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Jackknifed Liu-type estimator in the Conway-Maxwell Poisson regression model

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Abstract

Modelling of count data has been of extreme interest to researchers. However, in practice, count data is often identified with overdispersion or underdispersion. The Conway Maxwell Poisson regression model (CMPRE) has been proven powerful in modelling count data with a wide range of dispersion. In regression modeling, it is known that multicollinearity negatively affects the variance of the maximum likelihood estimator. To address this problem, shrinkage estimators, such as Liu and Liu-type estimators have been consistently verified to be attractive to decrease the effects of multicollinearity. However, these shrinkage estimators are considered biased estimators. In this study, the jackknife approach and its modified version are proposed for modeling count data with CMPRE. These two estimators are proposed to reduce the effects of multicollinearity and the biasedness of using the Liu-type estimator simultaneously. The results of Monte Carlo simulation and real data recommend that the proposed estimators were significant improvement relative to other competitor estimators, in terms of absolute bias and mean squared error with superiority to the modified jackknifed Liu-type estimator.

Keywords: Multicollinearity, Liu-type estimator, Conway-Maxwell-Poisson regression model, Jackknife estimator, Monte Carlo simulation.

1. Introduction

The count response variable is widely included in modeling some real data problems, such as social, automobile insurance claims, healthcare economics, physical sciences, and medical science

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[1, 2]. Specifically, count data regression model is used when the response variable under the study is discrete distributions representing counts and proportions [3, 4].

Consequently, the Poisson regression model is one of the most models that used in modeling count data. However, it assumes that the equidispersion property in which the variance which is a measure of dispersion is equal to the mean for Poisson distribution. This property is often not hold in real data resulting the incapability of fitting the Poisson regression model [5, 6, 7].

The Conway–Maxwell–Poisson (CMP) distribution which is introduced by Conway and Maxwell in 1962 [8] is a great tool to overcome the equidispersion issue. This is because CMP can model a wide range of dispersion. In addition, CMP belongs to an exponential family [9].

As in linear regression model, in contact with the Conway–Maxwell–Poisson regression model, it is supposed that no correlation were among the explanatory variables. In repetition, conversely, this assumption repeatedly not embraces, which directs to the problem of multicollinearity. In the attendance of multicollinearity, when estimating the regression coefficients for Conway–Maxwell–Poisson regression model using the maximum likelihood method, the estimated coefficients are regularly become not fixed with a high variance, and so low statistical significance. Shrinkage estimators, such as ridge estimator, have been recommended to overcome the problem of multicollinearity. The ridge - estimator [10] has been consistently demonstrated to be an attractive and alternative to the maximum likelihood estimator.

In this paper, the jackknifed ridge estimator and its modified version are proposed for modeling count data with CMPRE. These two estimators are proposed to reduce the effects of multicollinearity and the biasedness of using ridge estimator simultaneously.

2. Conway-Maxwell-Poisson regression model

In real application, count data have often been shown to exhibit overdispersion, meaning that the variance is greater than the mean, and have sometimes shown characteristics of underdispersion, meaning that the variance is less than the mean. The Conway–Maxwell–Poisson distribution (CMPD) offers a simple way to accommodate the overdispersion and underdispersion [11, 12]. The CMPD is an extension of the Poisson distribution with two parameters λ (centering parameter related to the observations mean) and θ (the shape parameter) [13]. Suppose $y \in$ ***Error! Bookmark not defined.*** is a random variable that follows a CMPD, then the probability mass function is defined as

$$P\ (\ Y = y\ ;\ \lambda\ , \theta) = \frac{\lambda^{y}}{(y!)^{\theta} Z\left(\lambda, \theta\right)}, \lambda > 1, \theta \geq 0, \tag{2.1}$$

where $Z\left(\lambda, \theta\right) = \sum_{s=0}^{\infty} \left(\lambda^{s}/(s!)^{\theta}\right)$ is a normalizing constant. The CMPD can model both underdispersed ($\theta > 1$) and overdispersed ($\theta < 1$) data.

According to Eq. (2.1), there is no closed form representation existing for the mean. This is because that the normalizing constant, $Z\left(\lambda, \theta\right)$, is an infinite series with no closed form representation [14]. Shmueli, et al. [15] used the asymptotic expression for $Z\left(\lambda, \theta\right)$ in Eq. (2.1) to express the mean and variance of the CMPD as

$$E(Y) \approx \lambda^{\frac{1}{\theta}} - \frac{\theta - 1}{2\theta},$$

$$Var(Y) \approx \frac{1}{\theta} \lambda^{\frac{1}{\theta}} \tag{2.2}$$

regression modeling in which the count responses may change depending on a set of explanatory variables, it is more convenient and interpretable to model the mean of the CMPD directly. By setting $\mu = \lambda^{\frac{1}{\theta}}$ [16], a re-parameterization of Eq. (2.1) to provide a clear centering parameter as

$$P\ (Y = y\ ;\ \mu\ ,\theta) = \left(\frac{\mu^y}{y!}\right)^{\theta}\frac{1}{S(\mu,\theta)},\tag{2.3}$$

where $S(\mu,\theta) = \sum_{n=0}^{\infty}(\mu^n/n!)^{\theta}$. Depending on Eq. (2.3) and in terms of generalized linear model framework, Conway–Maxwell–Poisson regression model (CMPR) can be formulated as

$$ln(\ \mu) = \beta_0 + \sum_{j=1}^{p}\beta_j\boldsymbol{x}_j,\tag{2.4}$$

$$ln(\ \theta) = \gamma_0 + \sum_{k=1}^{q}\gamma_k\boldsymbol{m}_j.\tag{2.5}$$

In Eqs. (2.4) and (2.5), $\boldsymbol{x}_j$and $\boldsymbol{m}_j$are expandatory variables, and there are assumed to be p covariates used in the link function and q covariates used in the shape link function. Assuming θ as a dispersion parameter and using single link function, Eq. (2.4), with $\eta = ln(\ \mu) = \boldsymbol{\beta x}$ as a linear predictor with log link, where $\boldsymbol{\beta}$ is the vector of regression coefficients including intercept, the log likelihood can be written a [5]

$$\ell(\beta) = \theta\sum_{i=1}^{n}y_i(\boldsymbol{\beta x}_i) - \theta\sum_{i=1}^{n}ln(\ y_i!) - \sum_{i=1}^{n}ln\left[S(\boldsymbol{\beta x}_i,\theta)\right].\tag{2.6}$$

Solving Eq. (2.6), the estimation of the regression parameters,$\boldsymbol{\beta}$, and the estimation of the dispersion parameter, θ, can be obtained as, respectively,

$$\frac{\partial\ell(\boldsymbol{\beta})}{\partial\boldsymbol{\beta}} = \sum_{i=1}^{n}(y_i\theta - \frac{\partial}{\partial\eta_i}ln[\ S(\eta_i,\theta)])x_{ij}\tag{2.7}$$

$$\frac{\partial\ell(\boldsymbol{\beta})}{\partial\theta} = \sum_{i=1}^{n}(-ln(\ y_i!) - \frac{\partial}{\partial\theta}log[\ S(\eta_i,\theta)])\tag{2.8}$$

Iterative reweighted least square (IRLS) is used to solve both Eq. (2.7) and Eq. (2.8). By fixing θ, the (MLE) of $\boldsymbol{\beta}$is by

$$\widehat{\boldsymbol{\beta}}_{\text{MLE}} = (\boldsymbol{X}^T\widehat{\boldsymbol{W}}\boldsymbol{X})^{-1}\boldsymbol{X}^T\widehat{\boldsymbol{W}}\widehat{\boldsymbol{u}},\tag{2.9}$$

where $\widehat{\boldsymbol{u}} = ln(\ \widehat{\mu}) + \frac{(y-\widehat{\mu})}{\widehat{\mu}^2}$ is a vector of the adjusted response variable, and $\widehat{\boldsymbol{W}}$ is a matrix of weights [14].

In the attendance of multicollinearity, the matrix $\boldsymbol{X}^T\widehat{\boldsymbol{W}}\boldsymbol{X}$ converts ill-conditioned foremost to high variance and instability of the MLE of the Conway–Maxwell–Poisson regression model. As a preparation, a ridge estimator of Hoerl and Kennard [10] for Conway–Maxwell–Poisson regression model (CMPRE) can be defined as

$$\widehat{\boldsymbol{\beta}}_{\text{CMPRE}} = (\boldsymbol{X}^{\ T}\widehat{\boldsymbol{W}}\ \boldsymbol{X} + k\boldsymbol{I}\ )^{-1}\boldsymbol{X}^T\widehat{\boldsymbol{W}}\boldsymbol{X}\widehat{\boldsymbol{\beta}}_{\text{MLE}} = (\ \boldsymbol{X}^{\ T}\ \widehat{\boldsymbol{W}}\ \boldsymbol{X} + k\ \boldsymbol{I})^{-1}\boldsymbol{X}^T\widehat{\boldsymbol{W}}\widehat{\boldsymbol{u}},\tag{2.10}$$

where $k > 0$.

The (MSE) of Eq. (2.9) can be found as

$$\mathrm{MSE}(\widehat{\boldsymbol{\beta}}_{\mathrm{MLE}}) = E(\widehat{\boldsymbol{\beta}}_{\mathrm{MLE}} - \widehat{\boldsymbol{\beta}})^T(\widehat{\boldsymbol{\beta}}_{\mathrm{MLE}} - \widehat{\boldsymbol{\beta}}) = \widehat{\theta} tr[(\boldsymbol{X}^T\widehat{\boldsymbol{W}}\boldsymbol{X})^{-1}] = \widehat{\theta}\sum_{j=1}^{p}\frac{1}{\lambda_j}, \tag{2.11}$$

where λ_j is the eigen value of the $\boldsymbol{X}^T\widehat{\boldsymbol{W}}\boldsymbol{X}$ matrix and $\widehat{\theta}$is the estimated dispersion parameter. On the other hand, the(MSE) of Eq. (2.10) can be gotten as

$$\mathrm{MSE}(\widehat{\boldsymbol{\beta}}_{\mathrm{CMPRE}}) = \widehat{\theta}\sum_{j=1}^{p}\frac{\lambda_j}{(\lambda_j+k)^2} + k^2\sum_{j=1}^{p}\frac{\alpha_j^2}{(\lambda_j+k)^2}, \tag{2.12}$$

where α_j is definite as the jth element of $\gamma\widehat{\boldsymbol{\beta}}_{\mathrm{MLE}}$and γ is the eigenvector of the$\boldsymbol{X}^T\widehat{\boldsymbol{W}}\boldsymbol{X}$.

3. The proposed estimator: Jackknifed Liu-type

In the situation of the linear model, Singh, Chaubey and Dwivedi [17] planned the Jackknife practice to improve the bias in generalized ridge estimator. The application and theoretical of the jackknife estimator have been considered by several authors [18, 1, 19, 20, 21, 22, 23, 24].

Most popular biased estimator is Liu (1993) which is adopted in CMPR is defined as follows [25]:

$$\widehat{\beta}_{CMPRLE} = (X'\hat{W}X + I)^{-1}(X'\hat{W}X + dI)\,\widehat{\beta}_{MLE} \tag{3.1}$$

Where $0 < d < 1$, where the MSE of $\widehat{\beta}_{CMPRLE}$ is lower than MSE of $\widehat{\beta}_{MLE}$ [26] which is equal:

$$\widehat{\beta}_{CMPRLE} = \sum_{j=1}^{J}\frac{(\lambda_j+d)^2}{\lambda_j(\lambda_j+I)^2} + (d-I\ )^2\sum_{j=1}^{J}\frac{\alpha_j^2}{(\lambda_j+I)^2} \tag{3.2}$$

Where α_j^2 is the j-th element of $\gamma\beta$, where γ is the eigen vector defined $X'\hat{W}\ X = \gamma'\Lambda\gamma$ and Λ is the diagonal matrix with elements equal to λ_j .

For the estimator $\widehat{\beta}_{CMPRLE}$ the matrix of cross products in Liu (1993) replaced with the matrix weights of cross products and the ordinary least square of β with ML estimator [27]. The MSE of $\widehat{\beta}_{CMPRLE}$ is

$$\begin{aligned} MSE(\widehat{\beta}_{CMPRLE}) &= E(\widehat{\beta}_{CMPRLE} - \beta)'(\widehat{\beta}_{CMPRLE} - \beta) \\ &= tr\left[(\widehat{\beta}_{MLE} - \beta)'(\widehat{\beta}_{MLE} - \beta)S'S\right] + \beta'k^2\ (X'\hat{W}X + kI)^{-2}\ \beta \end{aligned}$$

By taking the trace for the equation above, we have:

$$MSE(\widehat{\beta}_{CMPRLE}) = \sum_{j=1}^{J}\frac{(\lambda_j+d)^2}{\lambda_j(\lambda_j+I)^2} + (d-I)^2\sum_{j=1}^{J}\frac{\alpha_j^2}{(\lambda_j+I)^2} \tag{3.3}$$

$$MSE(\widehat{\beta}_{CMPRLE}) = \omega(d)_1 + \omega(d)_2$$

From equation (3.3) the MSE of $(\widehat{\beta}_{CMPRLE})$ is equal to $\omega(d)_1$ which is the variance and the biased part which is represent by $\omega(d)_2$.

To show that the $MSE(\widehat{\beta}_{CMPRLE}) < MSE(\beta_{MLE})$ we attractive the first derivative of equation (3.1) with respect to d as follows [25]:

$$\frac{\partial\left(MSE(\widehat{\beta}_{CMPRLE})\right)}{\partial(d)} = 2\sum_{j=1}^{J}\frac{\lambda_j + d}{\lambda_j(\lambda_j+I)^2} + 2(d-I)\sum_{j=1}^{J}\frac{\alpha_j^2}{(\lambda_j+I)^2} \tag{3.4}$$

Since $0 < d < 1$, by inserting $d = 1$ on equation (3.4), we have:

$$\frac{\partial\left(MSE(\widehat{\beta}_{CMPRLE})\right)}{\partial(d)} = 2\sum_{j=1}^{J}\frac{\lambda_j + 1}{\lambda_j(\lambda_j+I)^2} \tag{3.5}$$

$$= 2\sum_{j=1}^{J}\frac{1}{\lambda_j(\lambda_j+I)} \qquad where \lambda_j > 0 \tag{3.6}$$

The optimal value of the value d_j can be found by setting equation (3.4) to zero and solve for d_j ,then it may be show as:

$$d_j = \frac{\alpha_j^2 - 1}{\frac{1}{\lambda_j + \alpha_j^2}} \tag{3.7}$$

Liu upgraded by proposing Liu type to overcome the problem of sever multicollinearity, Liu type estimator is defined as follows [25]

$$\widehat{\beta}_{CMPRLTE} = (X'\hat{W}X + kI)^{-1}(X'\hat{W}X - dI)\widehat{\beta}_{MLE} \tag{3.8}$$

where $-\infty < d < \infty$ and $k \geq 0$. Liu type estimator has superior over ridge estimator [28]. Liu note that when ther exists sever mulicollinearity, the shrinkage ridge parameter may not fully address the illconditioning problem, Therfore he modified liu estimator and he has suggest Liu type estimator. The MSE of $\widehat{\beta}_{CMPRLTE}$ is

$$MSE(\widehat{\beta}_{CMPRLTE}) = \sum_{j=1}^{J}\frac{(\lambda_j - d)^2}{\lambda_j(\lambda_j+k)^2} + (d+k)^2\sum_{j=1}^{J}\frac{\alpha_j^2}{(\lambda_j+k)^2} \tag{3.9}$$

Let $M = (m_1, m_2, ..., m_p)$ and $\varLambda = diago(\lambda_1, \lambda_2, ..., \lambda_p)$, respectively, be the matrices of eigenvectors and eigenvalues of the a symmetric matrix $C = X'WX$ has an eigenvalues and eigenvectors decomposition $C = T\Lambda T'$, where T is an orthogonal matrix and Λ is a diagonal matrix. [28] proposed a new estimator for γ where this estimator is biased and it's called Liu-type estimators. It can be defined for CMPR, as follows:

$$\begin{aligned}
\widehat{\gamma}_{CMPRLTE}(\ k,\ d\ ) &= (\ \varLambda + k\ I)^{-1}(M'y - d\widehat{\gamma}_{CMPR}) \\
&= (\Lambda + kI)^{-1}(M'y - d\varLambda^{-1}M'y) \\
&= \left[I - (\varLambda + kI)^{-1}(k+d)\right]\widehat{\gamma}_{CMPR} \\
&= H(k,d)\widehat{\gamma}_{CMPR}
\end{aligned} \tag{3.10}$$

where $H(k,d) = (\Lambda + kI)^{-1}(\Lambda - dI)$.

By using [29], [30], [21] and [31] we proposed the jackknifed from$\widehat{\gamma}_{PLTE}$. [32] and [33] introduced the Jackknife method so as to reduce the value of the bias. [29] stated that with a few exceptions, where the jackknife technique can be applied to balanced models. The jackknifed estimator after some algebraic manipulations is obtained by deleting the i-th observation (m_i', y_i):

$$
\begin{aligned}
\widehat{\gamma}_{PLTE}(\ k,\ d\ ) =&(M'_{-i}\hat{W}_{-i}M_{-i}+kI)^{-1}(M'_{-i}\hat{W}_{-i}M_{-i}-dI)(M'_{-i}\hat{W}_{-i}M_{-i})^{-1}M'_{-i}y_{-i} \\
=&(A-M'_{-i}\hat{W}_{-i}M_{-i}+kI)^{-1}(M'y-m_iy_i) \\
=&\left[A^{-1}+\frac{A^{-1}m_iw_im_i'A^{-1}}{1-m_iA^{-1}m_i}\right] \\
=&A^{-1}M'y-A^{-1}m_iy_i\left[\frac{A^{-1}m_iw_im_i'A^{-1}}{1-m_i'A^{-1}m_i}M'y-\frac{A^{-1}m_iw_im_i'A^{-1}}{1-m_i'A^{-1}m_i}m_iy_i\right] \\
=&\widehat{\gamma}_{PLTE}(\ k,\ d\ )+A^{-1}m_iy_i\left[1+\frac{m_i'A^{-1}m_i}{1-m_i'A^{-1}m_i}\right]+\frac{A^{-1}m_iw_im_i'}{1-m_i'A^{-1}m_i}\widehat{\gamma}_{PLTE}(\ k,\ d\ ) \\
=&\widehat{\gamma}_{PLTE}(\ k,\ d\ )-A^{-1}m_i\frac{A^{-1}m_i(y_i-m_i'\widehat{\gamma}_{PLTE}(\ k,\ d\ ))}{1-m_i'A^{-1}m_i} \\
=&\widehat{\gamma}_{PLTE}-\frac{A^{-1}m_ie_i}{1-f_i}
\end{aligned}
\tag{3.11}
$$

where $m_i'$is the i-th row of the matrix M,$e_i = y_i - m_i'\widehat{\gamma}_{PLTE}(\ k,\ d\ )$ is the Liu-type residual, $M'_{-i}\hat{W}_{-i}M_{-i} = M'\hat{W}_{-i}M - m_i'\hat{W}_im_i'$,$M_{-i}y_{-i} = M'y - m_iy_i$ and$f_i = m_i'A^{-1}m_i$ is the reserve factor and $A^{-1} = (\Lambda + kI)^{-1}(I - d\Lambda^{-1}) = H(k, d)\Lambda^{-1}$.

The bias part and variance of $\widehat{\gamma}_{PLTE}(\ k,\ d\ )$ are obtained as, respectively,

$$Bias(\widehat{\gamma}_{PLTE}(\ k,\ d\ )) = (1 - H(k, d))^2\gamma \tag{3.12}$$

$$Cov(\widehat{\gamma}_{PLTE}(\ k,\ d\ )) = \sigma^2(2I - H(k, d))H(k, d)\Lambda^{-1}H(k, d)'(2I - H(k, d))' \tag{3.13}$$

The MSEMs of JPLTE and PLTE are given as follows

$$
\begin{aligned}
MSEM(\widehat{\gamma}_{PLTE}(\ k,\ d\ )) =&Cov(\widehat{\gamma}_{PLTE}(\ k,\ d\ ))+Bias(\widehat{\gamma}_{PLTE}(\ k,\ d\ ))Bias(\widehat{\gamma}_{PLTE}(\ k,\ d\ ))' \\
=&(2I-H(k,d))H(k,d)\Lambda^{-1}H
\end{aligned}
\tag{3.14}
$$

$$MSEM(\widehat{\gamma}_{JPLTE}) = H(k, d)\mathit{\Lambda}^{-1}H(k, d)' + (H(k, d) - I)\,\gamma\gamma'\,(H(k, d) - I) \tag{3.15}$$

4. Simulation study

Monte Carlo simulation study is conducted to evaluate the performance and comparison of our proposed estimators, JCMPRLTE with CMPRE and CMPRLTE under different conditions. The response variable of $n \in \{50, 150, 250\}$ observations from CMP regression model was made as $y_i \sim CMP(\mu_i, \theta)$, where $\mu_i = exp(\ \boldsymbol{x}_i^T\boldsymbol{\beta})$, $\boldsymbol{\beta} = (\beta_1, ..., \beta_p)$ with $\sum_{j=1}^{p}\beta_j^2 = 1$ and $\beta_1 = \beta_2 = ... = \beta_p$ [34]. Three different values of the dispersion parameter, θ, are considered to capture overdispersion ($\theta = 0.5$), equidispersion ($\theta = 1$) and underdispersion ($\theta = 2.5$) cases. The explanatory variables $\boldsymbol{x}_i^T = (x_{i1}, x_{i2}, ..., x_{in})$ have been produced from the following formulary

$$x_{ij} = (1 - \rho^2)^{1l2}w_{ij} + \rho w_{ip}, i = 1, 2, ..., n, j = 1, 2, ..., p, \tag{4.1}$$

where ρ denotes the correlation between the descriptive variables and w_{ij}'s are independent standard normal pseudo-random numbers. the number of the explanatory variables is considered as $p = 4$ and $p = 12$. moreover , because we are interested in the impact of multicollinearity, Degrees of correlation can be considered the most important, three values of the pairwise correlation are considered with $\rho = \{0.90, 0.95, 0.99\}$. The optimal value of k can be gotten by using Hoerl, Kannard and Baldwin [35] formula as

$$\hat{k} = \frac{\widehat{\theta} p}{\widehat{\alpha}^T \widehat{\alpha}}. \tag{4.2}$$

For a combination of the different values of n, θ, p, and ρ the produced data is recurrent 1000 times and the average absolute bias and average MSE is

$$\text{Bias } (\widehat{\boldsymbol{\beta}}) = \frac{1}{1000} \sum_{i=1}^{1000} \left| \widehat{\boldsymbol{\beta}} - \boldsymbol{\beta} \right|, \tag{4.3}$$

$$\text{MSE}(\widehat{\boldsymbol{\beta}}) = \frac{1}{1000} \sum_{i=1}^{1000} (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})^T (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}). \tag{4.4}$$

The average bias and MSE all the combination of n, θ, p, and ρ, are respectively in Tables 1 – 6. The best value of the average bias and MSE is tinted in bold.

As Tables 1,3, and 5 show, the proposed estimators, JCMPRLTE, achieved smaller averaged bias than CMPRE and CMPRLTE. On other hand, JCMPRLTE performances better than CMPRLTE. In general, this specifies that the Jackknifed estimator is significantly with decreasing the bias. In terms of MSE, it is evident from Tables 2, 4, and 6 that both JCMPRLTE and CMPRLTE are is quite better than the CMPRE with superiority to JCMPRLTE. Meanwhile, MLE estimator of CMPRE has the worst performance including others, which is significantly impacted by the multicollinearity.

Also, with respect to ρ, there is increasing in the bias and MSE values when the correlation degree increases irrespective the value of n, θ and p. Concerning the number of explanatory variables, it is easily seen that there is a negative impact on both bias and MSE, where there are increasing in their values when the p increasing from 4 variables to 12 variables. In Accumulation When the sample size n is increased, the bias & MSE values decrease when n increases, regardless the value of $\rho, \theta$and p. Regardless the dispersal parameter, both bias & MSE values are decreasing when θ increasing.

Table 1: Average bias values in case of overdispersion ($\theta = 0.5$)

n	p	ρ	CMPRLTE	JCMPRLTE
50	4	0.90	0.5431	**0.4291**
		0.95	0.5735	**0.4595**
		0.99	0.5851	**0.4711**
	12	0.90	0.6632	**0.5492**
		0.95	0.6936	**0.5796**
		0.99	0.7052	**0.5912**
150	4	0.90	0.3013	**0.1873**
		0.95	0.3317	**0.2177**
		0.99	0.3433	**0.2293**
	12	0.90	0.4214	**0.3074**
		0.95	0.4518	**0.3378**
		0.99	0.4634	**0.3494**
250	4	0.90	0.2501	**0.1361**
		0.95	0.2805	**0.1665**
		0.99	0.2921	**0.1781**
	12	0.90	0.3702	**0.2562**
		0.95	0.4008	**0.2867**
		0.99	0.4122	**0.2982**

Table 2: Average MSE values in case of overdispersion ($\theta = 0.5$)

n	p	ρ	CMPRE	CMPRLTE	JCMPRLTE
50	4	0.90	4.2922	3.9532	**3.8392**
		0.95	4.3422	4.0032	**3.8892**
		0.99	4.6082	4.2692	**4.1552**
	12	0.90	4.4122	4.0732	**3.9592**
		0.95	4.4622	4.1232	**4.0092**
		0.99	4.7282	4.3892	**4.2752**
150	4	0.90	4.0502	3.7112	**3.5972**
		0.95	4.1002	3.7612	**3.6472**
		0.99	4.3662	4.0272	**3.9132**
	12	0.90	4.1702	3.8312	**3.7172**
		0.95	4.2202	3.8812	**3.7672**
		0.99	4.4862	4.1472	**4.0332**
250	4	0.90	3.9992	3.6602	**3.5462**
		0.95	4.0492	3.7102	**3.5972**
		0.99	4.3152	3.9762	**3.8622**
	12	0.90	4.1192	3.7802	**3.6672**
		0.95	4.1692	3.8302	**3.7162**
		0.99	4.4352	4.0962	**3.9822**

Table 3: Averaged bias values in case of equidispersion ($\theta = 1$)

n	p	ρ	CMPRLTE	JCMPRLTE
50	4	0.90	0.5424	**0.4284**
		0.95	0.5728	**0.4588**
		0.99	0.5844	**0.4704**
	12	0.90	0.6625	**0.5485**
		0.95	0.6929	**0.5789**
		0.99	0.7045	**0.5905**
150	4	0.90	0.3006	**0.1866**
		0.95	0.331	**0.217**
		0.99	0.3426	**0.2286**
	12	0.90	0.4207	**0.3067**
		0.95	0.4511	**0.3371**
		0.99	0.4627	**0.3487**
250	4	0.90	0.2494	**0.1354**
		0.95	0.2798	**0.1658**
		0.99	0.2914	**0.1774**
	12	0.90	0.3695	**0.2555**
		0.95	0.3999	**0.2859**
		0.99	0.4115	**0.2975**

Table 4: Averaged MSE values in case of equidispersion ($\theta = 1$)

n	p	ρ	CMPRE	CMPRLTE	JCMPRLTE
50	4	0.90	4.2915	3.9525	**3.8385**
		0.95	4.3415	4.0025	**3.8885**
		0.99	4.6075	4.2685	**4.1545**
	12	0.90	4.4115	4.0725	**3.9585**
		0.95	4.4615	4.1225	**4.0085**
		0.99	4.7275	4.3885	**4.2745**
150	4	0.90	4.0495	3.7105	**3.5965**
		0.95	4.0995	3.7605	**3.6465**
		0.99	4.3655	4.0265	**3.9125**
	12	0.90	4.1695	3.8305	**3.7165**
		0.95	4.2195	3.8805	**3.7665**
		0.99	4.4855	4.1465	**4.0325**
250	4	0.90	3.9985	3.6595	**3.5455**
		0.95	4.0485	3.7095	**3.5965**
		0.99	4.3145	3.9755	**3.8615**
	12	0.90	4.1185	3.7795	**3.6665**
		0.95	4.1685	3.8295	**3.7155**
		0.99	4.4345	4.0955	**3.9815**

Table 5: Averaged bias values in case of underdispersion ($\theta = 2.5$)

n	p	ρ	CMPRLTE	JCMPRLTE
50	4	0.90	0.5417	**0.4277**
		0.95	0.5721	**0.4581**
		0.99	0.5837	**0.4697**
	12	0.90	0.6618	**0.5478**
		0.95	0.6922	**0.5782**
		0.99	0.7038	**0.5898**
150	4	0.90	0.2999	**0.1859**
		0.95	0.3303	**0.2163**
		0.99	0.3419	**0.2279**
	12	0.90	0.42	**0.306**
		0.95	0.4504	**0.3364**
		0.99	0.462	**0.348**
250	4	0.90	0.2487	**0.1347**
		0.95	0.2791	**0.1651**
		0.99	0.2907	**0.1767**
	12	0.90	0.3688	**0.2548**
		0.95	0.3992	**0.2852**
		0.99	0.4108	**0.2968**

Table 6: Averaged MSE values in case of underdispersion ($\theta = 2.5$)

n	p	ρ	CMPRE	CMPRLTE	JCMPRLTE
50	4	0.90	4.2906	3.9516	**3.8376**
		0.95	4.3406	4.0016	**3.8876**
		0.99	4.6066	4.2676	**4.1536**
	12	0.90	4.4106	4.0716	**3.9576**
		0.95	4.4606	4.1216	**4.0076**
		0.99	4.7266	4.3876	**4.2736**
150	4	0.90	4.0486	3.7096	**3.5956**
		0.95	4.0986	3.7596	**3.6456**
		0.99	4.3646	4.0256	**3.9116**
	12	0.90	4.1686	3.8296	**3.7156**
		0.95	4.2186	3.8796	**3.7656**
		0.99	4.4846	4.1456	**4.0316**
250	4	0.90	3.9976	3.6586	**3.5446**
		0.95	4.0476	3.7086	**3.5956**
		0.99	4.3136	3.9746	**3.8606**
	12	0.90	4.1176	3.7786	**3.6656**
		0.95	4.1676	3.8286	**3.7146**
		0.99	4.4336	4.0946	**3.9806**

5. Real Data Application

To compare the modeling performance of the CMPRLTE and JCMPRLTE, laminated plastic plywood data which was presented by Marcondes Filho and Sant'Anna [36]. This data was furthered analyzed by Mammadova, Özkale and Mathematics [37]. The response of interest is the number of defects per laminated plastic plywood area. The explanatory variables are volumetric shrinkage (x_1), assembly time (x_2), wood density (x_3), and drying temperature (x_4). Marcondes Filho and Sant'Anna [36] stated that multicollinearity exists between volumetric shrinkage and assembly time and between wood density and drying temperature.

Fitting Conway–Maxwell–Poisson regression model with log-link function gave the estimated dispersion parameter as $\widehat{\theta} = 0.9614$. the eigenvalues of the matrix $\boldsymbol{X}^T\widehat{\boldsymbol{W}}\boldsymbol{X}$ are obtained as 2.14×10^9,3.85×10^6, 2.42×10^5, 1.26×10^4, 1.29×10^3, 2.14×10^9,9.01×10^2, 4.71×10^2, 1.71×10^2, 5.93×10^1, 3.24×10^1, 2.77×10^1, 1.78×10^1, 9.56, and 1.23. The determined condition number $\text{CN} = \sqrt{\lambda_{\max}/\lambda_{\min}}$ of the matrix $\boldsymbol{X}^T\widehat{\boldsymbol{W}}\boldsymbol{X}$ is 8634.73 indicating that the severe multicollinearity issue is exist.

The estimated regression coefficients and MSE values for the CMPRE, CMPRLTE, and JCMPRLTE estimators are listed in Table 7. According to Table 7, it is obviously seen that the both CMPRLTE and JCMPRLTE shrinkage the value of the estimated coefficients efficiently. Also, in the MSE, there is an prominent reduction in favor of the JCMPRLTE. Specifically, it can be seen that the MSE of the JCMPRLTE estimator was about 39.66% and 15.11% lower than that of CMPRE and CMPRLTE estimators, respectively.

Table 7: The valub coefficients and MSE values for the four used estimators.

		Estimators		
		CMPRE	CMPRLTE	JCMPRLTE
$\widehat{\beta}_1$		4.916	3.271	3.183
	$\widehat{\beta}_2$	5.971	4.814	4.257
	$\widehat{\beta}_3$	3.805	2.226	2.058
	$\widehat{\beta}_4$	7.351	6.781	5.389
	MSE	6.775	4.816	**4.088**

6. Conclusions

The Conway–Maxwell–Poisson regression model is very popular statistical model to analyze data whose response variable are counts. This paper addresses issue of multicollinearity by integrating the Liu-type estimator with jackknife approach. Further, a Jackknifed Liu-type estimator was proposed. The Monte Carlo simulation studies shows the proposed estimators significant improvement relative to others, by MSE and bias. in real data application, compared to CMPRE and CMPRLTE, the JCMPRLTE can reduce the MSE.

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