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A new [0, 1] truncated inverse Weibull rayleigh distribution properties with application to COVID-19

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Abstract

In this paper, we introduce a new 2- parameters family of distributions named [0,1] Truncated Inverse Weibull - G family ([0,1] TIW-G) family, to generate new types of continues distributions. A special model namely, [0,1] Truncated Inverse Weibull Rayleigh distribution ([0,1] TIWR) distribution is considered and defined and some of the statistical properties are derived. Parameter's estimations using MLE method is provided and a simulation is given to determine the accuracy of the method used above. To demonstrate the utility of the distribution in nowday's applications, we explore and investigate the death rates of COVID-19 in Iraq in the period from 14 December 2020 to 30 April 2021.

Keywords: Rayleigh distribution, [0,1] Truncated, Inverse Weibull family, Maximum likelihood Estimation.

1. Introduction

The interest in generating new families of distribution has increased rapidly during the last few years. Researchers always look for new distributions in order to fit real life data. In most cases, these distributions have more flexibility than their predecessors and provide more fitness for empirical data. There are many ways to generate new families of distributions, one such way is adding new parameters to an already exist model. This procedure provides a better modeling of data and extends the study of the original distributions as well. Many new families had been developed in the previous few decades for example beta-G by Eugene et al. (2002) [11], Marshal-Olkin generated family (MO-G) by Marshal and Olkin (2007) [16] ,Kumaraswamy-G by Cordeiro and de Castro (2011) [9], log-gamma-G

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by Amini (2012) [8], exponentiated generalized-G by Cordeiro (2013) [10], Transformed-Transformer (T-X) by Alzaatreh (2013) [3], Kumaraswamy odd log-logistic-G by Alizadeh (2015) [5] and The Marshall-Olkin and Topp Leone-G family by Khaleel et al (2020) [15].
Eugene et al [11] introduce beta-G by using the logit of beta distribution. A random variable X has a beta-G cumulative distribution function (cdf) defined by:

$$F\left(x\right)=\frac{1}{B\ \left(\alpha,\beta\right)}\int_0^{G(x)}t^{\alpha-1}(1-t)^{\beta-1}dt;\alpha,\beta>0, \tag{1.1}$$

Where $G(x)$ is a cdf of another random variable. Many pioneering works had been established on this works.
Alzaatreh et al [7] has posted a new family of distributions named T-X family, its cdf has the form:

$$F\left(x;\alpha,\beta\right)=\int_a^{W(G(x;\alpha))}f(t;\beta)dt \tag{1.2}$$

Where the random variable $T\in[a,b]$ for $-\infty<a<b<\infty$. And the function $W\left(x\right)$ satisfies the conditions:

- $W\left(G\left(x;\alpha\right)\right)\in[a,b]$,
- $W\left(G\left(x;\alpha\right)\right)$ is differentiable and monotonically non-decreasing,
- $W\left(G\left(x;\alpha\right)\right)\longrightarrow a$ as $x\longrightarrow-\infty$ and $W\left(G\left(x;\alpha\right)\right)\longrightarrow b$ as $x\longrightarrow\infty$.

This family add one more parameter to a given distribution and renders it more flexible and richer model.
Salah et al. [2] defined a new truncated family of distribution named [0,1] truncated-G family by defining the new cdf M as:

$$M\left(x\right)=\frac{F(G\left(x\right))-F(0)}{F\left(1\right)-F(0)} \tag{1.3}$$

Where both F and G are cdf of another random variables.

In this paper we defined our new family of distribution by sets F equal to the cdf of inverse Weibull distribution in (1.3) namely:
$F\left(x;a,b\right)=\exp\left(-bx^{-a}\right);x>0$ and $a,b>0$. we call this family $[0,1]$ $TIW-G$ family.

The paper is divided as follows: in section 2 we give a definition of the cdf and pdf of the new family and then we consider a special model the: $[0,1]$ inverse Weibull Rayleigh distribution ([0,1] TIWR Distribution) along with the survival and hazard functions followed by a series representation of both cdf and pdf functions. Section 3 is devoted to the statistical properties such as moments, quantile function, median, entropies and stress-strength reliability. Section 4 addresses the estimations of parameters using ML method. In section 5 we define the order statistics of the distribution. Section 6 demonstrate the estimation study of the parameters using R software and finally section 7 we study some applications of our model to a life data. We wrapp up with conclusions in section 8.

2. $[0,1]$ $\boldsymbol{TIW-G}$ Family And The [0,1] Truncated Inverse Weibull Rayleigh Distribution ([0,1] TIWR Distribution)

As we mentioned above the cumulative distribution function (cdf) of the $[0,1]$ TIW-G family is given by sets F equal to the cdf of inverse Weibull distribution in (1.3) to get:

$$F_{[0,1]\ TIW-G} = \exp\left(-bG\left(x\right)^{-a}\right)/\exp(-b) \quad ; \quad x>0 \ and \ \ a,b>0 \tag{2.1}$$

A differentiation with respect to x gives the pdf:

$$f\left(x;a,b\right) = abg\left(x\right)\left[G\left(x\right)\right]^{-(a+1)}\exp\left(-bG\left(x\right)^{-a}\right)/\exp(-b) \quad ; \tag{2.2}$$

The cumulative distribution function (cdf) of the [0,1] TIWR distribution is given by set $G(x)$ and $g(x)$ equal to the cdf and pdf of the Rayleigh distribution respectively to obtain:

$$F\left(x;a,b,\lambda\right) = \frac{\exp(-b[1-\exp\left(-\lambda \mathrm{x}^2\right)\,]^{-a})}{\exp(-b)} \ ,\ x>0 \ and \ a,b,\lambda>0 \tag{2.3}$$

A differentiation with respect to x gives the probability distribution function (pdf):

$$f\left(x;a,b,\lambda\right) = \frac{2ab\lambda}{\exp(-b)} x\exp\left(-\lambda x^2\right)\ \left[1-\exp\left(-\lambda x^2\right)\right]^{\ -(a+1)}\ \exp(-b\big[1-\exp\left(-\lambda x^2\right)\ \big]^{-a}) \tag{2.4}$$

The survival and hazard function are given by the following equations:

$$S\left(x;a,b,\lambda\right) = 1-F(x;a,b,\lambda) = 1-\frac{\exp(-b[1-\exp\left(-\lambda x^2\right)\,]^{-a})}{\exp(-b)} \tag{2.5}$$

And,

$$H\left(x;a,b,\lambda\right) = \frac{f\left(x;a,b,\lambda\right)}{S\left(x;a,b,\lambda\right)}$$

$$H\left(x;a,b,\lambda\right) = \frac{\frac{2ab\lambda}{\exp(-b)} x\exp\left(-\lambda x^2\right)\ \left[1-\exp\left(-\lambda x^2\right]\right.^{\ -(a+1)}\ \exp(-b[1-\exp\left(-\lambda x^2\right)\,]^{-a})}{\left(1-\frac{\exp(-b[1-\exp(-\lambda x^2)\,]^{-a})}{\exp(-b)}\right)} \tag{2.6}$$

respectively. Here a and b are shape parameters while λ is a scale parameter.

For illustration purposes, the plots for the pdf and hazard function of [0,1] TIWR distribution for different values of parameter are presented in Figures 1 and 2 respectively. Many different shape for pdf we can see like right skewed, left skewed, and exponential. For the hazard function we can see also different shapes such as increase, decrease, bathtub, reverse J, and increase-decrease-increase.

Proposition 2.1. *The pdf given in* (1.2) *has the following expansion:*

$$f\left(x;a,b,\lambda\right) = \frac{2\lambda}{\exp(-b)}\sum_{i_1,i_2=0}^{\infty}\tau_{i_1,i_2} x\ \exp(-\lambda i_2 x^2) \tag{2.7}$$

Where,

$$\tau_{i_1,i_2} = \frac{(-1)^{i_1+1} b^{i_1}\Gamma\left(ai_1+i_2\right)}{i_1!\left(i_2-1\right)!\Gamma\left(ai_1\right)}.$$

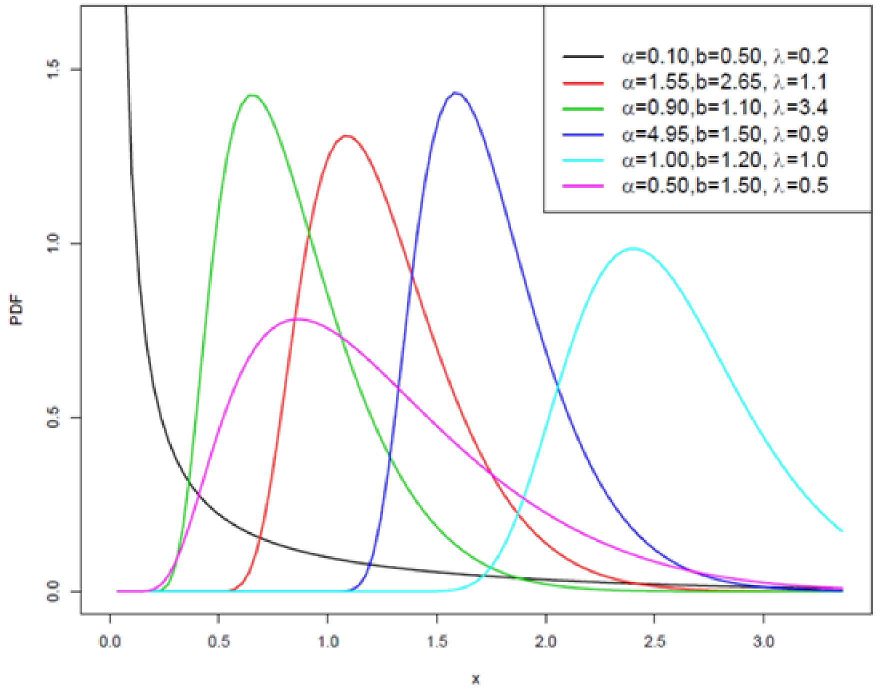

Figure 1: The pdf plot for [0,1] TIWR distribution with different parameters.

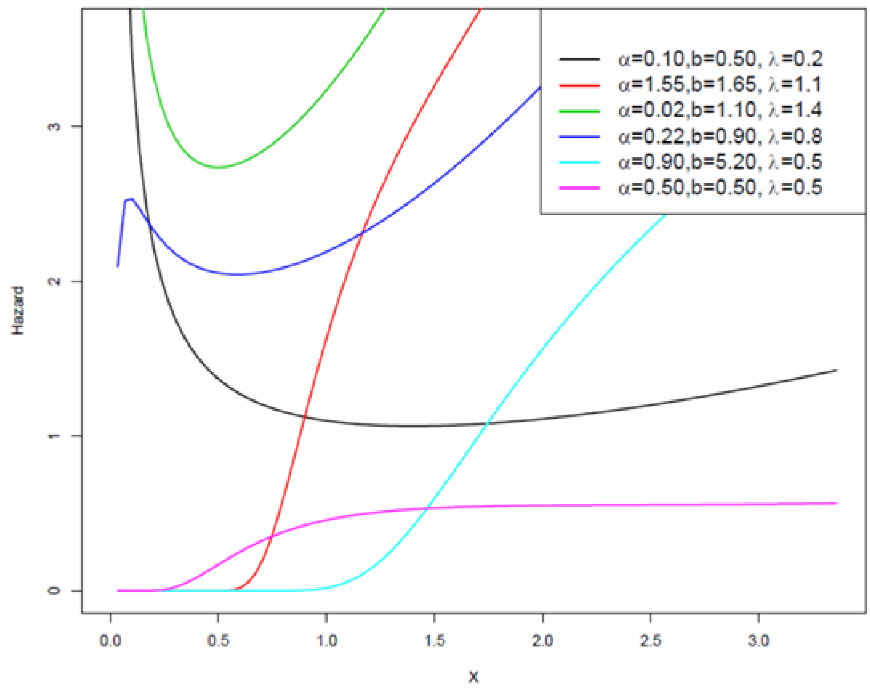

Figure 2: The hazard plot for [0,1] TIWR distribution with different parameters.

Proof . First, let us give an expansion for the cdf. Using the exponential Taylor series and the generalized binomial theorem we get:

$$F\left(x;a,b,\lambda\right)=\frac{\exp\left(-b[1-\exp\left(-\lambda x^2\right)]^{-a}\right)}{\exp\left(-b\right)}=\frac{1}{\exp(-b)}\sum_{i_1,i_2=0}^{\infty}\frac{(-1)^{i_1}b^{i_1}\Gamma(ai_1+i_2)}{i_1!i_2!\Gamma(ai_1)}\exp(-\lambda i_2x^2)$$

□

A differentiation with respect to x gives the desired result. It follows that the pdf of the $[0,1]$ IWR distribution can be expand as a linear combination of the pdf of the Weibull distribution. The next proposition has a useful application later.

Proposition 2.2. *For the pdf given in* (1.2) *the following holds:*

$$f(x;a,b,\lambda)^{\nu}=\left(\frac{2ab\lambda}{\exp\left(-b\right)}\right)^{\nu}\sum_{i_1,i_2=0}^{\infty}\frac{(-1)^{i_1}\ \nu^{i_1}b^{i_1}\Gamma(\nu\left(a+1\right)+ai_1+i_2)}{i_1!i_2!\Gamma(\nu\left(a+1\right)+ai_1)}x^{\nu}\exp(-\lambda\left(\nu+i_2\right)x^2) \tag{2.8}$$

Proof . The proof follows by raising (1.2) to ν and using the exponential Taylor series and the generalized binomial theorem to obtain the desired result. □

3. Statistical Properties

In this section we will provide some useful statistical properties of the $[0,1]$ TIWR distribution such as moments, quantile function, median, entropy and Stress-Strength Reliability.

3.1. r th Moments

Now, we consider the moments of the $[0,1]$ TIWR distribution. By making use of proposition 2.1 and formulas (3.326) of Gradshteyn and Ryzhik (1965) [12], we obtain:

$$\begin{aligned}E\left(x^n\right)&=\int_0^{+\infty}x^nf\left(x;a,b,\lambda\right)dx=\frac{2\lambda}{\exp\left(-b\right)}\sum_{i_1,i_2=0}^{\infty}\tau_{i_1,i_2}\int_0^{+\infty}x^{n+1}\exp\left(-\lambda i_2x^2\right)\,dx\\&=\frac{\Gamma(\gamma)}{\exp\left(-b\right)\ \lambda^{\gamma-1}}\sum_{i_1,i_2=0}^{\infty}\frac{\tau_{i_1,i_2}}{i_2^{\gamma}}\ ,\ \gamma=\frac{n+2}{2}.\end{aligned} \tag{3.1}$$

In particular,

$$\mu_1'=E\left(x\right)=\frac{\sqrt{\frac{\pi}{\lambda}}}{2\ \exp\left(-b\right)}\sum_{i_1,i_2=0}^{\infty}\frac{\tau_{i_1,i_2}}{i_2^{\frac{3}{2}}} \tag{3.2}$$

And,

$$\mu_2'=E\left(x^2\right)=\frac{1}{\exp\left(-b\right)\ \lambda}\sum_{i_1,i_2=0}^{\infty}\frac{\tau_{i_1,i_2}}{i_2^{2}} \tag{3.3}$$

And the variance can be obtained from the formulas above using, $\sigma^2=\mu_2'-\left(\mu_1'\right)^2$. The skewness and kurtosis are crucial in describing the statistical analysis of distributions. For the $[0,1]$ TIWR distribution, they are given by the following formulas [1]:

$$\mathrm{sk} = \frac{\mu_3'}{\left(\mu_2'\right)^{\frac{3}{2}}} = \frac{\frac{3\sqrt{\pi}}{4\exp(-b)\lambda^{\frac{3}{2}}}\sum_{i_1,i_2=0}^{\infty}\frac{\tau_{i_1,i_2}}{i_2^{\frac{5}{2}}}}{\left(\frac{1}{\exp(-b)\ \lambda}\right)^{\frac{3}{2}}\left[\sum_{i_1,i_2=0}^{\infty}\frac{\tau_{i_1,i_2}}{i_2^{2}}\right]^{\frac{3}{2}}} = \frac{3\sqrt{\pi}}{4\exp\left(\frac{b}{2}\right)}\left(\frac{\sum_{i_1,i_2=0}^{\infty}\frac{\tau_{i_1,i_2}}{i_2^{\frac{5}{2}}}}{\left[\sum_{i_1,i_2=0}^{\infty}\frac{\tau_{i_1,i_2}}{i_2^{2}}\right]^{\frac{3}{2}}}\right) \tag{3.4}$$

and,

$$\mathrm{kr} = \frac{\mu_4'}{\left(\mu_2'\right)^{2}} = \frac{\frac{2}{\exp(-b)\lambda^{2}}\sum_{i_1,i_2=0}^{\infty}\frac{\tau_{i_1,i_2}}{i_2^{3}}}{\left(\frac{1}{\exp(-b)\ \lambda}\right)^{2}\left[\sum_{i_1,i_2=0}^{\infty}\frac{\tau_{i_1,i_2}}{i_2^{2}}\right]^{2}} = \frac{2}{\exp(b)}\left(\frac{\sum_{i_1,i_2=0}^{\infty}\frac{\tau_{i_1,i_2}}{i_2^{3}}}{\left[\sum_{i_1,i_2=0}^{\infty}\frac{\tau_{i_1,i_2}}{i_2^{2}}\right]^{2}}\right). \tag{3.5}$$

respectively. Using the moments formula and Taylor expansion, the moments generating $M_x\left(t\right)$ function is given by:

$$\begin{aligned} M_X\left(t\right) &= E(\exp(tx)) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\ E\left(x^n\right) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{\Gamma\left(\gamma\right)}{\exp\left(-b\right)\ \lambda^{\gamma-1}}\sum_{i_1,i_2=0}^{\infty}\frac{\tau_{i_1,i_2}}{i_2^{\gamma}} \\ &= \frac{\varGamma\left(\gamma\right)}{\exp\left(-b\right)\ \lambda^{\gamma-1}}\sum_{i_1,i_2,n=0}^{\infty}\frac{t^n\tau_{i_1,i_2}}{n!i_2^{\gamma}}. \end{aligned} \tag{3.6}$$

Replacing t by (it) gives the characteristic function $\Psi_x\left(t\right)$:

$$\Psi_x\left(t\right) = \frac{\varGamma\left(\gamma\right)}{\exp\left(-b\right)\ \lambda^{\gamma-1}}\sum_{i_1,i_2,n=0}^{\infty}\frac{i^n t^n\tau_{i_1,i_2}}{n!i_2^{\gamma}}. \tag{3.7}$$

3.2. Quantile Function and Median

The quantile function $Q(p;a,b,\lambda)$ can be obtained by solving the equation $\frac{\exp\left(-b\left[1-\exp\left(-\lambda Q^2\right)\ \right]^{-a}\right)}{\exp(-b)} = p$ for Q. A simple arithmetic yield:

$$Q\left(p\right) = \sqrt{\frac{\ln\left(\frac{1}{1-\sqrt[a]{\frac{b}{b-\ln p}}}\right)}{\lambda}},\quad \lambda > 0 \tag{3.8}$$

Therefore, the median can be computed by letting $p=\frac{1}{2}$ in equation (3.3) to get [3]:

$$M_e = \sqrt{\frac{\ln\left(\frac{1}{1-\sqrt[a]{\frac{b}{b-\ln\frac{1}{2}}}}\right)}{\lambda}} \tag{3.9}$$

3.3. Entropy

The entropy is an essential concept in statistical analysis, it gives raise to the computations of the uncertainty inherited in the variable's outcomes and it has lots of applications in information theory. There are many measures of entropy but we will confine our self with only two [4]: Rényi entropy and Shannon entropy. Let X be a random variable with pdf $f(x;\zeta)$. Then, Rényi Entropy is given by:

$$R_\nu = \frac{1}{1-\nu}\log\left\{\int_{-\infty}^{+\infty} f(x,\zeta)^\nu dx\right\} \tag{3.10}$$

For the $[0,1]$ TIWR distribution, the Rényi Entropy is obtained from proposition 2.2 and formula (3.326) of Gradshteyn and Ryzhik:

$$\begin{aligned} R_\nu &= \frac{1}{1-\nu}\log\left\{\left(\frac{2ab\lambda}{\exp(-b)}\right)^\nu \sum_{i_1,i_2=0}^{\infty}\frac{(-1)^{i_1}\ \nu^{i_1}b^{i_1}\Gamma(\nu(a+1)+ai_1+i_2)}{i_1!i_2!\Gamma(\nu(a+1)+ai_1)}\int_0^{+\infty} x^\nu \exp(-\lambda(\nu+i_2)x^2)\ dx\right\} \\ &= \frac{1}{1-\nu}\log\left\{\left(\frac{2ab\lambda}{\exp(-b)}\right)^\nu \frac{\Gamma(\frac{\nu+1}{2})}{2\lambda^{\frac{\nu+1}{2}}}\sum_{i_1,i_2=0}^{\infty}\frac{(-1)^{i_1}\nu^{i_1}b^{i_1}\Gamma(\nu(a+1)+ai_1+i_2)}{i_1!i_2!\Gamma(\nu(a+1)+ai_1)}\frac{1}{2(\nu+i_2)^{\frac{\nu+1}{2}}}\right\} \end{aligned} \tag{3.11}$$

Shannon entropy is given by:

$$H = -\left\{\int_{-\infty}^{+\infty} log(\ f(x;\zeta)f(x;\zeta)dx\right\} \tag{3.12}$$

Using proposition 2.1, In our case we get:

$$\begin{aligned} H = -\Bigg\{\frac{2\lambda}{\exp(-b)}\sum_{i_1,i_2=0}^{\infty}\tau_{i_1,i_2}\int_0^{+\infty} &\log\left(\frac{2ab\lambda}{\exp(-b)}\right)x\exp\left(-\lambda i_2x^2\right) + \log(x)\,x\exp\left(-\lambda i_2x^2\right) \\ &- \lambda x^3\exp\left(-\lambda i_2 x^2\right) - (a+1)\log\left(1-\exp\left(-\lambda x^2\right)\right)x\exp\left(-\lambda i_2x^2\right) \\ &- b(\left[1-\exp\left(-\lambda x^2\right)\right]^{-a})x\exp\left(-\lambda i_2x^2\right))dx\Bigg\} \end{aligned}$$

Calculating the integrals above, we get:

$$\begin{aligned} I_1 &= \log\left(\frac{2ab\lambda}{\exp(-b)}\right)\int_0^{\infty} x\exp\left(-\lambda i_2x^2\right)\ dx = \log\left(\frac{2ab\lambda}{\exp(-b)}\right)\frac{1}{2\lambda i_2} \\ I_2 &= \int_0^{+\infty}\log(x)\ x\exp\left(-\lambda i_2x^2\right)\ dx = -\frac{1}{4\lambda i_2}\left(\boldsymbol{C}+\log(\lambda i_2)\right) \\ I_3 &= -\lambda\int_0^{+\infty} x^3\exp\left(-\lambda i_2x^2\right)\ dx = \frac{-1}{2\lambda i_2^2} \\ I_4 &= (a+1)\sum_{m_1=1}^{\infty}\frac{1}{m_1}\int_0^{+\infty} x\exp(-\lambda(m_1+i_2)x^2)\ dx = \frac{(a+1)}{2\lambda}\sum_{m_1=1}^{\infty}\frac{1}{m_1(m_1+i_2)} \\ I_5 &= -b\sum_{m_2=0}^{\infty}\frac{\Gamma(a+m_2)}{\mathrm{m_2!\Gamma(a)}}\int_0^{+\infty} x\exp\left(-\lambda(m_2+i_2)x^2\right)\ dx = -\frac{b}{2\lambda}\sum_{m_2=0}^{\infty}\frac{\Gamma(a+m_2)}{\mathrm{m_2!\Gamma(a)}}\frac{1}{(m_2+i_2)} \end{aligned}$$

Where $\boldsymbol{C}$ is Euler's constant and binomial expansion and Taylor expansion of $\ln(1-x)$ were used. Putting everything together, we obtain:

$$H=-\left\{\frac{2\lambda}{\exp\left(-b\right)}\sum_{i_1,i_2=0}^{\infty}\tau_{i_1,i_2}\left[\log\left(\frac{2ab\lambda}{\exp\left(-b\right)}\right)\frac{1}{2\lambda i_2}-\frac{1}{4\lambda i_2}\left(\boldsymbol{C}+\log\left(\lambda i_2\right)\right)-\frac{1}{2\lambda i_2^2}\right.\right.$$
$$\left.\left.+\frac{(a+1)}{2\lambda}\sum_{m_1=1}^{\infty}\frac{1}{m_1\left(m_1+i_2\right)}-\frac{b}{2\lambda}\sum_{m_2=0}^{\infty}\frac{\Gamma\left(a+m_2\right)}{m_2!\Gamma\left(\mathrm{a}\right)}\frac{1}{\left(m_2+i_2\right)}\right]\right\} \tag{3.13}$$

3.4. Stress-Strength Reliability

Stress-Strength analysis shows in many physical as well as statistical applications. Let $X_1\sim[0,1]$ TIWR with parameters a_1,b_1,λ_1 and $X_2\sim[0,1]$ TIWR with parameters a_2,b_2,λ_2 the stress-strength reliability of the $[0,1]-$TIWR distribution is given by [6]:

$$R=\int_0^{+\infty}f_1\left(x\right)F_2\left(x\right)dx \tag{3.14}$$

Using expansions of proposition 2.1 we obtain:

$$R=\int_0^{\infty}\left\{\frac{2\lambda_1}{\exp(-b_1)}\sum_{i_1,i_2=0}^{\infty}\frac{(-1)^{i_1+1}b_1^{i_1}\mathit{\Gamma}\left(a_1i_1+i_2\right)}{i_1!\left(i_2-1\right)!\mathit{\Gamma}\left(a_1i_1\right)}x\exp(-\lambda_1i_2x^2)\right.$$
$$\left.\times\frac{1}{\exp\left(-b_2\right)}\sum_{i_3,i_4}^{\infty}\frac{(-1)^{i_3}b_2^{i_3}\mathit{\Gamma}\left(a_2i_3+i_4\right)}{i_3!i_4!\mathit{\Gamma}\left(a_2i_3\right)}\exp(-\lambda_2i_4x^2)\right\}dx$$
$$=\frac{2\lambda_1}{\exp(-(b_1+b_2))}\sum_{i_1,i_2,i_3,i_4=0}^{\infty}\frac{(-1)^{i_1+i_3+1}b_1^{i_1}b_2^{i_3}\mathit{\Gamma}\left(a_1i_1+i_2\right)\mathit{\Gamma}\left(a_2i_3+i_4\right)}{i_1!\left(i_2-1\right)!i_3!i_4!\mathit{\Gamma}\left(a_1i_1\right)\mathit{\Gamma}\left(a_2i_3\right)}\int_0^{+\infty}x\exp(-(\lambda_1i_2+\lambda_2i_4)x^2)dx$$
$$=\frac{\lambda_1}{\exp(-(b_1+b_2))}\sum_{i_1,i_2,i_3,i_4=0}^{\infty}\frac{(-1)^{i_1+i_3+1}b_1^{i_1}b_2^{i_3}\mathit{\Gamma}\left(a_1i_1+i_2\right)\mathit{\Gamma}\left(a_2i_3+i_4\right)}{i_1!\left(i_2-1\right)!i_3!i_4!\mathit{\Gamma}\left(a_1i_1\right)\mathit{\Gamma}\left(a_2i_3\right)\lambda_1i_2+\lambda_2i_4)}. \tag{3.15}$$

4. Maximum Likelihood Estimation

In this section we consider the maximum likelihood estimator of the $[0,1]$ TIWR's parameters. Let x_1,x_2,...,x_n be the observed values of the $[0,1]-$TIWR distribution with unknown parameters a, b and λ.The total log-likelihood function for $[0,1]$ TIWR distribution has the form:

$$L\left(a,b,\lambda\right)=n\log2+n\log a+n\log b+n\log\lambda+b+\sum\log x_i-\lambda\sum x_i^2$$
$$-\left(a+1\right)\sum\log\left(1-\exp\left(-\lambda x_i^2\right)\right)-b\sum\left(1-\exp\left(-\lambda x_i^2\right)\right)^{-a} \tag{4.1}$$

Hence the MLEs that maximize $L(a,b,\lambda)$, must satisfy the following equations:

$$\begin{aligned}\frac{\partial}{\partial a}L(a,b,\lambda)=&\frac{n}{a}-\sum\log\left(1-\exp\left(-\lambda x^2\right)\right)\\&+b\sum\left(1-\exp\left(-\lambda x_i^2\right)\right)^{-a}\ln\left(1-\exp\left(-\lambda x_i^2\right)\right)=0\end{aligned}\tag{4.2}$$

$$\frac{\partial}{\partial b}L(a,b,\lambda)=\frac{n}{b}+1-\sum\left(1-\exp\left(-\lambda x_i^2\right)\right)^{-a}=0\tag{4.3}$$

$$\begin{aligned}\frac{\partial}{\partial \lambda}L(a,b,\lambda)=&\frac{n}{\lambda}-\sum x_i^2-(a+1)*\sum\frac{x_i^2\exp(-\lambda x_i^2)}{1-\exp\left(-\lambda x_i^2\right)}\\&+ab\sum x_i^2\exp\left(-\lambda x_i^2\right)\left(1-\exp\left(-\lambda x_i^2\right)\right)^{-(a+1)}=0\end{aligned}\tag{4.4}$$

Equation (4.3) gives the MLE of b as a function of a and λ :

$$\hat{b}=\hat{b}(a,\lambda)=\frac{n}{\sum\left(1-\exp\left(-\lambda x_i^2\right)\right)^{-a}-1}\tag{4.5}$$

Substituting (4.5) in equations (4.2) and (4.4) gives two equations satisfied by $\hat{a}$ and $\widehat{\lambda}$.The equations above have a very complicated form and cannot be solved algebraically. Therefore, we resort to numerical method to solve them. In most cases Newton- Raphson method is a good candidate.

5. Order Statistics

Order statistics play an important role in many aspects of statistical analysis and random variables behavior. In this section we briefly address the order statistic of the $[0,1]$ TIWR distribution. Let $X_1, X_2, \ldots, X_n$ be a random simple from $[0,1]$ TIWR. The pdf of the jth random simple is given by [14]:

$$f_{X(j)}(x)=\frac{n!}{(j-1)!\,(n-j)!}f(x)\,[F(x)]^{j-1}[1-F(x)]^{n-j},\, j=1,2,\ldots,n\tag{5.1}$$

Using some expansion techniques, we obtain the following expression:

$$f_{X(j)}(x)=\frac{n!}{(j-1)!\,(n-j)!}\frac{2ab\lambda}{\exp(-bj)}\sum_{i_1,i_2=0}^{\infty}\sum_{k=0}^{n-j}\binom{n-j}{k}s_{i_1,i_2,k}x\exp(-\lambda\,(i_2+1)\,x^2+kb)\tag{5.2}$$

Where,

$$s_{i_1,i_2,k}=\frac{(-1)^{i_1}b^{i_1}\,(j+k)^{i_1}\Gamma(a\,(i_1+1)+1+i_2)}{i_1 i_2\Gamma(a\,(i_1+1)+1)}$$

and $X_{(j)}$denotes the jth order statistic. Substitute $j=1$ and $j=n$ gives the pdf of the first and last order statistics respectively.

From the table 1. We that when we increase the value of a the value of mean will increase too and the value of variance decrease. When we increase the value of ***b***, the value of mean will increase too and the value of variance decrease. When we increase the value ofλ, the value of mean will decrease and the value of variance decrease.

Table 1: The Expected value and variance for the $[0,1]$ TIWR distribution with selected values of the parameters

$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{\lambda}$	$\boldsymbol{E(x)}$	$\boldsymbol{Var(x)}$
0.5	1	1	0.743501297	0.162640413
1	1	1	1.016115450	0.141072419
2	1	1	1.271932629	0.116206007
1	1	0.5	1.437004250	0.282144839
1	1	2	0.718502124	0.070536210
1	2	0.5	1.727403145	0.279461300
0.5	2	1	0.962277987	0.167548324

6. Simulation study

In this section we study the effectiveness of the MLE method for estimating the parameters of the $[0,1]$ TIWR distribution using a Monte Carlo simulation study With 250 replications.

Here we calculate the means of the parameter estimates, bias and Root-mean-square deviation (RMSE) using the R softwire. We generate N = 2500 samples of sizes n = 75, 150, 250 from $[0,1]$ TIWR distribution with three sets of parameters $(I : a = 1, b = 2, \lambda = 0.9)$, $(II : a = 0.5, b = 1, \lambda = 2)$, $(III : a = 1, b = 0.8, \lambda = 2)$. we note that the estimated biases decrease as the sample size increases. Furthermore, the RMSE tend to zero as we increase the sample size which reveals the effectiveness and consistence of the distribution compared with observed data. Finally, the means also decrease with respect to increasing sample size. The empirical results are given in Table 2.

Table 2: Means, Bias and RMSE for the $[0,1]$ TIWR distribution. parameters

Set	n	Means			Bias			RMSE		
		a	b	λ	a	b	λ	a	b	λ
I	75	1.083	2.054	0.911	0.083	0.054	0.011	0.312	0.868	0.141
	150	1.043	2.062	0.904	0.043	0.062	0.004	0.238	0.742	0.101
	250	1.023	2.067	0.905	0.023	0.067	0.005	0.183	0.632	0.083
II	75	0.521	1.115	2.082	0.021	0.115	0.082	0.136	0.608	0.489
	150	0.510	1.071	2.039	0.010	0.071	0.039	0.101	0.444	0.347
	250	0.506	1.053	2.034	0.006	0.053	0.034	0.080	0.354	0.280
III	75	1.039	0.912	2.064	0.039	0.112	0.064	0.254	0.537	0.457
	150	1.015	0.881	2.036	0.015	0.081	0.036	0.191	0.412	0.331
	250	1.007	0.859	2.030	0.007	0.059	0.030	0.148	0.328	0.276

7. Applications

In this section we give a real data application for the [0,1] TIWR distribution which also shows a better fitting compared with other distributions. The comparison includes their negative log-likelihood (NLL), Akaike Information Criteria (AIC), Consistent Akaike Information Criteria (CAIC), Bayesian Information Criteria (BIC), Hanan and Quinn Information Criteria (HQIC), Kolmogorov-Smirnov (KS) and Anderson Darling (AD) values.

The application we considered is the deaths rate of **COVID-19** in Iraq in 137 days from 14 December 2020 to 30 April 2021. The observations are:

11, 22, 14, 20, 10, 17, 10, 13, 15, 12, 7, 11, 12, 27, 11, 29, 30, 11, 5, 5, 10, 12, 9, 4, 8, 4, 14, 11, 5, 4, 7, 10, 3, 9, 9, 9, 6, 9, 7, 4, 5, 7, 10, 8, 6, 12, 5, 6, 10, 11, 11, 12, 8, 9, 8, 6, 4, 13, 7, 15, 6, 7, 12, 16, 12, 13, 27, 23, 16, 13, 27, 14, 18, 23, 22, 30, 25, 24, 30, 11, 24, 24, 22, 27, 26, 25, 23, 32, 37, 39, 33, 36, 41, 32, 21, 22, 30, 29, 33, 29, 20, 35, 37, 37, 37, 37, 30, 40, 33, 39, 33, 37, 34, 35, 34, 35, 37, 35, 44, 39, 40, 49, 30, 33, 33, 45, 34, 38, 30, 46, 43, 40, 46, 45, 44, 41, 32.

The results are demonstrated in table 3.

Table 3: Comparison of the results of data fitting between the [0,1] TIWR distributions and other distributions.

Distributions	**NLL**	**AIC**	**CAIC**	**BIC**	**HQIC**	**KS**	**A**
[0,1]TIW-R	**214.95**	**435.91**	**436.09**	**444.67**	**439.47**	**0.01955**	**2.5623**
[0,1]TEE-R	**218.10**	**442.21**	**442.39**	**450.97**	**445.77**	**0.02898**	**2.9352**
Beta-R	**217.07**	**440.14**	**440.32**	**448.90**	**443.70**	**0.02761**	**2.8297**
Kumaraswamy-R	**217.29**	**440.58**	**440.76**	**449.34**	**444.14**	**0.02624**	**2.8552**
EG-R	**217.29**	**440.58**	**440.76**	**449.34**	**444.14**	**0.02788**	**2.8546**
We-R	**217.98**	**441.96**	**442.14**	**450.72**	**445.52**	**0.02353**	**2.9190**
Go-R	**220.53**	**447.07**	**447.25**	**455.83**	**450.63**	**0.00012**	**2.8245**
MO-R	**218.72**	**441.44**	**441.53**	**447.28**	**443.81**	**0.01511**	**2.9457**
RAY	**220.58**	**443.17**	**443.20**	**446.09**	**444.36**	**0.00021**	**2.8830**

Our suggested distribution shows a significantly better fitting of data compared to the other distributions we considered here as it has the lowest values for the NLL, AIC, CAIC, BIC, HQIC, KS and A values.

The MLE estimations of the parameters is given in table 4.

Table 4: The parameters estimations using ML method.

Distribution	Estimations
[0,1]TIWR	$\widehat{\alpha} = 0.2297252$, $\widehat{\beta} = 2.2089408$,$\widehat{\lambda} = 0.1081969$
[0,1]TEER	$\widehat{\alpha} = 19.992322762$, $\widehat{\beta} = 0.784775493$, $\widehat{\lambda} = 0.007010988$
BE-R	$\widehat{\alpha} = 0.7397223$, $\widehat{\beta} = 0.4336251$, $\widehat{\lambda} = 0.3186380$
Ku-R	$\widehat{\alpha} = 0.7601221$, $\widehat{\beta} = 0.8637753$, $\widehat{\lambda} = 0.1559698$
EG-R	$\widehat{\alpha} = 2.84984935$, $\widehat{\beta} = 0.76713200$, $\widehat{\lambda} = 0.04671613$
We-R	$\widehat{\alpha} = 0.8572101$, $\widehat{\beta} = 2.0922150$, $\widehat{\lambda} = 0.3573170$
Go-R	$\widehat{\alpha} = 2.61430129$, $\widehat{\beta} = 0.09492149$, $\widehat{\lambda} = 0.05856151$
MO-R	$\widehat{\alpha} = 0.5528685$, $\widehat{\beta} = 0.1193882$
RAY	$\widehat{\alpha} = 0.1585135$

The histogram plot is presented in Figure 3 while the empirical cdf (ecdf) plot is presented in Figure 4 for **COVID-19** data. In Figure 3, the curve for the [0,1]TIWR distribution has the highest peak and fits the histogram of the dataset better than the other competing models. Figure 4 shows the ecdf plot. The plots in Figure 4 support the ones in Figure 3 as the cdf plot for the [0,1] TIWR distribution fits the Covid-19 dataset than the other competing models.

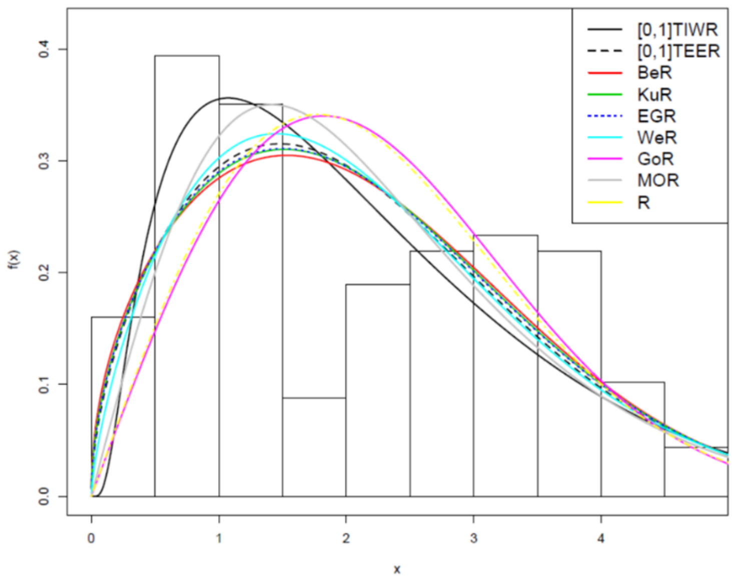

Figure 3: The pdf plot for all the competing models using the COVID-19 dataset.

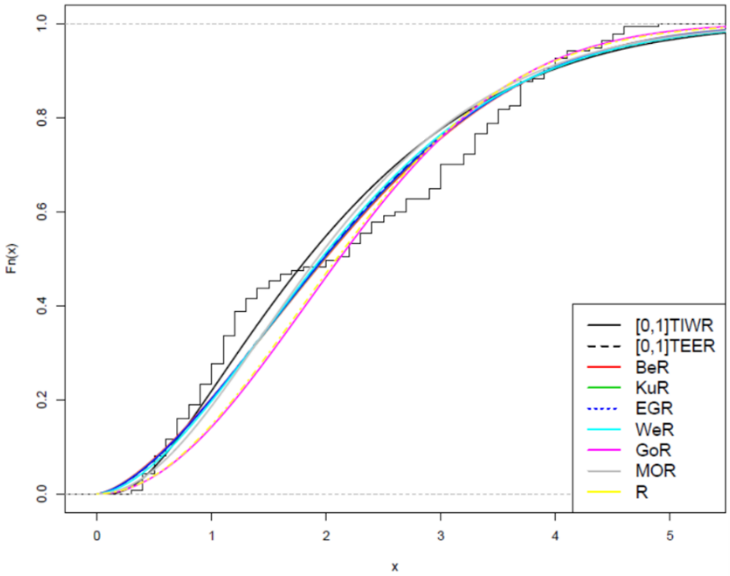

Figure 4: The ecdf plot for all the competing models using the COVID-19 dataset.

8. Conclusions

The [0,1] TIW-G family is a promising model for generating flexible continuous random variables and extends the study of other known distributions. It has the potential to provides a better fitting of data compared to other families. The special model ([0,1] TIWR distribution) seems to give a relatively accurate fitting for data. In this paper many useful result and properties were derived such as series expansion, moments, skewness, kurtosis, quantile function, median, entropies and many other results and statistical properties. We recommend a more extensive study given that our limited study concerns a new dataset and more studies could reveals more interesting results.

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