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New theoretical conditions for solving functional nonlinear equations by linearization then discretization

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Abstract

In this paper, we propose to solve nonlinear functional equations given in an infinite-dimensional Banach space by linearizing first and then discretizing the linear iterative equations. We establish new sufficient conditions which provide new criteria for dealing with convergence results. These conditions define a class of discretization schemes. Some numerical examples confirm the theoretical results by treating an integro-differential equation.

Keywords: Nonlinear equation, Iterative methods, Convergence, Newton-Kantorovich method.
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1. Introduction

Nonlinear functional equations are a fundamental tool for mathematical modeling derived from physical, chemical, or other discipline processes. The solutions of these equations are typically given in an infinite-dimensional Banach space, where an analytical determination is usually unavailable. A nonlinear equation is illustrated here as:

$$\text{Find } \varphi \in \mathcal{O} \subset \mathcal{X} : \quad F(\varphi) = 0, \tag{1.1}$$

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where $\mathcal{O}$ is a nonempty open set of an infinite-dimensional Banach space $\mathcal{X}$, F is a nonlinear operator from $\mathcal{O}$ into $\mathcal{X}$, φ is the solution, and 0 is the null vector of $\mathcal{X}$.

The linearization of equation (1.1) by the Newton-Kantorovich method generates a sequence $(\varphi^{(k)})_{k\geq 0} \subset \mathcal{O}$ given by the relation

$$\varphi^{(k+1)} = \varphi^{(k)} - F'(\varphi^{(k)})^{-1} F(\varphi^{(k)}), \quad k \geq 0, \quad \varphi^{(0)} \in \mathcal{O}, \tag{1.2}$$

where F' denotes the Fréchet derivative of F and $F'(x)^{-1}$ denote the inverse of $F'(x)$. To approximate the solution of equation (1.1), there are two options:

- **Option (A):** Discretizing equation (1.2) then solving numerically the corresponding finite dimensional linear problem.

- **Option (B):** Discretizing equation (1.1) then applying Newton method to the discrete nonlinear problem and then solving numerically the corresponding finite dimensional linear problem.

The authors of a recent work [1] investigate the use of options (A) and option (B) where they prove that these options are not in general equivalent, and option (A) is more efficient than option (B). Moreover, they show that under certain assumptions on the method of discretization, the iterated solution converges towards the exact solution, contrary to option (B), in which the iterated solution always converges towards the approximated vector (See also [2, 3, 4]).

In this paper, we apply option (A) where we prove that the iterated solution converges to the exact solution under new conditions on the discretization process. These new sufficient conditions are more applicable than those used in [3]. We show subsequently that these conditions generate variant schemes, precisely the Sloan scheme also established in [3]. However, we present a new scheme, called the interpolation Sloan discretization scheme, where we treat an integro-differential equation as a first numerical application.

The paper is organized as follows. In section 2, we present the main theoretical convergence results. In section 3, we illustrate these results with a numerical application showing the accuracy and efficiency of our algorithms.

2. Theoretical Convergence Study

Let $\mathcal{X}$ be a complex Banach space, where its norm is denoted by $\|\cdot\|$. The space $\mathrm{BL}(\mathcal{X})$ defines the Banach algebra of bounded linear operators from $\mathcal{X}$ to itself, where its norm is given by:

$$\forall A \in \mathrm{BL}(\mathcal{X}), \qquad \|A\| = \sup\Big\{\|Ax\| : \ \|x\| \leq 1\Big\}.$$

The main problem is set as

$$\text{Find } \varphi \in \mathcal{O} \subset \mathcal{X} : \quad F(\varphi) = 0, \tag{2.1}$$

where $\mathcal{O}$ is a nonempty open set, F is a nonlinear operator defined from $\mathcal{O}$ into $\mathcal{X}$, φ is the solution and 0 is the null vector of $\mathcal{X}$. Applying the option (A) on (2.1) with a general discretization scheme leads to the sequence $(\varphi_n^{(k)})_{k\geq 0}$ defined through the relation:

$$\varphi_n^{(k+1)} = \varphi_n^{(k)} - \Sigma_n(\varphi_n^{(k)}) F(\varphi_n^{(k)}), \quad k \geq 0, \quad \varphi^{(0)} \in \mathcal{O}, \tag{2.2}$$

where, for all $x \in \mathcal{O}$ and for $n \geq 1$, the operator $\Sigma_n(\cdot) : \mathcal{O} \to \mathrm{BL}(\mathcal{X})$ is a numerical approximation to $F'(x)^{-1}$.

The next priori convergence theorem constitutes the main result of this work, where we establish new sufficient conditions for the approximate operator $\Sigma_n(\cdot)$, to guarantee that the constructed sequence $(\varphi_n^{(k)})_{k\geq 0}$ given by Eq (2.2) converges to φ, the zeros of Eq (2.1), such that n is fixed in $\mathbb{N}^*$ and k tends to infinity.

Theorem 2.1. *Assume that there exist $\varphi \in \mathcal{O}$, $\mu > 0$, $R > 0$, $l > 0$, $\beta > 0$ and $\alpha \in]0, 1[$ such that:*

1. $F(\varphi) = 0$, $F'(\varphi)$ *is invertible and* $\|F'(\varphi)^{-1}\| \leq \mu$.
2. *The closed ball $B_R(\varphi)$ is included in $\mathcal{O}$, such that $F'(\cdot) : B_R(\varphi) \to \mathrm{BL}(\mathcal{X})$ is (l, α) Hölder continuous.*
3. *There exists $\gamma_n \in]0, \frac{1}{3}[$ such that*

$$\sup_{x,y\in B_r(\varphi)} \Big\| \Big(I - \Sigma_n(x)F'(x)\Big) \Big(I - \Sigma_n(y)F'(y)\Big) \Big\| \leq \gamma_n, \quad r = \min\Big\{R, \Big(\frac{1}{2\mu l}\Big)^{\frac{1}{\alpha}}\Big\}.$$

4. *Let*

$$\sup_{x\in B_R(\varphi)} \Big\|I - \Sigma_n(x)F'(x)\Big\| \leq \beta. \tag{2.3}$$

5. *The starting function $\varphi_n^{(0)}$ and the first approximation $\varphi_n^{(1)}$ are included in $B_{\rho_n}(\varphi)$ such that*

$$\rho_n = \min\Big\{r, \Big(\frac{1 - 3\gamma_n}{8l\mu(1+\beta)^2}\Big)^{\frac{1}{\alpha}}\Big\}, \quad \varphi_n^{(1)} = \varphi^{(0)} - \Sigma_n(\varphi^{(0)})F(\varphi^{(0)}).$$

Then, for all $k \geq 2$ we find that $\varphi_n^{(k)} \in B_{\rho_n}(\varphi)$, and

$$\|\varphi_n^{(k)} - \varphi\| \leq \rho_n\big(1 - \gamma_n\big)^k \to 0 \ \ as \ \ k \to +\infty.$$

Proof . We show first that, for all $x \in B_r(\varphi)$, the bounded operator $F'(x)$ is invertible and

$$\|F'(x)^{-1}\| \leq 2\mu.$$

Indeed, we can see that

$$F'(x) = F'(\varphi)\Big(I - F'(\varphi)^{-1}\big(F'(\varphi) - F'(x)\big)\Big).$$

Since

$$\Big\|F'(\varphi)^{-1}\Big(F'(\varphi) - F'(x)\Big)\Big\| \ \leq \ \mu\,\|F'(\varphi) - F'(x)\| \leq \mu\, l\, \|\varphi - x\|^\alpha \leq \mu l r^\alpha \leq \frac{1}{2}.$$

Then using Neumann series, we can find that

$$F'(x)^{-1} = \Big(I - F'(\varphi)^{-1}\big(F'(\varphi) - F'(x)\big)\Big)^{-1} F'(\varphi)^{-1},$$

and

$$\|F'(x)^{-1}\| \leq \mu \sum_{k=0}^{+\infty} \Big\|F'(\varphi)^{-1}\Big(F'(\varphi) - F'(x)\Big)\Big\|^k \leq 2\mu. \tag{2.4}$$

Now, regarding $\Sigma_n(x)$, we remark that

$$\forall x \in B_r(\varphi), \quad \Sigma_n(x) = \Big(I - G_n(x)\Big)F'(x)^{-1}, G_n(x) = I - \Sigma_n(x)F'(x).$$

Hence, according to (2.3) we conclude that

$$\big\|\Sigma_n(x)\big\| \leq 2\mu\big(1+\beta\big).$$

Next, from (2.2) we find that

$$\forall k \geq 0 \quad \varphi_n^{(k+1)} - \varphi = \varphi_n^{(k)} - \varphi - \Sigma_n(\varphi_n^{(k)})\Big(F(\varphi_n^{(k)}) - F(\varphi)\Big). \tag{2.5}$$

So, using the residual integral expression, we get:

$$F(\varphi_n^{(k)}) - F(\varphi) = \int_0^1 F'\big((1-t)\varphi_n^{(k)} + t\varphi\big)\,(\varphi_n^k - \varphi)dt,$$

then remplace this formula in (2.5), we find that

$$\varphi_n^{(k+1)} - \varphi = \int_0^1 \Big(I - \Sigma_n(\varphi_n^{(k)})F'\big((1-t)\varphi_n^{(k)} + t\varphi\big)\Big)\,(\varphi_n^{(k)} - \varphi)dt.$$

Thereafter, we add and subtract $F'(\varphi_n^{(k)})$ from $F'\big((1-t)\varphi_n^{(k)} + t\varphi\big)$, we obtain

$$\varphi_n^{(k+1)} - \varphi = I - \Sigma_n(\varphi_n^{(k)})F'(\varphi_n^{(k)})(\varphi_n^{(k)} - \varphi) + \Sigma_n(\varphi_n^{(k)})\int_0^1 \Big(F'\big((1-t)\varphi_n^{(k)} + t\varphi\big) - F'(\varphi_n^{(k)})\Big)\,(\varphi_n^{(k)} - \varphi)dt. \tag{2.6}$$

Now let's put

$$\begin{cases} G_n(\varphi_n^{(k)}) = I - \Sigma_n(\varphi_n^{(k)})F'(\varphi_n^{(k)}), \\ H_n(\varphi_n^{(k)}, \varphi) = \Sigma_n(\varphi_n^{(k)})\displaystyle\int_0^1 \Big(F'\big((1-t)\varphi_n^{(k)} + t\varphi\big) - F'(\varphi_n^{(k)})\Big)dt. \end{cases}$$

Then from (2.6), we can show for all $k \geq 1$

$$\begin{aligned} \varphi_n^{(k+1)} - \varphi &= G_n(\varphi_n^{(k)})\,(\varphi_n^{(k)} - \varphi) + H_n(\varphi_n^{(k)}, \varphi)\,(\varphi_n^{(k)} - \varphi), \\ \varphi_n^{(k)} - \varphi &= G_n(\varphi_n^{(k-1)})\,(\varphi_n^{(k-1)} - \varphi) + H_n(\varphi_n^{(k-1)}, \varphi)\,(\varphi_n^{(k-1)} - \varphi). \end{aligned}$$

So, we conclude that

$$\begin{aligned} \varphi_n^{(k+1)} - \varphi &= G_n(\varphi_n^{(k)})\Big(G_n(\varphi_n^{(k-1)})\,(\varphi_n^{(k-1)} - \varphi) + H_n(\varphi_n^{(k-1)}, \varphi)\,(\varphi_n^{(k-1)} - \varphi)\Big) \\ &\quad + H_n(\varphi_n^{(k)}, \varphi)\,(\varphi_n^{(k)} - \varphi). \end{aligned}$$

Supposing now that for $k \geq 1$, $\varphi_n^{(k-1)}, \varphi_n^{(k)} \in B_{\rho_n}(\varphi)$. Since, the ball $B_r(\varphi)$ is convex, then for $t \in [0,1]$, we can see that $(1-t)\varphi_n^{(k)} + t\varphi \in B_r(\varphi)$. Using the estimate,

$$\Big\|F'\big((1-t)\varphi_n^{(k)} + t\varphi\big) - F'(\varphi_n^{(k)})\Big\| \leq l\|\varphi_n^{(k)} - \varphi\|^{\alpha},$$

thus, we have three relations,

$$\Big\|G_n(\varphi_n^{(k)})G_n(\varphi_n^{(k-1)})\,(\varphi_n^{(k-1)} - \varphi)\Big\| \leq \gamma_n\|\varphi_n^{(k-1)} - \varphi\|,$$

$$\begin{aligned} \Big\|G_n(\varphi_n^{(k)})H_n(\varphi_n^{(k-1)}, \varphi)\,(\varphi_n^{(k-1)} - \varphi)\Big\| &\leq \Big(\beta\, 2\mu(1+\beta)\, l\|\varphi_n^{(k-1)} - \varphi\|^{\alpha}\Big)\|\varphi_n^{(k-1)} - \varphi\| \\ &\leq \Big(2\mu l(1+\beta)^2\|\varphi_n^{(k-1)} - \varphi\|^{\alpha}\Big)\|\varphi_n^{(k-1)} - \varphi\|, \end{aligned}$$

and

$$\begin{aligned}\left\|H_n(\varphi_n^{(k)},\varphi)\,(\varphi_n^{(k)}-\varphi)\right\| &\le \Big(2\mu(1+\beta)\,l\|\varphi_n^{(k)}-\varphi\|^{\alpha}\Big)\|\varphi_n^{(k)}-\varphi\| \\ &\le \Big(2\mu l(1+\beta)^2\|\varphi_n^{(k)}-\varphi\|^{\alpha}\Big)\|\varphi_n^{(k)}-\varphi\|.\end{aligned}$$

Then, we substitute these three relations in (2.7), we can find that

$$\begin{aligned}\|\varphi_n^{(k+1)}-\varphi\| &\le \Big(\gamma_n+2\mu l(1+\beta)^2\big(\|\varphi_n^{(k-1)}-\varphi\|^{\alpha}+\|\varphi_n^{(k)}-\varphi\|^{\alpha}\big)\Big)\Big(\|\varphi_n^{(k)}-\varphi\|+\|\varphi_n^{(k-1)}-\varphi\|\Big) \\ &\le \Big(\frac{1-\gamma_n}{2}\Big)\Big(\|\varphi_n^{(k)}-\varphi\|+\|\varphi_n^{(k-1)}-\varphi\|\Big)\le(1-\gamma_n)\rho_n.\end{aligned}$$

Since $1-\gamma_n<1$, the previous inequality implies that $\varphi_n^{(k+1)}\in B_{\rho_n}(\varphi)$ and that

$$\|\varphi_n^{(k)}-\varphi\|\le\rho_n\Big(1-\gamma_n\Big)^k\to 0,\ \text{ as }\ k\to+\infty,$$

which completes the proof. $\square$

We note that the hypothesis established in our Theorem 2.1 are more applicable than those of [3]. The following proposition shows that under adequate assumption, the first approximation function $\varphi_n^{(1)}$ can be found in $B_{\rho_n}(\varphi)$, if $\varphi^{(0)}\in B_{\rho_n}(\varphi)$, where

$$\varphi_n^{(1)}=\varphi^{(0)}-\Sigma_n(\varphi^{(0)})F(\varphi^{(0)}),\quad \rho_n=\min\Big\{r,\Big(\frac{1-3\gamma_n}{4l\mu(1+\beta)^2}\Big)^{\frac{1}{\alpha}}\Big\}.$$

Proposition 2.1. *Under the same assumptions of Theorem 2.1, if $\varphi^{(0)}\in B_{\rho_n}(\varphi)$ and for all $h\in\mathcal{X}$,*

$$\Sigma_n(\varphi^{(0)})h\longrightarrow F'(\varphi^{(0)})^{-1}\,h\quad \text{as } n\to\infty,$$

then for n large enough, $\varphi_n^{(1)}\in B_{\rho_n}(\varphi)$.

Proof . We define first the function $g\in\mathcal{X}$ such that

$$g=\varphi^{(0)}-F'(\varphi^{(0)})^{-1}F(\varphi^{(0)}),$$

then, using the same technics as in proof of Theorem 2.1, we can find that

$$\begin{aligned}g-\varphi &= (\varphi^{(0)}-\varphi)-F'(\varphi^{(0)})^{-1}\big(F(\varphi^{(0)})-F(\varphi)\big) \\ &= (\varphi^{(0)}-\varphi)-F'(\varphi^{(0)})^{-1}\int_0^1 F'((1-t)\varphi^{(0)}-t\varphi)(\varphi^{(0)}-\varphi)dt \\ &= (\varphi^{(0)}-\varphi)-F'(\varphi^{(0)})^{-1}\int_0^1 F'(\varphi^{(0)})(\varphi^{(0)}-\varphi)dt \\ &\quad -F'(\varphi^{(0)})^{-1}\int_0^1\Big(F'((1-t)\varphi^{(0)}-t\varphi)-F'(\varphi^{(0)})\Big)(\varphi^{(0)}-\varphi)dt \\ &= -F'(\varphi^{(0)})^{-1}\int_0^1\Big(F'((1-t)\varphi^{(0)}-t\varphi)-F'(\varphi^{(0)})\Big)(\varphi^{(0)}-\varphi)dt.\end{aligned}$$

So,

$$\|g-\varphi\|=\Big(2\mu l(1+\beta)\|\varphi^{(0)}-\varphi\|^{\alpha}\Big)\|\varphi^{(0)}-\varphi\|\le(1-\gamma_n)\rho_n\le\rho_n,$$

which implies that $g \in B_{\rho_n}(\varphi)$. On the other side, we can see that

$$g - \varphi_n^{(1)} = \Big(\Sigma_n(\varphi^{(0)}) - F'(\varphi^{(0)})^{-1}\Big) F(\varphi^{(0)}).$$

As, for all $h \in \mathcal{X}$, $\big(\Sigma_n(\varphi^{(0)}) - F'(\varphi^{(0)})^{-1}\big)h \longrightarrow 0$, thus for n large enough, $\varphi_n^{(1)} \in B_{\rho_n}(\varphi)$. $\square$ In the next section, we will prove that by applying the option (A) with the use of the Sloan discretization scheme, all the assumptions of Theorem 2.1 and of Proposition 2.1 are satisfied.

3. Sloan Discretization Scheme

Following [1, 3], we consider a Fréchet-differentiable nonlinear compact operator $K : \mathcal{O} \longrightarrow \mathcal{X}$, where its derivative is denoted by $T = K'$. The kind of the nonlinear equation to be treated is,

$$\text{Find } \varphi \in \mathcal{O} : \quad \varphi - K(\varphi) = f, \tag{3.1}$$

where f is given in $\mathcal{X}$. Let π_n be a projection given in a subspace of $\mathcal{X}$ such that

$$\forall h \in \mathcal{X}, \quad \pi_n h \longrightarrow h.$$

Using the Sloan projection discretization to the iterative equations issued from the linearization process, we get:

$$\text{Find } \varphi_n^{(k+1)} \in \mathcal{X} : \quad \varphi_n^{(k+1)} - T(\varphi_n^{(k)})\, \pi_n \varphi_n^{(k+1)} = g_n^{(k)}, \text{ where } g_n^{(k)} = K(\varphi_n^{(k)}) - T(\varphi_n^{(k)})\pi_n \varphi_n^k + f.$$

Now, we assume that equation (3.1) has a solution $\varphi \in \mathcal{O}$, $I - T(\varphi)$ is invertible and the operator $T(\cdot) : \mathcal{O} \to \mathrm{BL}(\mathcal{X})$ is $(l, \alpha)-$Hölder. In the sequel, we show that the hypotheses of Theorem 2.1 and of Proposition 2.1 are satisfied.

Let $\mu > 0$ and $R > 0$ be such that $B_R(\varphi) \subseteq \mathcal{O}$ and $\|\big(I - T(\varphi)\big)^{-1}\| \leq \mu$. We fix $r = \min\left\{R, \left(\frac{1}{2\mu l}\right)^{\frac{1}{\alpha}}\right\}$, we can show that, as in the proof of Theorem 2.1, for all $x \in B_r(\varphi)$, the operator $I - T(x)$ is invertible and $\|\big(I - T(x)\big)^{-1}\| \leq 2\mu$. We note that the discretization process is based upon the approximation

$$\forall x \in B_R(\varphi), \quad T_n(x) = T(x)\pi_n, \quad .$$

We state the first two lemmas which will be needed in the proof of our main results.

Lemma 3.1. *For all $x \in B_R(\varphi)$, if $M(x)$ is a compact operator in* $\mathrm{BL}(\mathcal{X})$*, then*

$$\lim_{n\to\infty} \sup_{x,y \in B_R(\varphi)} \Big\| \big(T_n(x) - T(x)\big)\, M(y) \Big\| = 0.$$

Proof . Since, for all $h \in \mathcal{X}$, $T_n(x)h = T(x)\pi_n h \to T(x)h$ and the set

$$W = \big\{\, M(x)h : x \in B_R(\varphi),\ h \in \mathcal{X},\ \|h\| = 1 \,\big\},$$

is relatively compact, then by the Banach-Steinhaus Theorem (See Corollary 9.2 of [5]), we find that

$$\lim_{n\to\infty} \sup_{x,y \in B_R(\varphi)} \Big\| \big(T_n(x) - T(x)\big)\, M(y) \Big\| = 0.$$

$\square$

Lemma 3.2. *For all $x \in B_R(\varphi)$ and for n large enough, there exists $C > 0$ such that*

$$\sup_{x \in B_R(\varphi)} \left\| \big(I - T(x)\big)^{-1} \big(T_n(x) - T(x)\big) \right\| \leq C.$$

Proof . For all $x \in B_R(\varphi)$, we remark that

$$\left\| \big(I - T(x)\big)^{-1} \big(T_n(x) - T(x)\big) \right\| \leq 2\mu \left\| \big(T_n(x) - T(x)\big) \right\| \leq 2\mu \left\| T(x) \big(\pi_n - I\big) \right\|.$$

As, for all $h \in \mathcal{X}$, $\pi_n h \longrightarrow h$ then the Uniform Boundedness Principle (See Theorem 9.1 of [5]) shows that there exists $C > 0$ such that

$$\sup_{x \in B_R(\varphi)} \left\| \big(I - T(x)\big)^{-1} \big(T_n(x) - T(x)\big) \right\| \leq C.$$

□

Proposition 3.1. *For all $x \in B_R(\varphi)$ and for n large enough, the approximate inverse $\Sigma_n(x) = \big(I - T_n(x)\big)^{-1}$, exists and is uniformly bounded on $B_R(\varphi)$.*

Proof . For all $x \in B_R(\varphi)$, we remark that

$$I - T_n(x) = \big(I - T(x)\big) \Big(I - \big(I - T(x)\big)^{-1} \big(T_n(x) - T(x)\big)\Big).$$

In addition,

$$\begin{aligned} \left\| \Big(\big(I - T(x)\big)^{-1} \big(T_n(x) - T(x)\big) \Big)^2 \right\| &\leq 2\mu \left\| \big(T_n(x) - T(x)\big) \big(I - T(x)\big)^{-1} T(x)(\pi_n - I) \right\| \\ &\leq 2\mu \left\| \big(T_n(x) - T(x)\big) \big(I - T(x)\big)^{-1} T(x) \right\|. \end{aligned}$$

Putting $M(x) = (I - T(x))^{-1} T(x)$, for all $x \in B_R(\varphi)$, then $M(x)$ is a compact operator. So, according to Lemma 3.1, we can find that for n large enough,

$$\left\| \Big(\big(I - T(x)\big)^{-1} \big(T_n(x) - T(x)\big) \Big)^2 \right\| \leq \frac{1}{2}.$$

Hence, using Neumann series, we get

$$\begin{aligned} \Sigma_n(x) &= \sum_{k=0}^{\infty} \Big(\big(I - T(x)\big)^{-1} \big(T_n(x) - T(x)\big) \Big)^k \big(I - T(x)\big)^{-1} \\ &= \Big(\sum_{k=0}^{\infty} \Big(\big(I - T(x)\big)^{-1} \big(T_n(x) - T(x)\big) \Big)^{2k} \\ &\quad + \sum_{k=0}^{\infty} \Big(\big(I - T(x)\big)^{-1} \big(T_n(x) - T(x)\big) \Big)^{2k+1} \Big) \big(I - T(x)\big)^{-1} \\ &= \Big(I + \big(I - T(x)\big)^{-1} \big(T_n(x) - T(x)\big) \Big) \sum_{k=0}^{\infty} \Big(\big(I - T(x)\big)^{-1} \big(T_n(x) - T(x)\big) \Big)^{2k} \big(I - T(x)\big)^{-1}. \end{aligned}$$

Therefore, according to Lemma 3.2, we obtain

$$\|\Sigma_n(x)\| \leq 2\mu(1 + C) \sum_{k=0}^{\infty} (\|((I - T(x))^{-1}(T_n(x) - T(x)))^2\|)^k \leq 4\mu(1 + C).$$

□

Proposition 3.2. *For all $x \in B_r(\varphi)$, we have*

1. *There exists $\beta > 0$ such that:*

$$\sup_{x\in B_r(\varphi)} \Big\| I + \Sigma_n(x)\big(I - T(x)\big)\Big\| \leq \beta.$$

2. *There exists $\gamma_n \in \Big]0, \frac{1}{3}\Big[$ such that:*

$$\sup_{x,y\in B_r(\varphi)} \Big\| \Big(I + \Sigma_n(x)\big(I - T(x)\big)\Big)\Big(I + \Sigma_n(y)\big(I - T(y)\big)\Big)\Big\| \leq \gamma_n.$$

Proof . Firstly, for all $x \in B_r(\varphi)$, we remark that

$$I - \Sigma_n(x)\big(I - T(x)\big) = \Sigma_n(x)\big(I - T_n(x) - I + T(x)\big) = \Sigma_n(x)\big(T(x) - T_n(x)\big).$$

Hence, there exists $\beta > 0$ such that

$$\sup_{x\in B_r(\varphi)} \Big\| I + \Sigma_n(x)\big(I - T(x)\big)\Big\| \leq \beta.$$

Secondly, for all $x, y \in B_r(\varphi)$, we put $G_n(x) = I + \Sigma_n(x)\big(I - T(x)\big)$, then

$$\begin{aligned} G_n(x)G_n(y) &= \Big(\Sigma_n(x)\big(T(x) - T_n(x)\big)\Big)\Big(\Sigma_n(y)\big(T(y) - T_n(y)\big)\Big) \\ &= \Sigma_n(x)\big(T(x) - T_n(x)\big)\,\Sigma_n(y)\,T(y)\,(I - \pi_n). \end{aligned}$$

Hence, there exists $C_1 > 0$ such that

$$\|G_n(x)G_n(y)\| \leq C_1 \|\big(T(x) - T_n(x)\big)\,\Sigma_n(y)\,T(y)\|.$$

Since,

$$\Sigma_n(y) - \big(I - T(y)\big)^{-1} = \Sigma_n(y)\big(T(y) - T_n(y)\big)\big(I - T(y)\big)^{-1}.$$

So, we conclude that for all $h \in \mathcal{X}$

$$\big(T(x) - T_n(x)\big)\,\Sigma_n(y)h \to 0.$$

Using Lemma 3.1, we can show that for n large enough, there exists $\gamma_n \in \Big]0, \frac{1}{3}\Big[$ such that

$$\sup_{x,y\in B_r(\varphi)} \|G_n(x)G_n(y)\| \leq \gamma_n.$$

This completes the proof. $\square$

3.1. The Numerical Implementation

In this section, we show that the numerical computation of $(\varphi_n^{(k)})_{k\geq 1}$ which is generated by the Sloan discretization scheme (for Sloan method see [6] and [7]), requires to solve an $n \times n$ linear system. In other words, by solving k linear systems of order n, we get an approximation $(\varphi_n^{(k)})_{k\geq 1}$ of any desired accuracy.

Defining a projection $\pi_n : \mathcal{X} \to \mathcal{X}$ spanned by an order basis

$$e_n = [e_{n,1}, \cdots, e_{n,n}] \in \mathcal{X}^n,$$

where $e_n^* = [e_{n,1}^*, \cdots, e_{n,n}^*] \in (\mathcal{X}^*)^n$ is an adjoint order basis to e_n. We denote by $\langle \, , \, \rangle$ the duality brackets between $\mathcal{X}$ and $\mathcal{X}^*$. We note that for all $x \in \mathcal{X}$

$$\pi_n x = \sum_{j=1}^{n} \langle x, e_{n,j}^* \rangle e_{n,j},$$

Remark that, for all $1 \leq j \leq n$

$$\pi_n e_{n,j} = e_{n,j}.$$

To simplify the description of matrices and linear combination we use the following notation: for all $\mathsf{x} \in \mathbb{C}^n$

$$e_n \mathsf{x} = \sum_{j=1}^{n} \mathsf{x}(j) e_{n,j}.$$

We recall that the Sloan discretization scheme is given by

$$\text{Find } \varphi_n^{(k+1)} \in \mathcal{O}: \quad \Big(I - T(\varphi_n^{(k)})\pi_n\Big)\varphi_n^{(k+1)} = g_n^{(k)}, \text{ where } g_n^{(k)} = -\pi_n T(\varphi_n^{(k)})\varphi_n^k + K(\varphi_n^{(k)}) + f.$$

So, if we put

$$\pi_n \varphi_n^{(k+1)} = \sum_{j=1}^{n} \mathsf{x}_n^{(k+1)}(j) e_{n,j},$$

then the function $\varphi_n^{(k+1)}$ is given by:

$$\varphi_n^{(k+1)} = T(\varphi_n^{(k)}) e_n \mathsf{x}_n^{(k+1)} + K(\varphi_n^{(k)}) - \pi_n T(\varphi_n^{(k)})\varphi_n^k + f,$$

where the unknown $\mathsf{x}_n^{(k+1)}$ is a column vector in $\mathbb{C}^n$ satisfying the linear system,

$$(I_n - M_n^{(k)})\mathsf{x}_n^{(k+1)} = b_n^{(k)},$$

where for $1 \leq i, j \leq n$:

$$M_n^{(k)}(i,j) = \langle T(\varphi_n^{(k)}) e_{n,j}, e_{n,i}^* \rangle, \quad b_n^{(k)}(j) = \langle K(\varphi_n^{(k)}), e_{n,j}^* \rangle - \langle T(\varphi_n^{(k)})\varphi_n^{(k)}, e_{n,j}^* \rangle + \langle f, e_{n,j}^* \rangle.$$

We note that the operators $T(x)$ and $K(x)$ are functional operators, hence the exact computations of $K(\varphi_n^{(k)})$ and $T(\varphi_n^{(k)})$ are almost always difficult. So, to go over this difficulty, in the next subsection, we propose a new interpolation scheme of the option (A), where the numerical implementation is much easier.

3.2. Interpolation Sloan discretization scheme

Using the same notations as in the previous sections, we define the new interpolation scheme of the option (A) as

$$\varphi_{n,m}^{(k+1)} = \varphi_{n,m}^{(k)} - \Sigma_n\big(\pi_m \varphi_{n,m}^{(k)}\big) F\big(\pi_m \varphi_{n,m}^{(k)}\big), \quad k \geq 0 \quad \varphi_{n,m}^{(0)} \in \mathcal{O},$$

where m, n are fixed in $\mathbb{N}$.

Theorem 3.3. *Under the same assumptions of Theorem 2.1, then for m large enough, we have*

$$\|\varphi_{n,m}^{(k)} - \varphi\| \leq \rho_n \Big(1 - \frac{2}{3}\gamma_n\Big)^k \to 0 \ \text{ as } \ k \to +\infty.$$

Proof . Remark that

$$\begin{aligned}\varphi_{n,m}^{(k+1)} &= \varphi_{n,m}^{(k)} - \Sigma_n\big(\varphi_{n,m}^{(k)}\big)F\big(\varphi_{n,m}^{(k)}\big) + \Sigma_n\big(\varphi_{n,m}^{(k)}\big)\Big(F\big(\varphi_{n,m}^{(k)}\big) - F\big(\pi_m\varphi_{n,m}^{(k)}\big)\Big)\\ &\quad+\Big(\Sigma_n\big(\varphi_{n,m}^{(k)}\big) - \Sigma_n\big(\pi_m\varphi_{n,m}^{(k)}\big)\Big)F\big(\pi_m\varphi_{n,m}^{(k)}\big),\end{aligned}$$

which implies that

$$\begin{aligned}\varphi_{n,m}^{(k+1)} - \varphi &= \varphi_{n,m}^{(k)} - \varphi - \Sigma_n(\varphi_{n,m}^{(k)})\Big(F(\varphi_{n,m}^{(k)}) - F(\varphi)\Big) + \Sigma_n(\varphi_{n,m}^{(k)})\Big(F(\varphi_{n,m}^{(k)}) - F(\pi_m\varphi_{n,m}^{(k)})\Big)\\ &\quad+\Big(\Sigma_n(\varphi_{n,m}^{(k)}) - \Sigma_n(\pi_m\varphi_{n,m}^{(k)})\Big)F(\pi_m\varphi_{n,m}^{(k)}),\end{aligned}$$

So, we use the same techniques as in the proof of Theorem 2.1, we find that

$$\begin{aligned}\|\varphi_{n,m}^{(k+1)} - \varphi\| &\le \frac{1}{2}(1-\gamma_n)\Big(\|\varphi_{n,m}^{(k)} - \varphi\| + \|\varphi_{n,m}^{(k-1)} - \varphi\|\Big) + \|\Sigma_n(\varphi_{n,m}^{(k)})\|\,\|F(\varphi_{n,m}^{(k)}) - F(\pi_m\varphi_{n,m}^{(k)})\|\\ &\quad+\|\Sigma_n(\varphi_{n,m}^{(k)}) - \Sigma_n(\pi_m\varphi_{n,m}^{(k)})\|\|F(\pi_m\varphi_{n,m}^{(k)})\|.\end{aligned}$$

Using the regularity and the boundedness of $\Sigma_n(x)$ and $F(x)$, then there exists $C > 0$ such that

$$\|\varphi_{n,m}^{(k+1)} - \varphi\| \le \frac{1}{2}(1-\gamma_n)\Big(\|\varphi_{n,m}^{(k)} - \varphi\| + \|\varphi_{n,m}^{(k-1)} - \varphi\|\Big) + C\|(\pi_m - I)\varphi_{n,m}^{(k)}\|.$$

As, $\pi_m h \to h$ as $m \to \infty$, for all $h \in \mathcal{X}$. Then we can obtain that for m large enough:

$$\|\varphi_{n,m}^{(k+1)} - \varphi\| \le \frac{1}{2}(1-\gamma_n)\Big(\|\varphi_{n,m}^{(k)} - \varphi\| + \|\varphi_{n,m}^{(k-1)} - \varphi\|\Big) + \frac{1}{3}\gamma_n\|\varphi_{n,m}^{(k)} - \varphi\|.$$

Hence, assuming that $\varphi_{n,m}^{(k)}$, $\varphi_{n,m}^{(k-1)} \in B_{\rho_n}(\varphi)$, wa can find that

$$\|\varphi_{n,m}^{(k+1)} - \varphi\| \le \rho_n(1 - \frac{2}{3}\gamma_n) \le \rho_n,$$

that implies that $\varphi_{n,m}^{(k+1)} \in B_{\rho_n}(\varphi)$ and

$$\|\varphi_{n,m}^{(k)} - \varphi\| \le \rho_n(1 - \frac{2}{3}\gamma_n)^k \to 0, \text{ as } k \to \infty.$$

The proof is complete. □

Defining the Sloan discretization scheme as before on the interpolation one, we get

$$\text{Find } \varphi_{n,m}^{(k)} \in \mathcal{X} : \quad \Big(I - T\big(\pi_m\varphi_{n,m}^{(k)}\big)\pi_n\Big)\varphi_{n,m}^{(k+1)} = g_{n,m}^{(k)},$$

where

$$g_{n,m}^{(k)} = -\pi_n T\big(\pi_m\varphi_{n,m}^{(k)}\big)\varphi_{n,m}^{(k)} + K\big(\pi_m\varphi_{n,m}^{(k)}\big) + f.$$

So, we put

$$\pi_n\varphi_{n,m}^{(k+1)} = \sum_{j=1}^{n} \mathsf{x}_{n,m}^{(k+1)}(j)e_{n,j},$$

then the function $\varphi_{n,m}^{(k+1)}$ is given by:

$$\varphi_{n,m}^{(k+1)} = T\big(\pi_m\varphi_{n,m}^{(k)}\big)e_n\mathsf{x}_{n,m}^{(k+1)} - \pi_n T\big(\pi_m\varphi_{n,m}^{(k)}\big)\varphi_{n,m}^{(k)} + K\big(\pi_m\varphi_{n,m}^{(k)}\big) + f,$$

where the unknown $\mathsf{x}_{n,m}^{(k+1)}$ is a column vector in $\mathbb{C}^n$ solving the linear system:

$$\big(I_n - M_{n,m}^{(k)}\big)\mathsf{x}_{n,m}^{(k+1)} = b_{n,m}^{(k)}, \tag{3.2}$$

where for $1 \le i, j \le n$:

$$M_{n,m}^{(k)}(i,j) = \big\langle T(\pi_m\varphi_{n,m}^{(k)})e_{n,j}, e_{n,i}^*\big\rangle, \quad b_{n,m}^{(k)}(j) = \big\langle K(\pi_m\varphi_{n,m}^{(k)}), e_{n,j}^*\big\rangle - \big\langle T(\pi_m\varphi_{n,m}^{(k)})\varphi_{n,m}^{(k)}, e_{n,j}^*\big\rangle + \big\langle f, e_{n,j}^*\big\rangle.$$

3.3. Numerical example

In this subsection, we establish an academic example showing the comportment of the Sloan discretization scheme with an interpolatory projection. Consider a Banach space $\mathcal{X} = \mathcal{C}^1([0,1],\mathbb{R})$ of all continuous differentiable functions defined on the interval $[0,1]$, which is equipped with the uniform norm

$$\|u\| = \max_{s\in[0,1]} |u(s)| + \max_{s\in[0,1]} |u'(s)|, \text{ for } u \in \mathcal{X}. \tag{3.3}$$

We consider the nonlinear integro-differential equation

$$\varphi(s) - \int_0^1 k(s,t,\varphi(t),\varphi'(t))dt = f(s), \quad s \in [0,1],$$

where

$$\left\| \begin{array}{l} k,\ \dfrac{\partial k}{\partial s} \text{ are both in } \mathcal{C}([0,1]\times[0,1]\times\mathbb{R}\times\mathbb{R}), \\ \\ f \text{ is given a function in } \mathcal{X}. \end{array} \right.$$

The projection π_n is built upon a uniform grid in $[0,1]$ given by:

$$h_n = \frac{1}{n-1},\ t_j = (j-1)h_n,\ 1 \le j \le n, \quad n \ge 2.$$

The canonical basis $e_n = [e_{n,1},\cdots,e_{n,n}] \in \mathcal{C}([0,1])$ is generated by the hat functions,

$$\begin{aligned}
e_{j,n}(t) &= \begin{cases} 1 - \dfrac{|t-t_j|}{h_n} & \text{for } t_{j-1} \le t \le t_{j+1}, \\ 0 & \text{otherwise.} \end{cases} \\
e_{1,n}(t) &= \begin{cases} \dfrac{t_2 - t}{h_n} & \text{for } t_1 \le t \le t_2, \\ 0 & \text{otherwise.} \end{cases} \\
e_{n,n}(t) &= \begin{cases} \dfrac{t - t_{n-1}}{h_n} & \text{for } t_{n-1} \le t \le t_n, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

The kernel k is defined by

$$k(s,t,x,y) = \lambda(\exp(t)\,x^2 - \exp(-t)\,y^2), \quad t,s \in [0,1],\ x,y \in \mathbb{R},$$

where λ is given in $\mathbb{R}$. The main nonlinear problem is set as: Find $\varphi \in \mathcal{X}$ such that

$$\varphi(s) = \lambda \int_0^1 \big(\exp(s)\,(\varphi(t))^2 - \exp(-s)\,(\varphi'(t))^2\big)dt + f(s), \quad s \in [0,1],$$
$$\varphi'(s) = \lambda \int_0^1 \big(\exp(s)\,(\varphi(t))^2 + \exp(-s)\,(\varphi'(t))^2\big)dt + f'(s), \quad s \in [0,1].$$

Applying the option (A) on the previous equations with the Sloan discretization scheme, we show that by solving k linear system of order $2n$ we get an approximation solution in which converges to the exact solution.

The next tables (Table 1 and Table 2) show the numerical results using two different starting function $\varphi^{(0)}$, where the function f is given by

$$f(s) = \sqrt{1+s} + \frac{\lambda}{4}\big(6\,\exp(s) - \log(2)\,\exp(-s)\big), \quad s \in [0,1],$$

and the exact solution φ is:

$$\varphi(s) = \sqrt{1+s}. \quad s \in [0,1].$$

The error formula is defined by:

$$E^k_{n,m} = \max_{1\le j\le n}\Big\{|\varphi(t_j) - \mathsf{y}^{(k)}_{\{n,m,j\}}|, |\varphi'(t_j) - \mathsf{z}^{(k)}_{\{n,m,j\}}|\Big\},$$

and

$$\mathsf{x}^{(k)}_{\{n,m\}} = (\mathsf{y}^{(k)}_{\{n,m\}}, \mathsf{z}^{(k)}_{\{n,m\}}) \in \mathbb{C}^{2n},$$

is calculated through the system (3.2).

Table 1: Numerical results with $\varphi^{(0)} = 0$ and $\lambda = 0.1$

n=10 m=10	$E^k_{n,m}$	n=100 m=100	$E^k_{n,m}$
k=2	4.141171392047060 e-01	k=2	4.141171392047060 e-01
k=6	1.731801805729800 e-02	k=6	1.722637261440900 e-02
k=10	3.348482261129049 e-04	k=10	2.880658280443038 e-04
k=14	4.876077139859270 e-05	k=14	6.617005265940890 e-06
k=18	5.589562884833477 e-05	k=18	4.648402012197295 e-07

Table 2: Numerical results with $\varphi^{(0)} = f$ and $\lambda = 0.1$

n=10 m=10	$E^k_{n,m}$	n=100 m=100	$E^k_{n,m}$
k=2	2.200328874691180 e-01	k=2	2.198300274707410 e-01
k=6	5.115776414219000 e-03	k=6	5.046035023891000 e-03
k=10	1.715094568838227 e-04	k=10	1.251126470973030 e-04
k=14	5.494202775446766 e-05	k=14	3.912001588446401 e-06
k=18	5.593171308415634 e-05	k=18	5.070256147154950 e-07

Conclusion

As a general conclusion, this paper shows new theoretical conditions that provide a new class of interpolation discretization scheme. The treatment of the nonlinear equation is starting first by linearizing, then discretizing, where the numerical results prove its efficiency and accuracy. However, there are a lot of open problems concerned with the resolution of nonlinear integral equations in particular the integral equation of the first type. therefore as a perspective, we will apply these methods of resolution on the nonlinear integral equations of the first types.

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