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Estimating coefficients for subclasses of meromorphic bi-univalent functions involving the polylogarithm function

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Abstract

In this paper, we introduce a new operator $\Omega_c\ g(z)$ associated with polylogarithm function, applying it on the subclasses $AH_{\Sigma^*_{\mathcal{B}}}(\gamma, k)$ of meromorphic starlike bi-univalent functions of order γ, and $AH_{\tilde{\Sigma}^*_{\mathcal{B}}}(\gamma, k)$ of meromorphic strongly starlike bi-univalent functions of order γ, also we find estimates on the coefficients $|b_0|$ and $|b_1|$ for functions in these subclasses.

Keywords: Analytic functions, univalent functions, Bi-univalent functions, Starlike functions, strongly starlike functions, polylogarithm function, Meromorphic functions and Coefficient estimates.

1. Introduction

Let A be the class of functions of the form:

$$f(x) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. We also denote by S the class of all functions in the normalized analytic function class A which are univalent in Δ. The well-known Koebe one-quarter theorem asserts that the function $f \in S$ has an inverse defined on disc $\Delta_\rho = \{ z \in \mathbb{C} : |z| < \rho \}$, $\left(\rho \geq \frac{1}{4}\right)$. Thus, the inverse of $f \in S$ is a univalent analytic function on the

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disc Δ_ρ. The function $f \in A$ is called bi-univalent in Δ if f^{-1} is also univalent in the whole disc Δ. The class σ of bi-univalent analytic functions was introduced in 1967 by Lewin [13] and he showed that, for every function $f \in \sigma$ of the form (1.1), the second coefficient of f satisfy the inequality $|a_2| < 1 : 51$. Subsequently, Brannan and Clunie [3] improved Lewin's result by showing $|a_2| \ \leq \sqrt{2}$. Later, Netanyahu [15] proved that $\max_{f\in\sigma} |a_2| \ = \frac{4}{3}$. Also, several authors such as Brannan and Taha [4], Taha [20] investigated subclasses of bi-univalent analytic functions and found estimates on the initial coefficients for functions in these subclasses. Ali et al. [2], Frasin and Aouf [6], Srivastava et al.[19], Juma and Aziz [9, 10], also introduced new subclasses of bi-univalent functions and found estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these classes.

Suzeini et al. [8] considered and studied the concept of bi-univalency for classes of meromorphic functions defined on $D = \{\ z \ : \ z \in \ \mathbb{C} \ and \ 1 \ < |z| < \infty\}$. For this purpose they denote by Σ the class of all meromorphic univalent functions g of the form

$$g(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n}, \tag{1.2}$$

defined on the domain D. Since $g \in \Sigma$ is univalent, it has an inverse g^{-1} that satisfy

$$g^{-1}\left(g\left(z\right)\right) = z \qquad \left(z \in D\right),$$

and

$$g\left(g^{-1}\left(\omega\right)\right) = \omega \quad \left(\beta < |\omega| < \infty,\ \beta > 0\right)$$

Furthermore, the inverse function g^{-1} has a series expansion of the form

$$g^{-1}\left(\omega\right) = \omega + \sum_{n=0}^{\infty} \frac{B_n}{\omega^n}, \tag{1.3}$$

where $\beta < |\omega| < \infty$. Analogous to the bi-univalent analytic functions, a function $g \in \Sigma$ is said to be meromorphic bi-univalent if $g^{-1} \in \Sigma$. The class of all meromorphic bi-univalent functions is denoted by $\Sigma_{\mathcal{B}}$. Estimates on the coefficients of meromorphic univalent functions were investigated in the literature; for example, Schiffer [16] obtained the estimate $|b_2| \leq \frac{2}{3}$ for meromorphic univalent functions $g \in \Sigma$ with $b_0 = 0$. In 1971, Duren [5] gave an elementary proof of the inequality $|b_k| \leq \frac{2}{k+1}$ on the coefficient of meromorphic univalent functions $g \in \Sigma$ with $b_n = 0$ for $1 \ \leq n < \frac{k}{2}$. For the coefficients of the inverse of meromorphic univalent functions, Springer [18] proved that

$$|B_3| \leq 1 \ and \ \left|B_3 + \frac{1}{2}B_1^2\right| \leq \frac{1}{2},$$

and conjectured that

$$|B_{2k-1}| \leq \frac{(2k-2)!}{k!\,(k-1)!} \quad (k = 1, 2, 3, \ldots).$$

In 1977, Kubota [12] has proved that the Springer conjecture is true for $k \ = \ 3, 4, 5$ and subsequently Schober [17] obtained sharp bounds for the coefficients B_{2k-1}, $1 \ \leq k \leq 7$, of the inverse of meromorphic univalent functions in D. Recently, Kapoor and Mishra [11] found the coefficient estimates for a class consisting of inverses of meromorphic starlike univalent functions of order γ in D.

Analogous to Liu and Srivastava work [14] and corresponding to a function $\varphi_c(z)$ given by

$$\varphi_k(z) = z^{-2}\mathrm{Li}_k(z) = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{1}{(n+2)^k} z^n, \qquad (k \geq 2) \tag{1.4}$$

where

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$$

which is also known as polylogarithm function. Now we define a function $\psi_k(z): \Sigma \to \Sigma$ as follow:

$$\psi_k(z) = z + \sum_{n=0}^{\infty} \frac{1}{(n+1)^k} \cdot \frac{1}{z^n} \quad (k \in \mathbb{N}), \tag{1.5}$$

which is introduced previously by Alhindi and Darus [1].
we consider a linear operator $\Omega_c\, g(z) : \Sigma \to \Sigma$ which is defined by the following Hadamard product (or convolution):

$$\Omega_k g(z) = \psi_k(z) * g(z) = z + \sum_{n=0}^{\infty} \frac{1}{(n+1)^k} \cdot \frac{b_n}{z^n} \tag{1.6}$$

In the present investigation, certain subclasses of meromorphic bi-univalent functions are introduced and estimates for the coefficients $|b_0|$ and $|b_1|$ of functions in these subclasses are obtained. These coefficients results are obtained by associating the given functions with the functions having positive real part. An analytic function p of the form $p(z) = 1 + c_1 z + c_2 z^2 \ldots$ is called a function with positive real part in Δ if $Re(p(z)) > 0$ for all $z \in \Delta$. The class of all functions with positive real part is denoted by $\mathbf{P}$.
The following lemma for functions with positive real part will be useful in the sequel.

Lemma 1.1. *[7] Theorem 3, p.80. The coefficients c_k of a function $p \in \boldsymbol{P}$ satisfy the sharp inequality $|c_k| \leq 2 \ (k \geq 1)$.*

2. Coefficients Estimates

In this section, certain subclasses like the subclass $AH_{\Sigma^*_{\mathcal{B}}}(\gamma, k)$ of the meromorphic bi-univalent functions associated with the linear operator $\Omega_k g(z)$ are introduced and estimates on the coefficients $|b_0|$ and $|b_1|$ for functions in these subclasses are obtained. The class of all meromorphic starlike bi-univalent functions of order α is denoted by $\Sigma^*_{\mathcal{B}}(\alpha)$.

Definition 2.1. *A function $g(z)$ given by (1.2) is said to be in the subclass $AH_{\Sigma^*_{\mathcal{B}}}(\gamma, k)$ if the following conditions are satisfied:*

$$Re\left(\frac{z\left(\Omega_k g(z)\right)'}{\Omega_k g(z)}\right) > \gamma \qquad (0 \leq \gamma < 1; k = 1, 2, 3, \ldots; z \in D), \tag{2.1}$$

and

$$Re\left(\frac{\omega\left(\Omega_k h(\omega)\right)'}{\Omega_k h(\omega)}\right) > \gamma \quad (0 \leq \gamma < 1; k = 1, 2, 3, \ldots; \omega \in D), \tag{2.2}$$

where the function $h(\omega)$ is the inverse of $g(z)$ given by (1.3).

Theorem 2.2. *Let the function $g(z)$ given by (1.2) be in the subclass $AH_{\Sigma^*_{\mathcal{B}}}(\gamma, k)$. Then*

$$|b_0| \leq 2(1-\gamma) \qquad \textit{and} \qquad |b_1| \leq (1-\gamma)\frac{\sqrt{1 + 4(1-\gamma)^2}}{2^{k-2}}.$$

Proof . Let $g(z)$ be the meromorphic starlike bi-univalent function of order γ given by (1.2). Then

$$\frac{z\left(\Omega_k g\left(z\right)\right)'}{\Omega_k g\left(z\right)} = 1 - \frac{b_0}{z} + \frac{b_0^2 - \frac{b_1}{2^{k-1}}}{z^2} - \frac{b_0^3 - \frac{3b_0 b_1}{2^k} + \frac{b_2}{3^{k-1}}}{z^3} + \dots \quad (z \in D) \tag{2.3}$$

Since $h(\omega) = g^{-1}(\omega)$ is the inverse of $g(z)$ whose series expansion is given in (1.3), and, since

$$\omega = g\left(h\left(\omega\right)\right) = g\left(g^{-1}\left(\omega\right)\right).$$

So, some calculations gives

$$B_0 = -b_0,\ B_1 = -b_1, B_2 = -b_2 - b_0 b_1 \quad and \quad B_3 = -\left(b_3 + 2b_0 b_2 + b_0^2 b_1 + b_1^2\right) \tag{2.4}$$

Using equations of (2.4) in (1.3), shows that the series expansion of the function $g^{-1}(\omega)$ becomes

$$h\left(\omega\right) = g^{-1}\left(\omega\right) = \omega - b_0 - b_1 \frac{1}{\omega} - \left(b_2 + b_0 b_1\right) \frac{1}{\omega^2} - \left(b_3 + 2b_0 b_2 + b_0^2 b_1 + b_1^2\right) \frac{1}{\omega^3} + \dots. \tag{2.5}$$

Using (2.5) we have

$$\frac{\omega\left(\Omega_k h\left(\omega\right)\right)'}{\Omega_k h\left(\omega\right)} = 1 + \frac{b_0}{\omega} + \frac{b_0^2 + \frac{b_1}{2^{k-1}}}{\omega^2} + \frac{b_0^3 + \frac{3b_0 b_1}{2^{k-1}} + \frac{b_2}{3^{k-1}}}{\omega^3} + \dots \quad (\omega \in D) \tag{2.6}$$

Since $g(z)$ is a bi-univalent meromorphic starlike function of order γ, there exist two functions p, q with positive real parts in D of the forms

$$P\left(z\right) = 1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \dots \quad (z \in D), \tag{2.7}$$

and

$$q\left(\omega\right) = 1 + \frac{d_1}{\omega} + \frac{d_2}{\omega^2} + \frac{d_3}{\omega^3} + \dots \quad (\omega \in D) \tag{2.8}$$

such that

$$\frac{z\left(\Omega_k g\left(z\right)\right)'}{\Omega_k g\left(z\right)} = \alpha + (1-\gamma)\, p\left(z\right), \tag{2.9}$$

and

$$\frac{\omega\left(\Omega_k h\left(\omega\right)\right)'}{\Omega_k h\left(\omega\right)} = \alpha + (1-\gamma)\, q\left(\omega\right) \tag{2.10}$$

Use of (2.7) in (2.9) shows that

$$\frac{z\left(\Omega_k g\left(z\right)\right)'}{\Omega_k g\left(z\right)} = 1 + \frac{(1-\gamma)\, c_1}{z} + \frac{(1-\gamma)\, c_2}{z^2} + \frac{(1-\gamma)\, c_3}{z^3} + \dots \quad (z \in D), \tag{2.11}$$

In view of the Equations (2.3) and (2.11), it is easy to see that

$$(1-\gamma)\, c_1 = -b_0 \tag{2.12}$$

and

$$(1-\gamma)\,c_2 = b_0^2 - \frac{b_1}{2^{k-1}} \tag{2.13}$$

Similarly, by making use of (2.6), (2.8) in (2.10) immediately yields

$$(1-\gamma)\,d_1 = b_0 \tag{2.14}$$

and

$$(1-\gamma)\,d_2 = b_0^2 + \frac{b_1}{2^{k-1}} \qquad (k=1,2,3,\dots) \tag{2.15}$$

Equations (2.12) and (2.14) together yields

$$c_1 = -d_1$$

and

$$b_0^2 = \frac{(1-\gamma)^2}{2}\left(c_1^1 + d_1^2\right) \tag{2.16}$$

Since $Re(p(z)) \;>\; 0$ in D, the function $p(\frac{1}{z}) \;\in \mathbf{P}$ and hence the coefficients c_k and similarly the coefficients d_k of the function q satisfy the inequality in Lemma 1 and this immediately yields the following estimates:

$$\left|b_0^2\right| = \frac{(1-\gamma)^2}{2}\left|c_1^1 + d_1^2\right| \leq 4\,(1-\gamma)^2\,.$$

Hence

$$|b_0| \leq 2\,(1-\gamma)\,.$$

Using Equations (2.13) and (2.15) yields

$$b_0^4 - \frac{b_1^2}{2^{2k-2}} = (1-\gamma)^2\,c_2 d_2,$$

or

$$\frac{b_1^2}{2^{2k-2}} = -\,(1-\gamma)^2\,c_2 d_2 + b_0^4.$$

By Lemma Refl1, the estimates $|c_2| = |d_2| \leq 2$ holds. This estimate together with the estimate of b_0 imply that

$$\frac{|b_1^2|}{2^{2k-2}} \leq 4\,(1-\alpha)^2 + 16\,(1-\gamma)^4.$$

Therefore

$$|b_1| \leq (1-\gamma)\,\frac{\sqrt{1+4\,(1-\gamma)^2}}{2^{k-2}}.$$

□

Corollary 2.3. *Let the function $g(z)$ given by (1.2) be in the subclass $N_{\Sigma^*_{\mathcal{B}}}(\alpha,k)$. Then*

$$|b_0| \leq 2\,(1-\gamma) \qquad \textit{and} \qquad |b_1| \leq (1-\gamma)\sqrt{(4\gamma^2 - 8\gamma + 5)} \tag{2.17}$$

The class of all meromorphic strongly starlike bi-univalent functions of order γ is denoted by $\widetilde{\Sigma}^*_{\mathcal{B}}(\gamma)$.

Definition 2.4. *A function $g(z)$ given by (1.2) is said to belong to the subclass $AH_{\widetilde{\Sigma}^*_{\mathcal{B}}}(\gamma, k)$ of bi-univalent strongly starlike meromorphic functions of order α, $0 < \gamma \leq 1$ if the following conditions are satisfied:*

$$\left|\arg\left(\frac{z\left(\Omega_k g\left(z\right)\right)'}{\Omega_k g\left(z\right)}\right)\right| < \frac{\gamma\pi}{2} \qquad \left(0 \leq \gamma < 1; k = 1, 2, 3, \ldots; z \in D\right), \tag{2.18}$$

and

$$\left|\arg\left(\frac{\omega\left(\Omega_k h\left(\omega\right)\right)'}{\Omega_k h\left(\omega\right)}\right)\right| < \frac{\gamma\pi}{2} \qquad \left(0 \leq \gamma < 1; k = 1, 2, 3, \ldots; \omega \in D\right), \tag{2.19}$$

where the function $h(\omega)$ is the inverse of $g(z)$ given by (1.3).

Theorem 2.5. *Let the function $g(z)$ given by (1.2) be in the subclass $AH_{\widetilde{\Sigma}^*_{\mathcal{B}}}(\gamma, k)$. Then*

$$|b_0| \leq 2\gamma \qquad \text{and} \qquad |b_1| \leq \sqrt{3}\left(2^k\gamma^2\right).$$

Proof . Consider the function $g \in AH_{\widetilde{\Sigma}^*_{\mathcal{B}}}(\gamma, k)$. Then, by Definition 2 of the subclass $AH_{\widetilde{\Sigma}^*_{\mathcal{B}}}(\gamma, k)$

$$\frac{z\left(\Omega_k g\left(z\right)\right)'}{\Omega_k g\left(z\right)} = \left(p\left(z\right)\right)^{\alpha}, \tag{2.20}$$

and

$$\frac{\omega\left(\Omega_k h\left(\omega\right)\right)'}{\Omega_k h\left(\omega\right)} = \left(q\left(\omega\right)\right)^{\gamma}, \tag{2.21}$$

where $\frac{z(\Omega_k g(z))'}{\Omega_k g(z)}$ is given by (2.3) and $p(z)$ is given in (2.7) so,

$$\begin{aligned} &1 - \frac{b_0}{z} + \frac{b_0^2 - \frac{b_1}{2^{k-1}}}{z^2} - \frac{b_0^3 - \frac{3b_0b_1}{2^k} + \frac{b_2}{3^{k-1}}}{z^3} + \ldots \\ &= 1 + \frac{\gamma c_1}{z} + \frac{\frac{1}{2}\gamma\left(\gamma - 1\right)c_1^2 + \gamma c_2}{z^2} + \frac{\frac{1}{6}\gamma\left(\gamma - 1\right)\left(\gamma - 2\right)c_1^3 + \gamma\left(\gamma - 1\right)c_1c_2 + \gamma c_3}{z^3} + \ldots. \end{aligned} \tag{2.22}$$

Equating the coefficients in both sides of equation (2.22) we get

$$\gamma c_1 = -b_0 \tag{2.23}$$

and

$$\frac{1}{2}\gamma\left(\gamma - 1\right)c_1^2 + \gamma c_2 = b_0^2 - \frac{b_1}{2^{k-1}} \tag{2.24}$$

Applying $q(\omega)$ from (2.8) and $\frac{\omega(\Omega_k h(\omega))'}{\Omega_k h(\omega)}$ from (2.6) in (2.21) we get

$$\begin{aligned} &1 + \frac{b_0}{\omega} + \frac{b_0^2 + \frac{b_1}{2^{k-1}}}{\omega^2} + \frac{b_0^3 + \frac{3b_0b_1}{2^{k-1}} + \frac{b_2}{3^{k-1}}}{\omega^3} + \ldots \\ &= 1 + \frac{\gamma d_1}{z} + \frac{\frac{1}{2}\gamma\left(\gamma - 1\right)d_1^2 + \gamma d_2}{z^2} + \frac{\frac{1}{6}\gamma\left(\gamma - 1\right)\left(\gamma - 2\right)d_1^3 + \gamma\left(\gamma - 1\right)d_1d_2 + \gamma d_3}{z^3} + \ldots. \end{aligned} \tag{2.25}$$

Equating the coefficients in both sides of equation (2.25) we get

$$\gamma d_1 = b_0 \tag{2.26}$$

and

$$\frac{1}{2}\gamma\left(\gamma-1\right)d_1^2+\gamma d_2 = b_0^2+\frac{b_1}{2^{k-1}} \tag{2.27}$$

Using the Equations (2.23) and (2.26), one gets

$$c_1 = -d_1$$

and

$$2b_0^2 = \gamma^2\left(c_1^2+d_1^2\right)$$

which implies

$$b_0^2 = \frac{\gamma^2}{2}\left(c_1^2+d_1^2\right). \tag{2.28}$$

By Lemma 1.1, $|c_1| = |d_1| \leq 2$ and using them in (2.28), it follows that

$$\left|b_0^2\right| = \frac{\gamma^2}{2}\left|c_1^2+d_1^2\right| \leq b_0^2 = \frac{\gamma^2}{2}\left(\left|c_1^2\right|+\left|d_1^2\right|\right) \leq 4\gamma^2.$$

Hence

$$|b_0| \leq 2\gamma.$$

Equations (2.24) and (2.27) together yield

$$2b_0^4+2\frac{b_1^2}{2^{2k-2}} = \frac{1}{4}\gamma^2\left(\gamma-1\right)^2\left(c_1^4+d_1^4\right)+\gamma^2\left(c_2^2+d_2^2\right)+\gamma^2\left(\gamma-1\right)\left(c_1^2c_2+d_1^2d_2\right). \tag{2.29}$$

In view of (2.28), the previous equation becomes

$$\begin{aligned} b_1^2 = {} & \frac{2^{2k-2}}{8}\gamma^2\left(\gamma-1\right)^2\left(c_1^4+d_1^4\right)+\frac{2^{2k-2}\gamma^2\left(c_2^2+d_2^2\right)}{2}+\frac{2^{2k-2}\gamma^2\left(\gamma-1\right)\left(c_1^2c_2+d_1^2d_2\right)}{2} \\ & -\frac{2^{2k-2}\gamma^4}{8}\left(c_1^4+d_1^4\right)-\frac{2^{2k-2}\gamma^4}{4}c_1^2d_1^2. \end{aligned}$$

Lemma 1.1 again gives the estimates $|c_i| = |d_i| \leq 2$ for $i = 1, 2$, and using these in the above equation immediately yields

$$\left|b_1^2\right| \leq 2^{2k}\gamma^2\left(\gamma-1\right)^2+2^{2k}\gamma^2+2^{2k+1}\gamma^2\left(\gamma-1\right)+2^{2k}\gamma^4+2^{2k}\gamma^4 = 3\left(2^{2k}\gamma^4\right)$$

This shows that

$$|b_1| \leq \sqrt{3}\left(2^k\gamma^2\right).$$

□

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