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Generalized solution of Schrödinger equation with singular potential and initial data

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Abstract

This paper proved the existence and uniqueness of the solution of the Schrödinger equation with singular potential and initial data in the Colombeau algebra $\mathcal{G}_e$.

Keywords: Schrödinger equation, Singular potential, initial data, Colombeau algebra, potentials.
2010 MSC: 46F10; 46S10; 35A27.

1. Introduction

The optimal solution for overcoming the problems that Schwartz theory of distributions is concerned with was offered by Colombeau (1984, 1985) ([1], [2]). He constructed an associative differential algebra of generalized functions $\mathcal{G}(\mathbb{R})$, which contains the space $\mathcal{D}'(\mathbb{R})$ of distributions as subspace and the algebra of $\mathcal{C}^\infty - functions$ as subalgebra. This theory of generalized functions of Colombeau actually generalizes the theory of Schwartz distributions: these new Colombeau generalized functions can be differentiated in the same way as distributions, but where multiplication and other nonlinear operations are concerned. It is significant that the result of these operations always exists in this algebra as Colombeau generalized functions. These new generalized functions are very much related to the distributions, in the sense that their definition may be considered as a natural evolution of the Schwartz definition of distributions.

The notion of 'association' in $\mathcal{G}(\mathbb{R})$ is a faithful generalization of the equality of distributions, and again enables us to interpret results in terms of distributions.

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Due to all these properties, Colombeau's theory has found extensive applications in different natural sciences and engineering, especially in fields where products of distributions with coinciding singularities are considered.

This paper solves the following nonlinear Schödinger equation with singular potential and initial data in Colombeau algebra of generalized functions $\mathcal{G}^e$ which allows multiplication of distributions and solution of nonlinear problems with singularities and proved the association of the solution.

$$\begin{cases} \frac{1}{i}\partial_t u(t,x) - \Delta u(t,x) + v(x)u(t,x) = 0 \\ v(x) = \delta(x), \quad u(0,x) = \delta(x) \end{cases}$$

This paper is divided into five parts. After the introduction, we give some basic preliminaries such as notations and definitions of the objects we will work with. We also introduce different spaces of Colombeau algebra of generalized functions. In the third section, we proved the existence and uniqueness of the solution of Schrödinger equation with singular potential and initial data in the Colombeau algebra $\mathcal{G}$. In the final section, we study the association.

2. Preliminaries

Similar to [2], we use the following notations:

$$\mathcal{A}_q = \left\{ \varphi \in \mathcal{D}(\mathbb{R}^n) / \int_{\mathbb{R}^n} \varphi(x)dx = 1, \int_{\mathbb{R}^n} x^\alpha \varphi(x)dx = 0 \quad \text{for} \quad 1 \le |\alpha| \le q \right\}$$

$q = 1, 2, \ldots$

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right) \quad \text{for} \quad \varphi \in \mathcal{D}(\mathbb{R}^n)$$

and $\mathcal{E}(\mathbb{R}^n) = \{u : \mathcal{A}_1 \times \mathbb{R}^n \to \mathbb{C}/$ with $u(\varphi, x)$ is $\mathcal{C}^\infty$ to the second variable $x\}$

$$u(x, \varphi_\varepsilon) = u_\varepsilon(x) \quad \forall \varphi \in \mathcal{A}_1$$

$$\begin{aligned} \mathcal{E}_M(\mathbb{R}^n) = \{ & (u_\varepsilon)_{\varepsilon>0} \subset \mathcal{E}(\mathbb{R}^n) / \forall K \subset\subset \mathbb{R}^n, \forall \alpha \in \mathbb{N}_0^n, \exists N \in \mathbb{N} \quad \text{such that} \\ & \sup_{x\in K} |D^\alpha u_\varepsilon(x)| = \mathcal{O}\left(\varepsilon^{-N}\right) \text{ as } \varepsilon \to 0\} \\ \mathcal{N}(\mathbb{R}^n) = \{ & (u_\varepsilon)_{\varepsilon>0} \subset \mathcal{E}(\mathbb{R}^n) / \forall K \subset\subset \mathbb{R}^n, \forall \alpha \in \mathbb{N}_0^n, \forall p \in \mathbb{N} \quad \text{such that} \\ & \sup_{x\in K} |D^\alpha u_\varepsilon(x)| = \mathcal{O}(\varepsilon^p) \quad \text{as} \quad \varepsilon \to 0\} \end{aligned}$$

The Colombeau algebra is defined as a factor set $\mathcal{G}(\mathbb{R}^n) = \mathcal{E}_M(\mathbb{R}^n)/\mathcal{N}(\mathbb{R}^n)$, where the elements of the set $\mathcal{E}_M(\mathbb{R}^n)$ are moderate while the elements of the set $\mathcal{N}(\mathbb{R}^n)$ are negligible.

We denote: [7]

$$\begin{aligned} \mathcal{E}_M^e(\mathbb{R}) = \{ & (u_\varepsilon)_{\varepsilon>0} \subset \mathcal{E}(\mathbb{R}) / \forall K \subset\subset \mathbb{R}, \forall \alpha \in \mathbb{R}_+ \cup \{0\}, \exists N \in \mathbb{N} \quad \text{such that} \\ & \sup_{x\in K} |D^\alpha u_\varepsilon(x)| = \mathcal{O}\left(\varepsilon^{-N}\right) \text{ as } \varepsilon \to 0\} \\ \mathcal{N}^e(\mathbb{R}) = \{ & (u_\varepsilon)_{\varepsilon>0} \subset \mathcal{E}(\mathbb{R}) / \forall K \subset\subset \mathbb{R}, \forall \alpha \in \mathbb{R}_+ \cup \{0\}, \forall p \in \mathbb{N} \quad \text{such that} \\ & \sup_{x\in K} |D^\alpha u_\varepsilon(x)| = \mathcal{O}(\varepsilon^p) \quad \text{as} \quad \varepsilon \to 0\}. \end{aligned}$$

The extended Colombeau algebra of generalized functions is the factor set:

$$\mathcal{G}^e(\mathbb{R}) = \mathcal{E}_M^e(\mathbb{R}) / \mathcal{N}^e(\mathbb{R}).$$

A fractional integral is defined by: [8]

$$I^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)d\tau \quad \alpha>0.$$

The fractional derivative of order $\alpha>0$ in the Caputo sense is defined by [8],

$$D^{\alpha}f(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t\frac{f^{(m)}(\tau)d\tau}{(t-\tau)^{\alpha+1-m}} \quad , \quad m-1<\alpha<m.$$

Let (f_ε) be a representative of $F\in\mathcal{G}$, then

$$D^{\alpha}f_\varepsilon(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{f'(\tau)}{(t-\tau)^{\alpha}}d\tau \quad 0<\alpha<1$$

$$\begin{aligned}\sup_{t\in[0,T]}|D^{\alpha}f_\varepsilon(t)| &\leq \frac{1}{\Gamma(1-\alpha)}\sup_{t\in[0,T]}\left|\int_0^t\frac{f'(\tau)d\tau}{(t-\tau)^{\alpha}}\right|\\
&\leq \frac{1}{\Gamma(1-\alpha)}\|f'\|_{L^\infty([0,T])}\sup_{t\in[0,T]}\int_0^t\frac{d\tau}{(t-\tau)^{\alpha}}d\tau\\
&\leq \frac{1}{\Gamma(1-\alpha)}\varepsilon^{-N}\frac{T^{1-\alpha}}{1-\alpha}\\
&\leq C_{\alpha,T}\varepsilon^{-N}.\end{aligned}$$

In general, from [8], for $m-1<\alpha<m$, we have

$$\begin{aligned}\sup_{t\in[0,T]}|D^{\alpha}f_\varepsilon(t)| &\leq \frac{1}{\Gamma(m-\alpha)}\sup_{t\in[0,T]}\int_0^t\frac{\left|f^{(m)}(\tau)\right|}{(t-\tau)^{\alpha+1-m}}d\tau\\
&\leq \frac{1}{\Gamma(m-\alpha)}\left\|f^{(m)}\right\|_{L^\infty([0,T])}\sup_{t\in[0,T]}\int_0^t\frac{1}{(t-\tau)^{\alpha+1-m}}d\tau\\
&\leq \frac{1}{\Gamma(m-\alpha)}\varepsilon^{-N}\frac{T^{m-\alpha}}{m-\alpha}\\
&\leq C_{\alpha,T}\varepsilon^{-N}.\end{aligned}$$

The constant $C_{\alpha,T}$ depends on two parameters α and T.

Definition 1. *[2] Let $G_1,G_2\in\mathcal{G}(\mathbb{R}^n)$ and $G_{1,\varepsilon},G_{2,\varepsilon}$ their representatives respectively. We say that $G_1,G_2\in\mathcal{G}(\mathbb{R}^n)$ are associated, and we write $G_1\approx G_2$, if for every $\varphi\in\mathcal{D}(\mathbb{R}^n)$,*

$$\lim_{\varepsilon\to0}\int_{\mathbb{R}^n}(G_{1,\varepsilon}-G_{2,\varepsilon})\varphi(x)dx=0.$$

3. Existence and uniqueness

Consider the nonlinear Schrödinger equation with singular potential and initial data:

$$\begin{cases}\frac{1}{i}\partial_t u(t,x)-\Delta u(t,x)+v(x)u(t,x)=0\\ v(x)=\delta(x), \quad u(0,x)=\delta(x)\end{cases} \tag{3.1}$$

For the Dirac measure, we will apply regularization:

$$v_\varepsilon(x)=\delta_\varepsilon(x)=(\phi_\varepsilon(x))=|\ln\varepsilon|^{cn}\phi\left(x|\ln\varepsilon|^{c}\right); \quad c>0$$
$$x\in\mathbb{R}^n,\phi\in\mathbb{A}_1,\phi(x)\geq0.$$

For the initial data, we use

$$u_{0,\varepsilon}(x)=|\ln\varepsilon|^{an}\phi\left(x|\ln\varepsilon|^{a}\right), \quad a>0, \quad x\in\mathbb{R}^n, \quad \phi\in\mathbb{A}_1, \quad \phi(x)\geq0.$$

3.1. Existence and uniqueness in the Colombeau algebra

Theorem 1. *The regularization of equation (3.1), defined as*

$$\begin{cases} \frac{1}{i}\partial_t u_\varepsilon(t,x) - \Delta u_\varepsilon(t,x) + v_\varepsilon(x)u_\varepsilon(t,x) = 0 \\ v_\varepsilon(x) = \delta_\varepsilon(x), \quad u_{0,\varepsilon}(x) = \delta_\varepsilon(x) \end{cases} \tag{3.2}$$

where v_ε and $u_{0,\varepsilon}$ are regularized of v and u_0, respectively. Then, the problem (3.2) has a unique solution in $\mathcal{G}\left(\mathbb{R}^+ \times \mathbb{R}^n\right)$.

Proof . From [4], the integral solution of the equation (3.2) is

$$u_\varepsilon(t,x) = \int_{\mathbb{R}^n} S_n(t,x-y)u_{0,\varepsilon}(y)dy + \int_0^t \int_{\mathbb{R}^n} S_n(t-\tau,x-y)v_\varepsilon(y)u_\varepsilon(\tau,y)dyd\tau,$$

where $S_n(t,x)$ is the heat kernel. Then

$$\|u_\varepsilon(t,.)\|_{L^\infty(\mathbb{R}^n)} \le \|S_n(t,x-.)\|_{L^1} \|u_{0,\varepsilon}\|_{L^\infty(\mathbb{R}^n)} + \int_0^t \|S_n(t-\tau,x-.)\|_{L^1} \|v_\varepsilon(.)\|_{L^\infty} \|u_\varepsilon(\tau,.)\|_{L^\infty(\mathbb{R}^n)} d\tau$$
$$\|u_\varepsilon(t,.)\|_{L^\infty(\mathbb{R}^n)} \le C\|u_{0,\varepsilon}\|_{L^\infty(\mathbb{R}^n)} + C\|v_\varepsilon(.)\|_{L^\infty(\mathbb{R}^n)} \int_0^t \|u_\varepsilon(\tau,.)\|_{L^\infty(\mathbb{R}^n)} d\tau.$$

By Gronwall inequality, we have

$$\|u_\varepsilon(t,.)\|_{L^\infty(\mathbb{R}^n)} \le C|\ln\varepsilon|^{an} \exp\left(CT|\ln\varepsilon|^{bn}\right)$$

Then there exists $N>0$, such that

$$\|u_\varepsilon(t,.)\|_{L^\infty(\mathbb{R}^n)} \le C\varepsilon^{-N}.$$

For the first derivative to $x_i, i \in \{1,\ldots,n\}$, we obtain

$$\partial_{x_i} u_\varepsilon = \int_{\mathbb{R}^n} S_n(t,x-y)\partial_{y_i}u_{0,\varepsilon}(y)dy + \int_0^t \int_{\mathbb{R}^n} S_n(t-\tau,x-y)\left(\partial_{y_i}v_\varepsilon(y)u_\varepsilon(\tau,y) + v_\varepsilon(y)\partial_{y_i}u_\varepsilon(\tau,y)\right)dyd\tau$$
$$\|\partial_{x_i}u_\varepsilon(t,.)\|_{L^\infty(\mathbb{R}^n)} \le \|S_n(t,x-.)\|_{L^1}\|\partial_{y_i}u_{0,\varepsilon}\|_{L^\infty(\mathbb{R}^n)} + \int_0^t \|S_n(t-\tau,x-.)\|_{L^1}$$
$$\times \left(\|\partial_{y_i}v_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\|u_\varepsilon\|_{L^\infty} + \|v_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\|\partial_{y_i}u_\varepsilon(\tau,.)\|_{L^\infty(\mathbb{R}^n)}\right)d\tau$$

$$\|\partial_{x_i}u_\varepsilon(t,.)\|_{L^\infty(\mathbb{R}^n)} \le C|\ln\varepsilon|^{a(n+1)} + C\int_0^t |\ln\varepsilon|^{b(n+1)}||u_\varepsilon||_{L^\infty} + |\ln\varepsilon|^{bn}||\partial_{y_i}u_\varepsilon(\tau,.)||_{L^\infty(\mathbb{R}^n)}d\tau$$

$$\|\partial_{x_i}u_\varepsilon(t,.)\|_{L^\infty(\mathbb{R}^n)} \le C\left(|\ln\varepsilon|^{a(n+1)} + T|\ln\varepsilon|^{b(n+1)}||u_\varepsilon||_{L^\infty}\right) + C|\ln\varepsilon|^{bn}\int_0^t ||\partial_{y_i}u_\varepsilon(\tau,.)||_{L^\infty(\mathbb{R}^n)}d\tau.$$

By Gronwall inequality, we have:

$$\|\partial_{x_i}u_\varepsilon(t,.)\|_{L^\infty(\mathbb{R}^n)} \le C\left(|\ln\varepsilon|^{a(n+1)} + T|\ln\varepsilon|^{b(n+1)}||u_\varepsilon||_{L^\infty}\right)\exp\left(CT|\ln\varepsilon|^{bn}\right).$$

By the previous step there exists $N>0$, such that

$$\|\partial_{x_i}u_\varepsilon(t,.)\|_{L^\infty(\mathbb{R}^n)} \le C\varepsilon^{-N}.$$

For the second derivative for $y_i, j \in \{1,\ldots,n\}$, we obtain

$$\partial_{x_i}\partial_{x_j}u_\varepsilon(t,x) = \int_{\mathbb{R}^n} S_n(t,x-y)\left(\partial_{y_i}\partial_{y_j}u_{0,\varepsilon}(y)dy + \int_0^t \int_{\mathbb{R}^n} S_n(t-\tau,x-y)\left(\partial_{y_i}\partial_{y_j}v_\varepsilon(y)u_\varepsilon(\tau,y)\right.\right.$$

$$+\partial_{y_j} v_\varepsilon(y)\partial_{y_i} u_\varepsilon(\tau, y) + \partial_{y_i} v_\varepsilon(y)\partial_{y_j} u_\varepsilon(\tau, y) + v_\varepsilon(y)\partial_{y_i}\partial_{y_j} u_\varepsilon(\tau, y))dyd\tau.$$

By Gronwall inequality:

$$\left\|\partial_{x_i}\partial_{x_j} u_\varepsilon(t,.)\right\|_{L^\infty(\mathbb{R}^n)} \leq C(|\ln\varepsilon|^{a(n+2)} + |\ln\varepsilon|^{b(n+1)} \left\|u_\varepsilon\right\|_{L^\infty} + |\ln\varepsilon|^{b(n+1)} \left\|\partial_{y_i} u_\varepsilon\right\|_{L^\infty}$$

$$+|\ln\varepsilon|^{b(n+1)}\|\partial_{y_j} u_\varepsilon\|_{L^\infty}) \exp\left(CT|\ln\varepsilon|^{bn}\right).$$

By the previous step there exists $N > 0$, such that

$$\left\|\partial_{x_i}\partial_{x_j} u_\varepsilon(t,.)\right\|_{L^\infty(\mathbb{R}^n)} \leq C\varepsilon^{-N}.$$

For uniqueness.

Suppose that there exist two solutions $u_{1,\varepsilon}(t,.), u_{2,\varepsilon}(t,.)$ to problem (3.2), then:

$$\frac{1}{i}\partial_t \left(u_{1,\varepsilon}(t,x) - u_{2,\varepsilon}(t,x)\right) - \Delta\left(u_{1,\varepsilon}(t,x) - u_{2,\varepsilon}(t,x)\right) + v_\varepsilon(x)\left(u_{1,\varepsilon}(t,x) - u_{2,\varepsilon}(t,x)\right)$$

$$= N_\varepsilon(t,x) u_{1,\varepsilon}(0,x) - u_{2,\varepsilon}(0,x) = N_{0,\varepsilon}(x)$$

Then

$$u_{1,\varepsilon}(t,x) - u_{2,\varepsilon}(t,x) = \int_{\mathbb{R}^n} S_n(t, x-y) N_{0,\varepsilon}(y)dy + \int_0^t \int_{\mathbb{R}^n} S_n(t-\tau, x-y) v_\varepsilon(y)\left(u_{1\varepsilon}(\tau,y) - u_{2\varepsilon}(\tau,y)\right) dyd\tau$$

$$+ \int_0^t \int_{\mathbb{R}^n} S_n(t-\tau, x-y) N_\varepsilon(\tau, y) dyd\tau$$

$$\|u_{1,\varepsilon}(t,.) - u_{2,\varepsilon}(t,.)\|_{L^\infty}(\mathbb{R}^n) \leq \|S_n(t, x-.)\|_{L^1} \|N_{0,\varepsilon}(y)\|_{L^\infty(\mathbb{R}^n)}$$

$$+ \|S_n(t, x-.)\|_{L^1} \int_0^t \|v_\varepsilon(.)\|_{L^\infty(\mathbb{R}^n)} \|u_{1\varepsilon}(\tau,.) - u_{2\varepsilon}(\tau,.)\|_{L^\infty(\mathbb{R}^n)} d\tau + \|S_n(t, x-.)\|_{L^1} \|N_\varepsilon\|_{L^\infty}$$

$$\|u_{1,\varepsilon}(t,.) - u_{2,\varepsilon}(t,.)\|_{L^\infty(\mathbb{R}^n)} \leq C\left(\|N_{0,\varepsilon}(y)\|_{L^\infty(\mathbb{R}^n)} + \|N_\varepsilon\|_{L^\infty}\right)$$

$$+C\|v_\varepsilon(\cdot)\|_{L^\infty(\mathbb{R}^n)} \int_0^t \|u_{1\varepsilon}(\tau,.) - u_{2\varepsilon}(\tau,.)\|_{L^\infty(\mathbb{R}^n)} d\tau$$

By Gronwall inequality:

$$\|u_{1,\varepsilon}(t,.) - u_{2,\varepsilon}(t,.)\|_{L^\infty(\mathbb{R}^n)} \leq C\left(\|N_{0,\varepsilon}(y)\|_{L^\infty(\mathbb{R}^n)} + \|N_\varepsilon\|_{L^\infty}\right) \times \exp\left(CT\|v_\varepsilon(.)\|_{L^\infty(\mathbb{R}^n)}\right)$$

Then

$$\|u_{1,\varepsilon}(t,.) - u_{2,\varepsilon}(t,.)\|_{L^\infty(\mathbb{R}^n)} \leq C\varepsilon^q \quad \forall q.$$

Then, the problem (3.2) has a unique solution in $\mathcal{G}\left(\mathbb{R}^+ \times \mathbb{R}^n\right)$. $\square$

3.2. Existence and uniqueness in the extension of Colombeau algebra

Theorem 2. *The regularization of equation (3.1), defined as :*

$$\begin{cases} \frac{1}{i}\partial_t u_\varepsilon(t,x) - \Delta u_\varepsilon(t,x) + v_\varepsilon(x)u_\varepsilon(t,x) = 0 \\ v_\varepsilon(x) = \delta_\varepsilon(x), \quad u_{0,\varepsilon}(x) = \delta_\varepsilon(x) \end{cases} \tag{3.3}$$

where v_ε and $u_{0,\varepsilon}$ are regularized of v and u_0, respectively.
Then, the problem (3.3) has a unique solution in $\mathcal{G}^e(\mathbb{R}^+ \times \mathbb{R}^n)$.

Proof . We shall prove only the fractional part since the entire part is already proved in theorem (1).
Consider the fractional derivative $D^\alpha; 0 < \alpha < 1$, without loss of generality. The same holds for $m-1 < \alpha < m$; $m \in \mathbb{N}$.
Take the fractional derivative to the spatial variable to equation (3.3):

$$\begin{aligned} D^\alpha(u_\varepsilon(t,x)) &= \int_{\mathbb{R}^n} S_n(t,x-y)D^\alpha u_{0,\varepsilon}(y)dy + \int_0^t\int_{\mathbb{R}^n} S_n(t-\tau,x-y)D^\alpha v_\varepsilon(y)u_\varepsilon(\tau,y)dyd\tau \\ &+ \int_0^t\int_{\mathbb{R}^n} S_n(t-\tau,x-y)v_\varepsilon(y)D^\alpha u_\varepsilon(\tau,y)dyd\tau \end{aligned}$$

$$\begin{aligned} \|D^\alpha(u_\varepsilon(t,.))\|_{L^\infty(\mathbb{R}^n)} \le{}& \|S_n(t,x-.)\|_{L^1}\|D^\alpha u_{0,\varepsilon}(.)\|_{L^\infty(\mathbb{R}^n)} \\ &+ \|S_n(t-\tau,x-.)\|_{L^1}\int_0^t \|D^\alpha v_\varepsilon(\cdot)\|_{L^\infty(\mathbb{R}^n)}\|u_\varepsilon(\tau,.)\|_{L^\infty(\mathbb{R}^n)}d\tau \\ &+ \|S_n(t-\tau,x-.)\|_{L^1}\int_0^t \|v_\varepsilon(y)\|_{L^\infty(\mathbb{R}^n)}\|D^\alpha u_\varepsilon(\tau,.)\|_{L^\infty(\mathbb{R}^n)}d\tau \end{aligned}$$

$$\begin{aligned} \|D^\alpha(u_\varepsilon(t,))\|_{L^\infty(\mathbb{R}^n)} \le{}& C\left(\|D^\alpha u_{0,\varepsilon}(\cdot)\|_{L^\infty(\mathbb{R}^n)} + T\|D^\alpha v_\varepsilon(\cdot)\|_{L^\infty(\mathbb{R}^n)}\|u_\varepsilon\|_{L^\infty}\right) \\ &+ C\|v_\varepsilon(.)\|_{L^\infty(\mathbb{R}^n)}\int_0^t\|D^\alpha u_\varepsilon(\tau,)\|_{L^\infty(\mathbb{R}^n)}d\tau. \end{aligned}$$

By Gronwall inequality

$$\begin{aligned} \|D^\alpha(u_\varepsilon(t,.))\|_{L^\infty(\mathbb{R}^n)} \le{}& C\left(\|D^\alpha u_{0,\varepsilon}(\cdot)\|_{L^\infty(\mathbb{R}^n)} + T\|D^\alpha v_\varepsilon(\cdot)\|_{L^\infty(\mathbb{R}^n)}\|u_\varepsilon\|_{L^\infty}\right) \\ &\times \exp\left(CT\|v_\varepsilon(\cdot)\|_{L^\infty(\mathbb{R}^n)}\right) \quad (3) \end{aligned}$$

by theorem (1) and the estimate (3):

$$\|D^\alpha(u_\varepsilon(t,))\|_{L^\infty(\mathbb{R}^n)} \le C\left(C_{\alpha,T}|\ln\varepsilon|^{a(n+1)} + TC_{\alpha,T}|\ln\varepsilon|^{b(n+1)}||u_\varepsilon||_{L^\infty}\right)\exp\left(CT||v_\varepsilon(\cdot)\|_{L^\infty(\mathbb{R}^n)}\right)$$

Then there exists $N > 0$, such that:

$$\|D^\alpha(u_\varepsilon(t,.))\|_{L^\infty(\mathbb{R}^n)} \le C\varepsilon^{-N}$$

It follows moderation for the fractional derivatives in the space $\mathcal{G}^e(\mathbb{R}^+ \times \mathbb{R}^n)$.

For uniqueness,

Take D^α,$0<\alpha<1$.

$D^\alpha\left(u_{1,\varepsilon}(t,x)-u_{2,\varepsilon}(t,x)\right)=\int_{\mathbb{R}^n}S_n(t,x-y)D^\alpha N_{0,\varepsilon}(y)dy$

$$+\int_0^t\int_{\mathbb{R}^n}S_n(t-\tau,x-y)D^\alpha v_\varepsilon(y)\left(u_{1\varepsilon}(\tau,y)-u_{2\varepsilon}(\tau,y)\right)dyd\tau$$

$$+\int_0^t\int_{\mathbb{R}^n}S_n(t-\tau,x-y)v_\varepsilon(y)D^\alpha\left(u_{1\varepsilon}(\tau,y)-u_{2\varepsilon}(\tau,y)\right)dyd\tau$$

$$+\int_0^t\int_{\mathbb{R}^n}S_n(t-\tau,x-y)D^\alpha N_\varepsilon(\tau,y)dyd\tau$$

$$+\left\|S_n(t-\tau,x-.)\right\|_{L^1}\int_0^t\left\|D^\alpha N_\varepsilon(\tau,.)\right\|_{L^\infty(\mathbb{R}^n)}d\tau$$

$||D^\alpha\left(u_{1,\varepsilon}(t,.)-u_{2,\varepsilon}(t,.)\right)||_{L^\infty(\mathbb{R}^n)}\le C||N_{0,\varepsilon}||_{L^\infty(\mathbb{R}^n)}$

$$+T||D^\alpha v_\varepsilon(.)||_{L^\infty(\mathbb{R}^n)}||u_{1\varepsilon}-u_{2\varepsilon}||_{L^\infty}d\tau+||D^\alpha N_\varepsilon||_{L^\infty}$$

$$+C\left\|v_\varepsilon\right\|_{L^\infty(\mathbb{R}^n)}\int_0^t\left\|D^\alpha\left(u_{1\varepsilon}(\tau,.)-u_{2\varepsilon}(\tau,.)\right)\right\|_{L^\infty(\mathbb{R}^n)}d\tau$$

By Gronwall inequality

$$\begin{aligned}\left\|D^\alpha\left(u_{1,\varepsilon}(t,.)-u_{2,\varepsilon}(t,.)\right)\right\|_{L^\infty}\le{}& C\left(N_{0,\varepsilon}\left\|_{L^\infty(\mathbb{R}^n)}+T\right\|D^\alpha v_\varepsilon(.)\left\|_{L^\infty(\mathbb{R}^n)}\right\|u_{1\varepsilon}-u_{2\varepsilon}\left\|_{L^\infty}d\tau+\right\|D^\alpha N_\varepsilon\|_{L^\infty}\right)\\ &\times\exp\left(CT\left\|v_\varepsilon\right\|_{L^\infty(\mathbb{R}^n)}\right)\end{aligned}$$

By theorem (1)

$$D^\alpha\left(u_{1,\varepsilon}(t,.)-u_{2,\varepsilon}(t,.)\right)\|_{L^\infty(\mathbb{R}^n)}\le C\varepsilon^q,\quad\forall q$$

□

4. Association

Let w_1 be a solution to the problem:

$$\begin{cases}\frac{1}{i}\partial_t w_1(t,x)-\triangle w_1(t,x)=0\\ w_1(0,x)=\delta(x)\end{cases}$$

And w_2 be a solution of the problem:

$$\begin{cases}\frac{1}{i}\partial_t w_2(t,x)-\Delta w_2(t,x)+v(x)w_{2,\varepsilon}(t,x)=0\\ v(x)=\delta(x),\quad w_2(0,x)=0\end{cases}$$

Proposition 1. *The generalized solution u of the problem (3.2) is associated with w_1+w_2.*

Proof . Let $w_{1,\varepsilon}$ be the classical solution to:

$$\begin{gathered}\frac{1}{i}\partial_t w_{1,\varepsilon}(t,x)-\Delta w_{1,\varepsilon}(t,x)=0\\ w_{1,\varepsilon}(0,x)=\delta_\varepsilon(x)\end{gathered}$$

And $w_{2,\varepsilon}$ the classical solution to:

$$\frac{1}{i}\partial_t w_{2,\varepsilon}(t,x)-\Delta w_2(t,x)+v_\varepsilon(x)\left(w_{2,\varepsilon}(t,x)+m(t,x)\right)=0$$

$$v_\varepsilon(x)=\delta(x),\quad w_{2,\varepsilon}(0,x)=0.$$

Then

$$\frac{1}{i}\partial_t\left(u_\varepsilon-w_{1,\varepsilon}-w_{2,\varepsilon}\right)-\Delta(u_\varepsilon(t,x)-w_{1,\varepsilon}(t,x)-w_{2,\varepsilon}(t,x))+v_\varepsilon\left(u_\varepsilon(t,x)-w_{2,\varepsilon}(t,x)-m(t,x)\right)=0$$

$$u_\varepsilon(0,x)-w_{1,\varepsilon}(0,x)-w_{2,\varepsilon}(0,x)=0$$

$$(u_\varepsilon(t,x)-w_{1,\varepsilon}(t,x)-w_{2,\varepsilon}(t,x))=\int_0^t\int_{\mathbb{R}^n}S_n(t-\tau,x-y)v_\varepsilon(y)(u_\varepsilon(\tau,y)-w_{2,\varepsilon}(\tau,y)-m(\tau,y))dyd\tau$$

$$(u_\varepsilon(t,x)-w_{1,\varepsilon}(t,x)-w_{2,\varepsilon}(t,x))=\int_0^t\int_{\mathbb{R}^n}S_n(t-\tau,x-y)v_\varepsilon(y)\left(u_\varepsilon(\tau,y)-w_{1,\varepsilon}(\tau,y)-w_{2,\varepsilon}(\tau,y)\right)dyd\tau$$

$$+\int_0^t\int_{\mathbb{R}^n}S_n(t-\tau,x-y)v_\varepsilon(y)\left(w_{1,\varepsilon}(\tau,y)-m(\tau,y)\right)dyd\tau$$

$$||u_\varepsilon(t,.)-w_{1,\varepsilon}(t,.)-w_{2,\varepsilon}(t,.)||_{L^\infty(\mathbb{R}^n)}\le\int_0^t||S_n(t-\tau,x-.)||_{L^1}||v_\varepsilon(\cdot)||_{L^\infty(\mathbb{R}^n)}||(w_{1,\varepsilon}(\tau,.)-m(\tau,.))||_{L^\infty}d\tau$$

$$+\int_0^t||S_n(t-\tau,x-.)||_{L^1}||v_\varepsilon(.)||_{L^\infty(\mathbb{R}^n)}||u_\varepsilon(\tau,.)-w_{1,\varepsilon}(\tau,.)-w_{2,\varepsilon}(\tau,.)||_{L^\infty(\mathbb{R}^n)}d\tau$$

$$||u_\varepsilon(t,.)-w_{1,\varepsilon}(t,.)-w_{2,\varepsilon}(t,.)||_{L^\infty(\mathbb{R}^n)}\le C||v_\varepsilon(.)||_{L^\infty(\mathbb{R}^n)}\int_0^t||w_{1,\varepsilon}(\tau,.)-m(\tau,.)||_{L^\infty(\mathbb{R}^n)}d\tau$$

$$+C||v_\varepsilon(.)||_{L^\infty(\mathbb{R}^n)}\int_0^t||u_\varepsilon(\tau,.)-w_{1,\varepsilon}(\tau,.)-w_{2,\varepsilon}(\tau,.)||_{L^\infty(\mathbb{R}^n)}d\tau$$

By Gronwall inequality

$$\|u_\varepsilon(t,.)-w_{1,\varepsilon}(t,.)-w_{2,\varepsilon}(t,.)\|_{L^\infty(\mathbb{R}^n)}\le\left[C\left\|v_\varepsilon(\cdot)\right\|_{L^\infty(\mathbb{R}^n)}\int_0^t\left\|(w_{1,\varepsilon}(\tau,.)-m(\tau,.))\right\|_{L^\infty(\mathbb{R}^n)}d\tau\right]$$

$$\times\exp\left(CT\left\|v_\varepsilon(\cdot)\right\|_{L^\infty(\mathbb{R}^n)}\right)$$

By applying the limit, we have

$$u\approx w_1+w_2$$

This completes the proof of the proposition. □

5. Conclusion:

In this work, we have transformed the problem of Schrödinger in colombeau algebra and proved the existence and uniqueness of the solution to this problem. This is significant because its a fundamental equation in quantum mechanics. It describes the evolution over time of a massive non relativistic particle, and thus fulfils the same role as the fundamental relation of dynamics in classical mechanics.

As an example of what we demonstrated earlier, we will look at the description of electronic transport in semiconductor devices of nanometric size (MOSFET, RTD, waveguides, ...). These devices are the essential components of today's electronics industry. Because of their small size, reaching nanometric scales, quantitative effects begin to play an important role, such as effect, interference, quantification, etc. Classical models (Newton's equation among others) are no longer valid and the quantum approach (Schrödinger's equation) becomes necessary. In this approach, Especially when applying very high potential from what we know (v=δ), the evolution of particles (electrons or protons) in an electric field can be written using the "Schödinger equation with singular potential and initial data".

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