca, S. and Vladimir, N., 2018. An axiomatic/asymptotic evaluation of the best theories for free vibration of laminated and sandwich shells using non-polyomial functions. *Engineering Structures*. *172*, pp. 1011-1024.

[41] Cong, P.H., Khanh,N.D., Khoa, N.D. and Duc, N.D., 2018. New approach to investigate nonlinear dynamic response of sandwich auxetic double curves shallow shells using TSDT. *Composite Structures*, *185*, pp. 455-465.

[42] Li, H., Pang, F., Wang,X. Du, Y. and Chen, H., 2018. Free vibration analysis for composite laminated doubly–curved shells of revolution by a semi analytical method. *Composite Structures*, *201*, pp. 86-111.

[43] Pang, F., Li, H., Wang, X., Miao, X. and Li, S., 2018. A semi-analytical method for the free vibration of doubly-curved shells of revolution. *Computers and Mathematics with Applications*, *75*, pp. 3249-3268.

[44] Kiani, Y. Dimitri, R. and Tornabene, F., 2018. Free vibration of FG-CNT reinforced composite skew cylindrical shells using the Chebyshev-Ritz formulation. *Composites part B*, *147*, pp. 169-177.

[45] Sayyad, A.S. and Ghugal, Y.M., 2019. Static and free vibration analysis of laminated composite and sandwich spherical shells using a generalized higher-order shell theory, *Composite Structures*, 219, pp. 129–146.

[46] Draiche, K., Tounsi, A., Mahmoud S.R., 2016. A refined theory with stretching effect for the flexure analysis of laminated composite plates, *Geomechanics and Engineering*, 11(5), pp. 671–690.

[47] Allam, O., Draiche, K., Bousahla, A.A, Bourada, F., Tounsi, A., Benrahou, K.H., Mahmoud, S.R., Adda Bedia, E.A. and Tounsi, A., 2020. A generalized 4-unknown refined theory for bending and free vibration analysis of laminated composite and sandwich plates and shells. *Computers and Concrete*, *26(2)*, pp. 185–201.

[48] Zine, A., Tounsi, A., Draiche, K., Sekkal, M., and Mahmoud, S.R., 2018. A novel higher-order shear deformation theory for bending and free vibration analysis of isotropic and multilayered plates and shells. *Steel and Composite Structures*, *26(2)*, pp. 125–137.

[49] Arefi, M., 2018a. Analysis of a doubly curved piezoelectric nano shell: Nonlocal electro-elastic bending solution. *European Journal of Mechanics / A Solids, 70*, pp. 226-237.

[50] Arefi, M., 2018b. Nonlocal free vibration analysis of a doubly curved piezoelectric nano shell. *Steel and Composite Structures*, *27(4)*, pp. 479-493.

[51] Arefi, M., 2019. Third-order electro-elastic analysis of sandwich doubly curved piezoelectric micro shells. *Mechanics Based Design of Structures and Machines*, *49(6)*, pp. 781-810.

[52] Arefi, M., 2020a. Size-dependent electro-elastic analysis of three-layered piezoelectric doubly curved nano shell. Mechanics of Advanced Materials and Structures, *27(23)*, pp. 1945-1965.

[53] Arefi, M., 2020b. Size-dependent bending behavior of three-layered doubly curved shells: Modifies couple stress formulation. *Journal of Sandwich Structures and Materials*, *22(7)*, pp. 2210-2249.

[54] Arefi, M. and Zenkour, A. M., 2016a. A simplified shear and normal deformations nonlocal theory for bending of functionally graded piezomagnetic sandwich nanobeams in magneto-thermo-electric environment. *Journal of Sandwich Structures and Materials*, *18(5)*, pp. 624-651.

[55] Arefi, M. and Zenkour, A. M., 2016b. Free vibration analysis of a three-layered microbeam based on strain gradient theory and three-unknown shear and normal deformation theory. *Steel and Composite Structures*, *26(4)*, pp. 421-437.

[56] Arefi, M. and Rabczuk, T., 2019. A nonlocal higher order shear deformation theory for electro-elastic analysis of a piezoelectric doubly curved nano shell. *Composites Part B*, *168*, pp. 496-510.

[57] Arefi, M. and Elyas, M. R. B., 2019. Electro-elastic displacement and stress analysis of the piezoelectric doubly curved shells resting on winkler's foundation subjected to applied voltage. *Mechanics of Advanced Materials and Structures*, *26(23)*, pp. 1981-1994.

[58] Arefi, M., Mohammad-Rezaei Bidgoli, E. and Civalek, O., 2020. Bending response of FG composite doubly curved nanoshells with thickness stretching via higher-order sinusoidal shear theory. *Mechanics Based Design of Structures and Machines*, pp.1-29.

[59] Arefi, M. and Amabili, M., 2021. A comprehensive electro-magneto-elastic buckling and bending analysis of three-layered doubly curved nanoshell, based on nonlocal three-dimensional theory. *Composite Structures*, 257, pp. 113100.

[60] Draiche, K., Bousahla, A.A, Tounsi, A., Alwabli. A.S., Tounsi, A., and Mahmoud, S. R., 2019. Static analysis of laminated reinforced composite plates using a simple first-order shear deformation theory. *Computers and Concrete*, *24(4)*, pp. 369-378.

[61] Belbachir, N., Bourada, M., Draiche, K., Tounsi, A., Bourada, F., Bousahla, A. A., Mahmoud, S. R., 2020. Thermal flexural analysis of anti-symmetric cross-ply laminated plates using a four variable refined theory. *Smart Structures and Systems*, *25(4)*, pp. 409-422.

[62] Belbachir, N., Draich, K., Bousahla, A.A., Bourada, M., Tounsi, A. and Mohammadimehr, M., 2019. Bending analysis of anti-symmetric cross-ply laminated plates under nonlinear thermal and mechanical loadings. *Steel and Composite Structures*, *33(1)*, pp. 81-92.

[63] Abualnour, M., Chikh, A., Hebali, H., Kaci, A., Tounsi, A, Bousahla, A. A. and Tounsi, A., 2019. Thermomechanical analysis of antisymmetric laminated reinforced composite plates using a new four variable trigonometric refined plate theory. *Computers and Concrete*, *24(6)*, pp. 489-498.

[64] Sahla, M., Saidi, H., Draiche, H., Bousahla, A.A., Bourada, F., and Tounsi, A., 2019. Free vibration analysis of angle-ply laminated composite and soft core sandwich plates. *Steel and Composite Structures*, *33(5)*, pp. 663-679.

[65] Carrera, E., 2005. Transverse normal strain effects on thermal stress analysis of homogeneous and layered plates. *AIAA Journal, 43(10)*, pp. 2232-2242.

[66] Sayyad, A.S. and Naik, N.S., 2019. New displacement model for accurate prediction of transverse shear stresses in laminated and sandwich rectangular plates. *ASCE, Journal of Aerospace Engineering*, *32(5)*, pp. 1-12.

[67] Naik, N.S. and Sayyad, A.S., 2018a. 2D analysis of laminated composite and sandwich plates using a new fifth-order plate theory, *Latin American journal of Solids and Structures*, *15(9)*, pp. 1-27.

[68] Naik, N.S. and Sayyad, A.S., 2018b. 1D analysis of laminated composite and sandwich plates using a new fifth-order plate theory. *Latin American journal of Solids and Structures*, *15*, pp. 1-17.

[69] Naik, N.S. and Sayyad, A.S., 2019. An accurate computational model for thermal analysis of laminated composite and sandwich plates, *Journal of Thermal Stresses*, *42(5)*, pp. 559-579.

[70] Ghumare, S.M. and Sayyad, A.S., 2017. A new fifth-order shear and normal deformation theory for static elastic buckling of P-FGM beams. *Latin American Journal of Solids and Structures*, *14*, pp. 1893-1911.

[71] Ghumare, S.M. and Sayyad, A.S., 2019a. Nonlinear hygro-thermo-mechanical analysis of functionally graded plates using a fifth-order plate theory. *Arabian Journal for Science and Engineering*, *44*, pp. 8727-8745.

[72] Ghumare, S.M. and Sayyad, A.S., 2019b. A new quasi-3D model for functionally graded plates. *Journal of Applied and Computational Mechanics*, *5(2)*, pp. 367-380.

[73] Shinde B. M, and Sayyad A. S., 2020. Analysis of laminated and sandwich spherical shells using a new higher order theory, *Advances in Aircraft and Spacecraft Sciences*, *7 (1)*, pp. 19-40.