Int. J. Nonlinear Anal. Appl. 13 (2022) 1, 2999–3010
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.6031

The Topp Leone Flexible Weibull distribution: An extension of the Flexible Weibull distribution

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(Communicated by Madjid Eshaghi Gordji)

Abstract

A new probability model for positively skewed datasets like economic data, medical, engineering and other sciences was developed in this paper. The new distribution is named the Topp-Leone Flexible Weibull distribution and it was generated using the Topp-Leone-G family of distributions. This new distribution has three parameters and it is very flexible in fitting several and different datasets. Its basic mathematical properties were studied and the method of maximum likelihood estimation was used for the estimation of model parameters. A real life dataset was used to illustrate the flexibility of the distribution and it was found that the new model provides a better fit to real life datasets than the Topp Leone Burr XII, Topp Leone Lomax and Exponentiated Generalized Flexible Weibull distributions.

Keywords: Flexible Weibull distribution, Mathematical Statistics, Topp-Leone family of distributions

1. Introduction

The purpose of generating new compound distributions from known families of distributions is to extend the baseline distributions and develop them by adding one or more shape parameters. These compound distributions have however been found to be better than the parent distribution in terms of flexibility and modeling capability. There are several families of distributions that can be used for this purpose but in this research, the interest is on the Topp-Leone-G family of distributions. The Topp-Leone-G family of distributions is relatively new, and it has only one extra parameter.

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Received: June 2021 *Accepted:* November 2021

By this, the resulting composite distribution will only have three parameters; two from the baseline distribution (Flexible Weibull distribution) and one from the Topp-Leone-G family of distributions. Weibull distribution is one of the most important probability models in the theory of statistics and other fields of sciences; it has some distributions like the Exponential distribution and Rayleigh distribution as sub-models. The model is appropriate for modeling real life phenomena with monotonous failure rates [2]. One of the important models derived from the Weibull distribution is the Flexible Weibull distribution; it has many applications in statistics as well as life testing, circuit studies and reliability analysis [6].
In the past, there have been attempts to expand this distribution, for instance, the Exponentiated Flexible Weibull extension distribution [10], Exponentiated Generalized Flexible Weibull extension distribution [13], Beta Flexible Weibull distribution [8], Exponential Flexible Weibull extension distribution [9], and Exponentiated Generalized Flexible Weibull extension distribution [2], are some of the notable works in the literature. However, the modeling capacity of the Flexible Weibull distribution can be increased when a more flexible family of distributions is used; this is one of the problems this present research addresses.
The Topp-Leone-G family of distribution was developed by [17] and it has been demonstrated in the works of [4], [1], [16], and [14] to be a powerful alternative to some other families of distributions. The Topp-Leone distribution itself which was introduced by Topp and Leone has recently received attention by other notable authors like [11], [15], [3],[12],[5] and [7].
This paper has been structured to develop the Topp-Leone Flexible Weibull distribution, establish its various properties, estimate its unknown parameters and to demonstrate its strength using a real life application.

2. The Topp-Leone Flexible Weibull (TLFW) Distribution

Let Y denote a random variable, the cumulative distribution function (cdf) of the baseline distribution (Flexible Weibull) distribution with parameters ω and v is given by:

$$G\left(y,\omega,v\right)=1-e^{-e^{vy-\frac{\omega}{y}}};\qquad y>0,\omega>0,v>0 \tag{2.1}$$

Differentiating the expression in Equation (2.1) gives the probability density function (pdf) of the Flexible Weibull distribution as:

$$g\left(y,\omega,v\right)=\left(v+\frac{\omega}{y^2}\right)e^{vy-\frac{\omega}{y}}.e^{-e^{vy-\frac{\omega}{y}}};\qquad y>0,\omega>0,v>0 \tag{2.2}$$

Also, the cdf of the Topp-Leone-G family of distributions is:

$$F\left(y,\lambda\right)=\left\{1-\left[1-G\left(y\right)\right]^2\right\}^{\lambda};\qquad y\in\Re,\lambda>0 \tag{2.3}$$

Its corresponding pdf is:

$$f\left(y,\lambda\right)=2\lambda g\left(y\right)\left\{1-\left[1-G\left(y\right)\right]\right\}^{\lambda-1};\qquad y\in\Re,\lambda>0 \tag{2.4}$$

where λ is the extra shape parameter introduced by the Topp-Leone-G family of distributions. To develop the cdf of the Topp-Leone Flexible Weibull (TLFW) distribution, the expressions in Equations (2.1) and (2.2) are substituted into Equation (2.3) as follows:

$$F\left(y,\lambda,\omega,v\right)=\left[1-\left(e^{-e^{vy-\frac{\omega}{y}}}\right)^2\right]^{\lambda};\qquad y>0,\lambda>0,\omega>0,v>0 \tag{2.5}$$

Its corresponding pdf is:

$$f\left(y,\lambda,\omega,v\right)=2\lambda\left(vy+\frac{\omega}{y^{2}}\right)e^{vy-\frac{\omega}{y}}.e^{-e^{vy-\frac{\omega}{y}}}\left[1-\left(1-e^{-e^{vy-\frac{\omega}{y}}}\right)\right]\left[-\left(e^{-e^{vy-\frac{\omega}{y}}}\right)^{2}\right]^{\lambda-1} \tag{2.6}$$

Figure 1 illustrates some possible forms of the pdf of the Topp Leone Flexible Weibull distribution by taking different values of the distribution parameters (λ, ω, v).

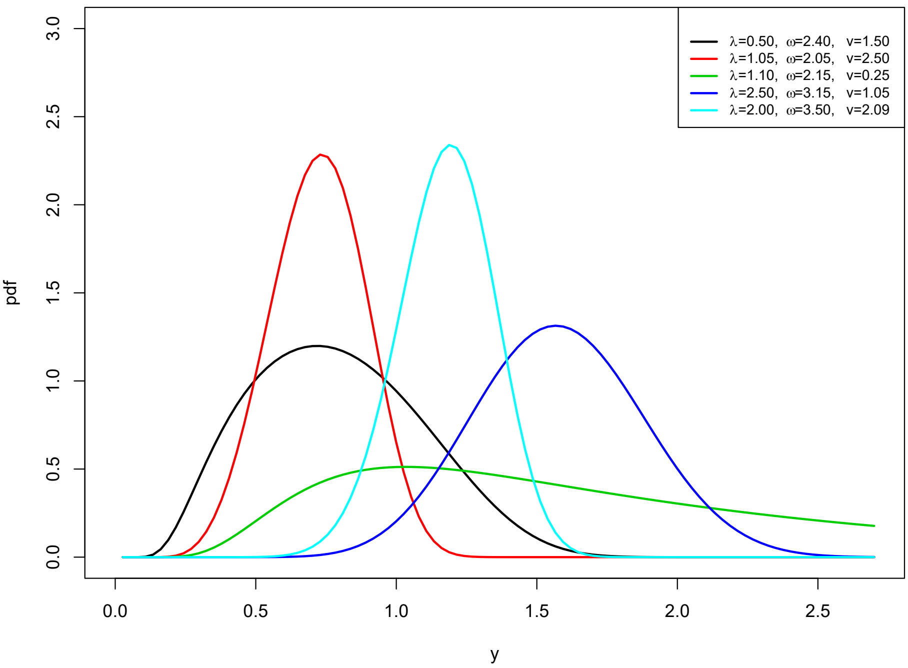

Figure 1: PDF plot of the Topp Leone Flexible Weibull distribution

The pdf of the TLFW distribution has a unimodal shape and is rightly skewed.

2.1. Survival Function

The survival function can be obtained for the Topp Leone Flexible Weibull distribution using the following relationship:

$$S\left(y,\lambda,\omega,v\right)=1-F\left(y,\lambda,\omega,v\right) \tag{2.7}$$

The expression for the survival function of the Topp Leone Flexible Weibull distribution now becomes:

$$S\left(y,\lambda,\omega,v\right)=1-\left[1-\left(e^{-e^{vy-\frac{\omega}{y}}}\right)^{2}\right]^{\lambda};\qquad y>0,\lambda>0,\omega>0,v>0 \tag{2.8}$$

2.2. The Failure rate or Hazard function

The hazard or failure rate can be obtained from the following relationship:

$$h\left(y,\lambda,\omega,v\right)=\frac{f\left(y,\lambda,\omega,v\right)}{S\left(y,\lambda,\omega,v\right)} \tag{2.9}$$

Hence, the failure rate for the Topp Leone Flexible Weibull distribution is:

$$h\left(y,\lambda,\omega,v\right)=\frac{2\lambda\left(vy+\frac{\omega}{y^2}\right)e^{vy-\frac{\omega}{y}}.e^{-e^{vy-\frac{\omega}{y}}}\left[1-\left(1-e^{-e^{vy-\frac{\omega}{y}}}\right)\right]\left[-\left(e^{-e^{vy-\frac{\omega}{y}}}\right)^2\right]^{\lambda-1}}{1-\left[1-\left(e^{-e^{vy-\frac{\omega}{y}}}\right)^2\right]^{\lambda}} \tag{2.10}$$

Figure 2 illustrates some possible forms of the hazard function of the Topp Leone Flexible Weibull distribution by taking some different values of the parameters

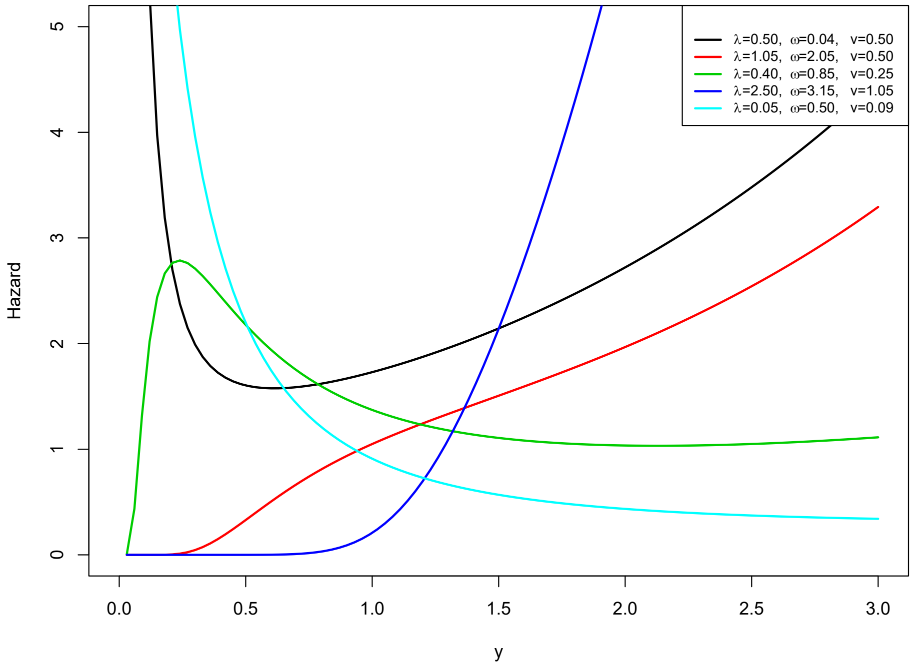

Figure 2: Plot for the Hazard function of Topp Leone Flexible Weibull distribution

It can be seen that the Topp Leone Flexible Weibull distribution has a bathtub, inverted bathtub and inverse J shapes.

2.3. Reversed hazard function

This can be obtained using the following relationship:

$$r\left(y,\lambda,\omega,v\right)=\frac{f\left(y,\lambda,\omega,v\right)}{F\left(y,\lambda,\omega,v\right)} \tag{2.11}$$

So, the reversed hazard function of the Topp Leone Flexible Weibull distribution becomes:

$$r\left(y,\lambda,\omega,v\right)=\frac{2\lambda\left(vy+\frac{\omega}{y^2}\right)e^{vy-\frac{\omega}{y}}.e^{-e^{vy-\frac{\omega}{y}}}\left[1-\left(1-e^{-e^{vy-\frac{\omega}{y}}}\right)\right]\left[-\left(e^{-e^{vy-\frac{\omega}{y}}}\right)^2\right]^{\lambda-1}}{\left[1-\left(e^{-e^{vy-\frac{\omega}{y}}}\right)^2\right]^{\lambda}} \tag{2.12}$$

2.4. The odd function

The odd function can be obtained using the following relationship:

$$O\left(y,\lambda,\omega,v\right)=\frac{F\left(y,\lambda,\omega,v\right)}{S\left(y,\lambda,\omega,v\right)} \tag{2.13}$$

So, the odd function for the Topp Leone Flexible Weibull distribution becomes:

$$O\left(y,\lambda,\omega,v\right)=\frac{\left[1-\left(e^{-e^{vy-\frac{\omega}{y}}}\right)^{2}\right]^{\lambda}}{1-\left[1-\left(e^{-e^{vy-\frac{\omega}{y}}}\right)^{2}\right]^{\lambda}} \tag{2.14}$$

2.5. Quantile Function

This can easily be obtained as the inverse of the cdf as follows:

$$Q\left(y\right)=u=F^{-1}\left(y\right) \tag{2.15}$$

This way, the quantile function for the Topp Leone Flexible Weibull distribution is obtained as follows:

$$Q\left(y,\lambda,\omega,v\right)=u=F^{-1}\left(y,\lambda,\omega,v\right)$$

$$u=\left[1-\left(e^{-e^{vy-\frac{\omega}{y}}}\right)^{2}\right]^{\lambda}$$

$$y=\frac{\ln\left[\ln\left(1-u^{\frac{1}{2\lambda}}\right)\right]\pm\sqrt{\left\{-\ln\left[\ln\left(1-u^{\frac{1}{2\lambda}}\right)\right]\right\}^{2}+4v\omega}}{2v} \tag{2.16}$$

2.6. Expansion for the probability density function

Expanding the pdf of a probability model is very useful in developing and studying some mathematical properties which otherwise would have required prolonged algebraic processes. Recall from Equation (2.6) that:

$$f\left(y,\lambda,\omega,v\right)=2\lambda\left(vy+\frac{\omega}{y^{2}}\right)e^{vy-\frac{\omega}{y}}.e^{-e^{vy-\frac{\omega}{y}}}\left[1-\left(1-e^{-e^{vy-\frac{\omega}{y}}}\right)\right]\left[-\left(e^{-e^{vy-\frac{\omega}{y}}}\right)^{2}\right]^{\lambda-1}$$

By making use of the following sequence; $\left[1-\left(e^{-e^{vy-\frac{\omega}{y}}}\right)^{2}\right]^{\lambda-1}$ then,

$$f\left(y,\lambda,\omega,v\right)=2\lambda\left(vy+\frac{\omega}{y^{2}}\right)e^{vy-\frac{\omega}{y}}.e^{-2e^{vy-\frac{\omega}{y}}}\sum_{k=0}^{\infty}\frac{\left(-1\right)^{k}\Gamma\left(\lambda\right)}{\Gamma\left(\lambda-k\right).k!}\left[\left(e^{-e^{vy-\frac{\omega}{y}}}\right)\right]^{2k}$$

Also by the series expansion of $e^{\left[2(k+1)e^{vy-\frac{w}{y}}\right]}$, the pdf of Topp-Leone Flexible Weibull distribution is expanded as;

$$f\left(y,\lambda,v,w\right)=2\lambda\left(v+\frac{w}{y^{2}}\right)\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{\left(-1\right)^{k}\Gamma\left(\lambda\right)\left[2\left(k+1\right)\right]^{i}}{\Gamma\left(\lambda-k\right)k!i!}e^{(i+1)\left(vy-\frac{w}{y}\right)} \tag{2.17}$$

2.7. Moments

Moments play important roles in finding and measuring some statistical characteristics such as finding flattening and spacing, coefficient of variation, standard deviation and other characteristics. The rth moment of the Topp Leone Flexible Weibull distribution can be obtained using the following relationship;

$$\mu_r = E\left(Y^r\right) = \int_0^\infty y^r f\left(y, \lambda, v, w\right) dy$$

Using the expanded pdf in Equation (2.17),

$$\mu_r = \int_0^\infty y^r 2\lambda \left(v + \frac{w}{y^2}\right) \sum_{k=0}^\infty \sum_{i=0}^\infty \frac{(-1)^k \Gamma(\lambda)\left[2(k+1)\right]^i}{\Gamma(\lambda-k)\, k! i!} e^{(i+1)\left(vy - \frac{w}{y}\right)} dy$$

$$\mu_r = 2\lambda \sum_{k=0}^\infty \sum_{i=0}^\infty \frac{(-1)^k \Gamma(\lambda)\left[2(k+1)\right]^i}{\Gamma(\lambda-k)\, k! i!} \int_0^\infty y^{r-k} \left(v + \frac{w}{y^2}\right) e^{(i+1)\left(vy - \frac{w}{y}\right)} dy$$

$$\mu_r = 2\lambda \sum_{k=0}^\infty \sum_{i=0}^\infty \frac{(-1)^k \Gamma(\lambda)\left[2(k+1)\right]^i}{\Gamma(\lambda-k)\, k! i!} \int_0^\infty v y^{r-k} e^{\left[i+1\left(vy - \frac{w}{y}\right)\right]} dy + \int_0^\infty w y^{r-k-1} e^{\left[i+1\left(vy - \frac{w}{y}\right)\right]} dy$$

$$\mu_r = 2\lambda \sum_{k=0}^\infty \sum_{i=0}^\infty \frac{(-1)^k \Gamma(\lambda)\left[2(k+1)\right]^i}{\Gamma(\lambda-k)\, k! i!} \left[\frac{\Gamma(r-k+1)}{v^{r-k}(i+1)^{r-k+1}} + \frac{w\Gamma(r-k-1)}{v^{r-k-1}(i+1)^{r-k-1}}\right] \quad (2.18)$$

2.8. Moment Generating Function

For a random variable Y, the moment generating function (mgf) is given by the relationship;

$$M_Y(t) = \int_0^\infty e^{ty} f(y)\, dy$$

Thus, the mgf for the Topp Leone Flexible Weibull distribution is obtained as follows;

$$M_Y(t) = \int_0^\infty e^{ty} f\left(y, \lambda, w, v\right) dy$$

By expressing e^{ty}in form of a sequence;

$$M_Y(t) = \sum_{j=0}^\infty \frac{t^j}{j!} \int_0^\infty y^j f\left(y, \lambda, w, v\right) dy$$

This turns to;

$$M_Y(t) = \sum_{j=0}^\infty \frac{t^j}{j!} \mu_r$$

Where $\mu_r$is as expressed in Equation (2.18), then;

$$M_Y(t) = 2\lambda \sum_{k=0}^\infty \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(-1)^k \Gamma(\lambda)\left[2(k+1)\right]^i t^j}{\Gamma(\lambda-k)\, k! i! j!} \left[\frac{\Gamma(r-j+1)}{v^{r-j}(i+1)^{r-j+1}} + \frac{w\Gamma(r-j-1)}{v^{r-j-1}(i+1)^{r-j-1}}\right] \quad (2.19)$$

2.9. Maximum Likelihood Estimation

Here, the parameters of the Topp Leone Flexible Weibull distribution are estimated using the method of maximum likelihood. Let $y_1, y_2, ..., y_n$ be random samples distributed according to the Topp Leone Flexible Weibull distribution, the likelihood function is obtained by the relationship;

$$L = \prod_{i=1}^{n} f(y, \lambda, w, v)$$

Using the expression in Equation (2.6), then;

$$L = \prod_{i=1}^{n} \left\{ 2\lambda \left(vy_i + \frac{w}{y_i^2} \right) e^{vy_i - \frac{w}{y_i}} e^{-e^{vy_i - \frac{w}{y_i}}} \left[1 - \left(1 - e^{-e^{vy_i - \frac{w}{y_i}}} \right) \right] \left[1 - \left(e^{-e^{vy_i - \frac{w}{y_i}}} \right)^2 \right]^{\lambda - 1} \right\}$$

By taking the natural logarithm, the log-likelihood function is obtained as;

$$\begin{aligned} \log L =& n\log(2) + n\log(\lambda) + \sum_{i=1}^{n} \log\left(vy_i + \frac{w}{y_i^2} \right) + \sum_{i=1}^{n} \left(vy_i - \frac{w}{y_i} \right) + \sum_{i=1}^{n} \left(e^{vy_i - \frac{w}{y_i}} \right) \\ &+ \sum_{i=1}^{n} \log\left[1 - \left(1 - e^{e^{vy_i - \frac{w}{y_i}}} \right) \right] + (\lambda - 1) \sum_{i=1}^{n} \log\left[1 - \left(e^{-e^{vy_i - \frac{w}{y_i}}} \right)^2 \right] \end{aligned} \quad (2.20)$$

The estimate of each of the parameters can therefore be obtained when the first partial derivative of the log-likelihood function for each of the parameters is taken, equated to zero and solved simultaneously. It is good to note that the solution cannot be obtained in closed form. This can however be resolved by solving numerically using available software like R and other sophisticated software.

3. Applications

A real life application is presented in this section to demonstrate the usefulness of the Topp Leone Flexible Weibull distribution. Comparisons are made with the Topp Leone Burr XII (TLBXII), Topp Leone Lomax (TLLo), Exponentiated Generalized Flexible Weibull (EGFW) distributions with respect to their negative log-likelihood (NLL), Akaike Information Criteria (AIC), Consistent Akaike Information Criteria (CAIC), Bayesian Information Criteria (BIC), Hanan and Quinn Information Criteria (HQIC), Kolmogorov Smirnov (KS) and Anderson Darling (AD) values.

The dataset used reports the failure times of 40 of the lifting engines for giant machines. The dataset has previously been analyzed by [18] to predict the failure times of these engines. The observations are:

1.6, 2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.0, 5.1, 5.3, 5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3, 6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 8.0, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.0

The result is presented in Table 1.

Table 1: Table of result

Distributions	NLL	AIC	CAIC	BIC	HQIC	KS	A
TLBXII	98.974	203.948	204.615	209.015	205.7805	0.5177	3.0343
TLLo	90.800	187.601	188.268	192.668	189.4335	0.2942	1.8622
EGFW	80.373	168.746	169.888	175.501	171.1886	0.0426	0.3273
TLFW	79.776	165.553	166.220	170.620	167.3858	0.0355	0.2926

The newly developed TLFW displays a very good potential in Table 1 as it has the lowest values for the NLL, AIC, CAIC, BIC, HQIC, KS statistic and A statistic.
The maximum likelihood estimates for the parameters are provided in Table 2.

Table 2: Parameter estimates

Distributions	Estimates
TLBXII	$\hat{\lambda} = 65.0528, \hat{w} = 1.1795, \hat{v} = 1.0157$
TLLo	$\hat{\lambda} = 10.4177, \hat{w} = 0.0157, \hat{v} = 15.4623$
EGFW	$\hat{\lambda} = 0.0901, \hat{w} = 1.5062, \hat{v} = 0.4211, \hat{\alpha} = 1.7334$
TLFW	$\hat{\lambda} = 0.1917, \hat{w} = 0.6394, \hat{v} = 51.8049$

To further validate the results obtained, the histogram plot of the dataset with the distributions compared is presented in Figure 1.

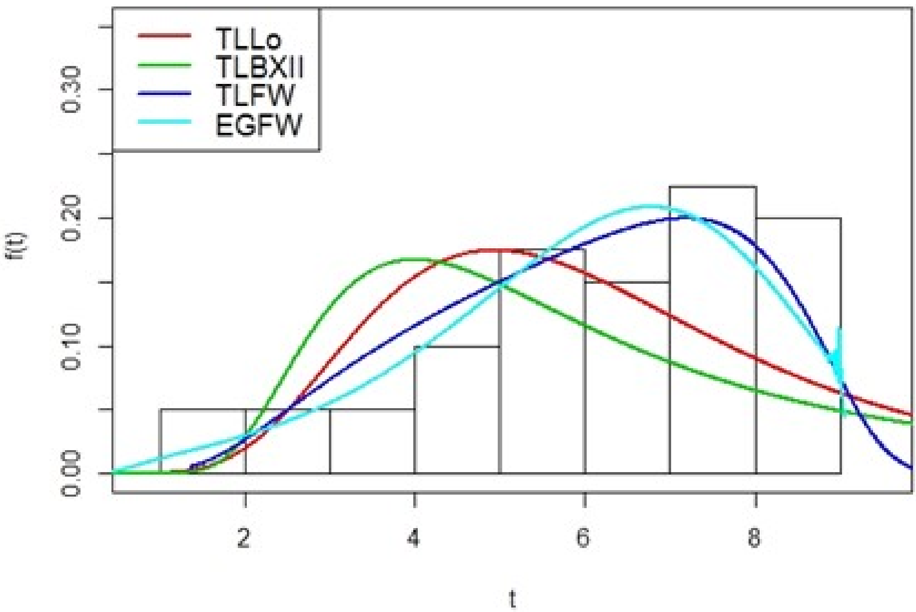

Figure 3: Histogram plot of the dataset with the compared distributions

The corresponding empirical cdf plot is presented in Figure 4.

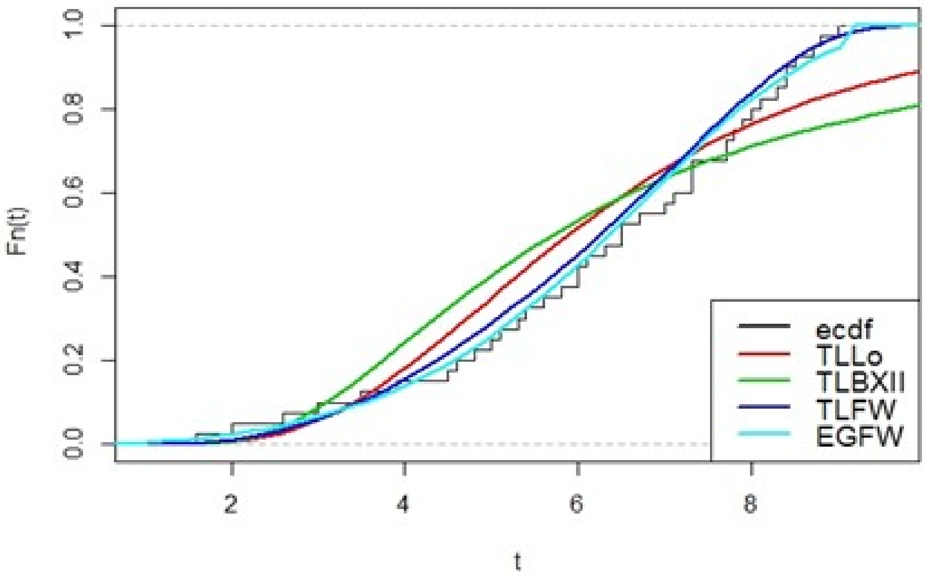

Figure 4: Empirical cdf of the dataset with the compared distributions

Figures 3 and 4 show that the TLFW distribution fits the dataset better than the other distributions. The probability-probability (p-p) plots for each of the distributions with respect to the dataset used are presented in Figures 5 to 8.

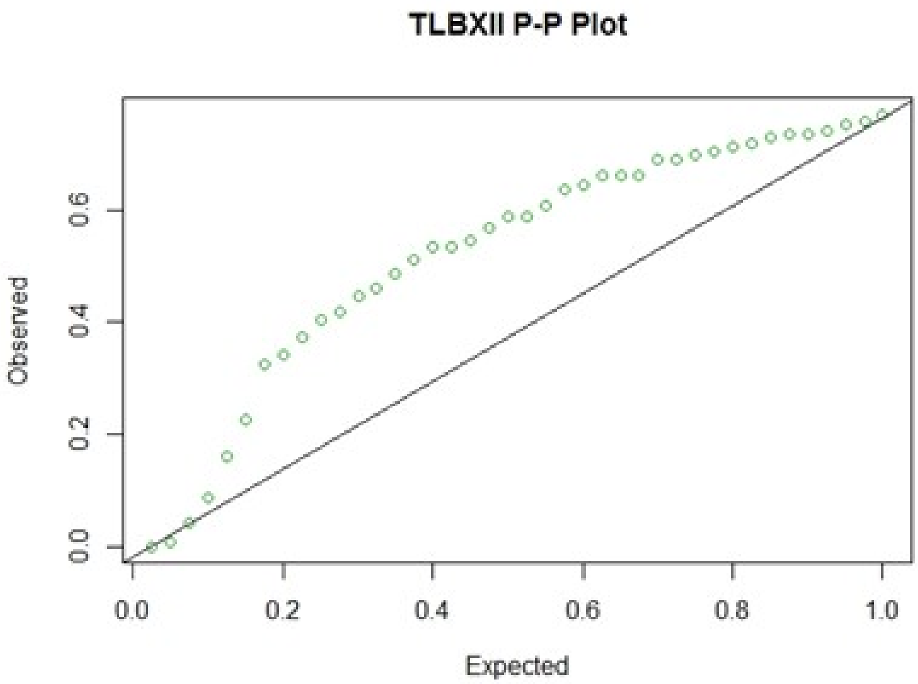

Figure 5: p-p plot for TLBXII distribution

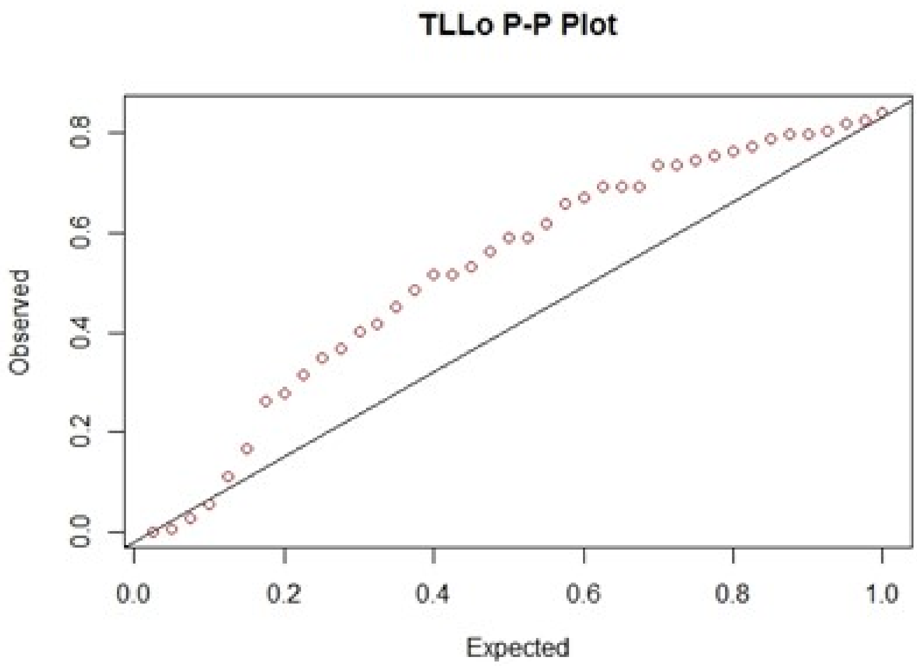

Figure 6: p-p plot for TLLo distribution

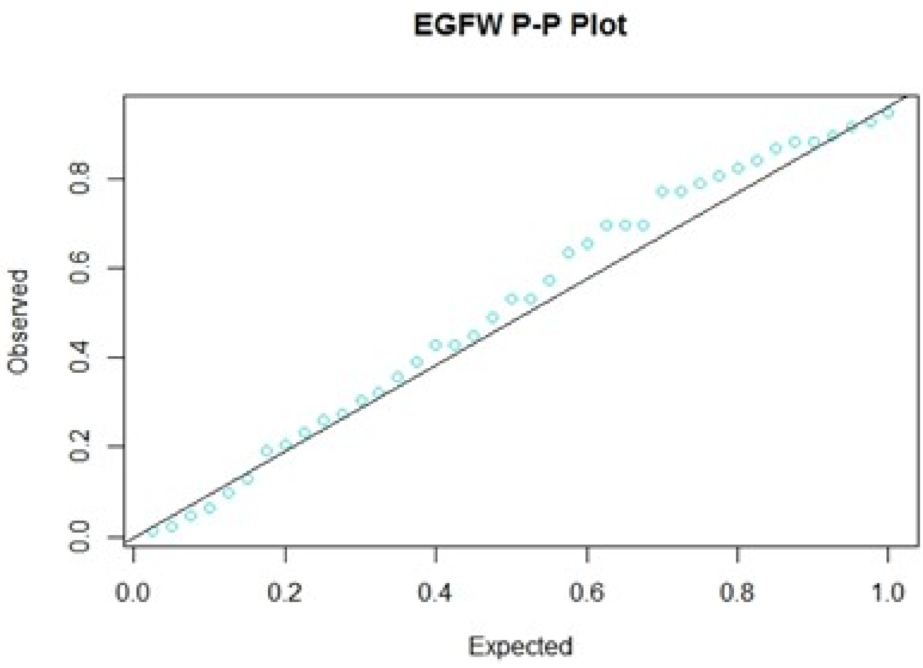

Figure 7: p-p plot for EGFW distribution

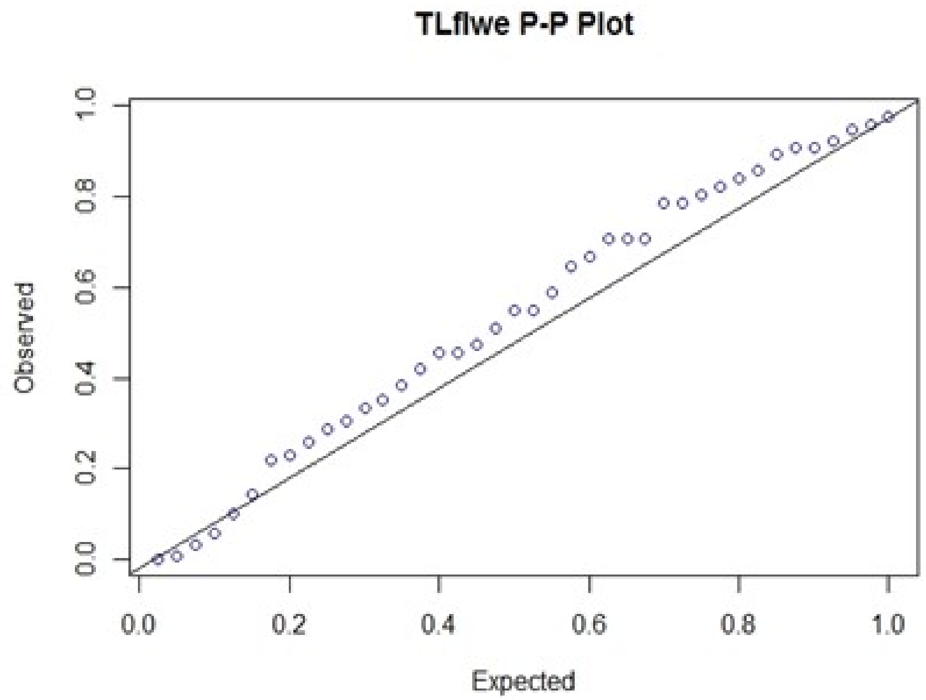

Figure 8: p-p plot for TLFW distribution

The plots in Figure 5 to 8 support the results in Table 1; showing that the TLFW distribution fits the dataset better the other compared distributions.

4. Conclusions

The Topp Leone Flexible Weibull distribution has been successfully defined and studied in this paper. The model is positively skewed and unimodal in shape, its various statistical properties were also obtained. The model is characterized by high flexibility; an application to real life dataset reveals that the model is a strong competitor as it fits better than the Topp Leone Burr XII, Topp Leone Lomax and Exponentiated Generalized Flexible Weibull distributions. The model can also be applied to other real life datasets.

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