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Existence and uniqueness results to a fractional q-difference coupled system with integral boundary conditions via topological degree theory

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Abstract

This paper aims to highlight existence and uniqueness results for a coupled system of nonlinear fractional q-difference subject to nonlinear more general four-point boundary conditions are treated. Our analysis relies on two approaches, the topological degree for condensing maps via a priori estimate method and the Banach contraction principle fixed point theorem. Finally, Two examples illustrating the effectiveness of the theoretical results are presented.

Keywords: fractional differential equations system; fractional q-derivative; topological degree theory; condensing maps; existence and uniqueness.
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1. Introduction

Fractional calculus and q-calculus is a branch of mathematics, witch deals with the generalization of integration and differentiation of integer order to any order. It is known that fractional calculus is used for a better description of phenomena having both discrete and continuous behaviors, and applying in different sciences and engineering such as mechanics, electricity, biology, control theory, signal and image processing. Fractional q-difference equations initiated at the beginning of the nineteenth century [2, 19] has received significant attention in recent years [10, 23]. In addition, propose

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more general results that involve initial and boundary value problems of q-difference and fractional q-difference equations, has also been gaining prominence [19, 9, 22, 23, 10, 3, 26, 39, 14, 21, 40, 8, 41, 42].

The topological methods proved to be a powerful tool in the study of various problems which appear in nonlinear analysis. Particularly, the a priori estimate method (or the method of a priori bounds) has been often used in order to prove the existence of solutions for some boundary value problems for nonlinear differential equations or nonlinear partial differential equations, for details about usefulness of coincidence degree theory approach for condensing maps in the study for the existence of solutions of certain integral equations, the reader can be referred to [11, 12, 15, 27, 36, 37, 38, 43, 45].

In this paper, we show existence of solutions for nonlinear fractional difference equations by applying a fixed point theorem due to Isaia [27], which was obtained via coincidence degree theory for condensing maps.

So we are mainly concerned with the existence results for the following fractional q-difference system of the form

$$\begin{cases} \mathcal{D}_q^{q_1} u_1(\tau) = \mathcal{F}_1(\tau, u_1(\tau), u_2(\tau)), \\ \qquad\qquad\qquad\qquad\qquad\qquad , \tau \in \mathrm{J} := [0,1], \\ \mathcal{D}_q^{q_2} u_2(\tau) = \mathcal{F}_2(\tau, u_1(\tau), u_2(\tau)), \end{cases} \tag{1.1}$$

with the fractional boundary conditions

$$\begin{cases} u_1(0) = a_1 \mathcal{I}_q^{\beta_1} u(\eta_1),\ 0 < \eta_1 < 1,\ \beta_1 > 0, \\ u_1(1) = b_1 \mathcal{I}_q^{\alpha_2} u(\sigma_1),\ 0 < \sigma_1 < 1,\ \alpha_1 > 0, \\ u_2(0) = a_2 \mathcal{I}_q^{\beta_2} u(\eta_2),\ 0 < \eta_2 < 1,\ \beta_2 > 0, \\ u_2(1) = b_2 \mathcal{I}_q^{\alpha_2} u(\sigma_2),\ 0 < \sigma_2 < 1,\ \alpha_2 > 0. \end{cases} \tag{1.2}$$

For all $i = 1, 2$, $\mathcal{D}_q^{q_i}$ is the fractional q-derivative of the Caputo type of order $1 < q_i \leq 2$, and $\mathcal{F} : \mathrm{J} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a given continuous function, $a_i, b_i, i = 1, 2$ are suitably chosen real constants.

The rest of the article is organized as follows. In Sect. 2, we introduce some important notions about fractional difference operators and topological degree theory, while Sect. 3 contains our main existence results for problem (1.1). Finally, we provide an example to illustrate the applicability of the developed results.

2. Preliminaries

In this section, we introduce notations, definitions and preliminary facts which are used throughout the paper.

Let $\mathcal{U} = C([0,1], \mathbb{R})$ be the Banach space of all continuous functions endowed with the norm

$$\|u\|_\infty = \sup\{\|u(\tau)\| : \quad \tau \in \mathrm{J}\}.$$

Then the product space $\mathcal{C} := \mathcal{U} \times \mathcal{V}$ defined by $\mathcal{C} = \{(u, v) : u \in \mathcal{U}, v \in \mathcal{V}\}$ is also a Banach space under the norm

$$\|(u, v)\|_{\mathcal{C}} = \|u\|_\infty + \|v\|_\infty.$$

Let $\mathfrak{M}_{\mathcal{U}}$ represents the class of all bounded mappings in $\mathcal{U}$.

In what follow, we recall some elementary definitions and properties related to fractional q-calculus. For $a \in \mathbb{R}$, we put

$$[a]_q = \frac{1-q^a}{1-q}.$$

The q-analogue of the power $(a-b)^n$ is expressed by

$$(a-b)^{(0)} = 1, \qquad (a-b)^{(n)} = \prod_{k=0}^{n-1}\big(a - bq^k\big), \quad a, b \in \mathbb{R}, n \in \mathbb{N}.$$

In general,

$$(a-b)^{(\alpha)} = a^\alpha \prod_{k=0}^{\infty}\left(\frac{a-bq^k}{a-bq^{k+\alpha}}\right), \quad a, b, \alpha \in \mathbb{R}.$$

Definition 2.1. *[29] The q-gamma function is given by*

$$\Gamma_q(\alpha) = \frac{(1-q)^{(\alpha-1)}}{(1-q)^{\alpha-1}}, \quad \alpha \in \mathbb{R} - \{0, -1, -2, \ldots\}.$$

The q-gamma function satisfies the classical recurrence relationship

$$\Gamma_q(1+\alpha) = [\alpha]_q \Gamma_q(\alpha).$$

Definition 2.2. *[29] For any $\alpha, \beta > 0$, the q-beta function is defined by*

$$B_q(\alpha, \beta) = \int_0^1 \omega^{(\alpha-1)}(1-q\omega)^{(\beta-1)} d_q\omega, \quad q \in (0,1),$$

where the expression of q-beta function in terms of the q-gamma function is

$$B_q(\alpha, \beta) = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}.$$

Definition 2.3. *[29] Let $\omega : \mathrm{J} \to \mathbb{R}$ be a suitable function. We define the q-derivative of order $n \in \mathbb{N}$ of the function by $\mathcal{D}_q^0\omega(\tau) = \omega(\tau)$,*

$$\mathcal{D}_q\omega(\tau) := \mathcal{D}_q^1\omega(\tau) = \frac{\omega(\tau) - \omega(qt)}{(1-q)\tau}, \quad \tau \neq 0, \qquad \mathcal{D}_q\omega(0) = \lim_{\tau\to 0}\mathcal{D}_q\omega(\tau),$$

and

$$\mathcal{D}_q^n\omega(\tau) = \mathcal{D}_q\mathcal{D}_q^{n-1}\omega(\tau), \quad \tau \in I, n \in \{1, 2, \ldots\}.$$

Set $I_\tau := \{tq^n : n \in \mathbb{N}\} \cup \{0\}$.

Definition 2.4. *[29] For a given function $\omega : I_\tau \to \mathbb{R}$, the expression defined by*

$$\mathcal{I}_q\omega(\tau) = \int_0^\tau \omega(s)\, d_qs = \sum_{n=0}^{\infty} \tau(1-q)q^n\omega\big(tq^n\big),$$

is called q-integral, provided that the series converges.
We note that $\mathcal{D}_q\mathcal{I}_q\omega(\tau) = \omega(\tau)$, while if ω is continuous at 0, then

$$\mathcal{I}_q\mathcal{D}_q\omega(\tau) = \omega(\tau) - \omega(0).$$

Definition 2.5. *[3] The integral of a function $\omega : \mathrm{J} \to \mathbb{R}$ defined by*

$$\mathcal{I}_q^0 \omega(\tau) = \omega(\tau),$$

and

$$\mathcal{I}_q^\alpha \omega(\tau) = \int_0^\tau \frac{(\tau - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \omega(s)\, d_q s, \quad \tau \in \mathrm{J}.$$

is called Riemann-Liouville-fractional q-integral of order $\alpha \in \mathbb{R}_+$

Lemma 2.6. *[34]*
Let $\alpha \in \mathbb{R}_+$ and $\beta \in (-1, \infty)$. One has

$$\mathcal{I}_q^\alpha \tau^\beta = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\alpha+\beta+1)} \tau^{\alpha+\beta}, \ \ \beta \in (-1, \infty), \ \alpha \geq 0, \ \tau > 0.$$

In particular, if $\omega \equiv 1$, then

$$\mathcal{I}_q^\alpha 1(\tau) = \frac{1}{\Gamma_q(1+\alpha)} \tau^{(\alpha)}, \quad \text{for all } \tau > 0.$$

Definition 2.7. *[35] The Riemann-Liouville fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $\omega : \mathrm{J} \to \mathbb{R}$ is defined by $\mathcal{D}_q^0 \omega(\tau) = \omega(\tau)$ and*

$$\begin{aligned}\mathcal{D}_q^\alpha \omega(\tau) &= \mathcal{D}_q^{[\alpha]} \mathcal{I}_q^{[\alpha]-\alpha} \omega(\tau)\\ &= \frac{1}{\Gamma_q(n-\alpha)} \int_0^\tau \frac{\omega(s)}{(\tau - qs)^{\alpha-n+1}} d_q s.\end{aligned}$$

where $[\alpha]$ is the integer part of α.

Lemma 2.8. *[25] Let $\alpha > 0$ and $n \in \mathbb{N}$ where $[\alpha]$ denotes the integer part of α. Then, the following fundamental identity holds*

$$_{RL}\mathcal{I}_q^\alpha \, _{RL}\mathcal{D}_q^n \omega(\tau) = \, _{RL}\mathcal{D}_q^n \, _{RL}\mathcal{I}_q^\alpha \omega(\tau) - \sum_{k=0}^{\alpha-1} \frac{\tau^{\alpha-n+k}}{\Gamma_q(\alpha+k-n+1)} (\mathcal{D}_q^k h)(0).$$

Definition 2.9. *[35] The Caputo fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $\omega : \mathrm{J} \to \mathbb{R}$ is defined by*

$$^C\mathcal{D}_q^\alpha \omega(\tau) = \mathcal{I}_q^{[\alpha]-\alpha} \mathcal{D}_q^{[\alpha]} \omega(\tau), \quad \tau \in \mathrm{J}.$$

We put by convention

$$^C\mathcal{D}_q^0 \omega(\tau) = \omega(\tau).$$

Lemma 2.10. *[35] Let $\alpha \in \mathbb{R}_+$. Then the following equality holds:*

$$\mathcal{I}_q^\alpha \, ^C\mathcal{D}_q^\alpha \omega(\tau) = \omega(\tau) - \sum_{k=0}^{[\alpha]-1} \frac{\tau^k}{\Gamma_q(1+k)} \mathcal{D}_q^k \omega(0).$$

In particular, if $\alpha \in (0, 1)$, then

$$\mathcal{I}_q^\alpha \, ^C\mathcal{D}_q^\alpha \omega(\tau) = \omega(\tau) - \omega(0).$$

Lemma 2.11. *[34] Let u be a function defined on* J *and suppose that* α, β *are two real nonnegative numbers. Then the following hold:*

$$\begin{aligned}\mathcal{I}_q^{\alpha}\mathcal{I}_q^{\beta}\omega(\tau) =&\mathcal{I}_q^{\alpha+\beta}\omega(\tau) = \mathcal{I}_q^{\beta}\mathcal{I}_q^{\alpha}\omega(\tau),\\ \mathcal{D}_q^{\alpha}\mathcal{I}_q^{\alpha}\omega(\tau) =& \omega(\tau).\end{aligned}$$

We start this section by introducing some necessary definitions and basic results required for further developments.

We state here the results given below from [4, 20].

Definition 2.12. *The mapping* $\kappa : \mathfrak{M}_{\mathcal{U}} \to [0, \infty)$ *for Kuratowski measure of non-compactness is defined as:*

$$\kappa(\mathcal{B}) = \inf\Big\{\varepsilon > 0 : \mathcal{B} \textit{ can be covered by finitely many sets with diameter} \leq \varepsilon\Big\}.$$

The Kuratowski measure of noncompactness satisfies some properties.

(1) $\mathcal{A} \subset \mathcal{B} \Rightarrow \kappa(\mathcal{A}) \leq \kappa(\mathcal{B})$,
(2) $\kappa(\mathcal{A}) = 0$ if and only if $\mathcal{A}$ is relatively compact,
(3) $\kappa(\mathcal{A}) = \kappa(\overline{\mathcal{A}}) = \kappa(conv(\mathcal{A}))$, where $\overline{\mathcal{A}}$ and conv($\mathcal{A}$) represent the closure and the convex hull of $\mathcal{A}$ respectively,
(4) $\kappa(\mathcal{A} + \mathcal{B}) \leq \kappa(\mathcal{A}) + \kappa(\mathcal{B})$,
(5) $\kappa(\lambda\mathcal{A}) = |\lambda|\kappa(\mathcal{A}), \lambda \in \mathbb{R}$.

Definition 2.13. *Let* $\mathcal{T} : \mathcal{A} \longrightarrow \mathcal{U}$ *be a continuous bounded map and* $\mathcal{A} \subset \mathcal{U}$. *The operator* $\mathcal{T}$ *is said to be* κ*-Lipschitz if we can find a constant* $\ell \geq 0$ *satisfying the following condition,*

$$\kappa(\mathcal{T}(\mathcal{B})) \leq \ell\kappa(\mathcal{B}), \textit{ for every } \mathcal{B} \subset \mathcal{A}.$$

Moreover, $\mathcal{T}$ *is called strict* κ*-contraction if* $\ell < 1$.

Definition 2.14. *The function* $\mathcal{T}$ *is called* κ*-condensing if*

$$\kappa(\mathcal{T}(\mathcal{B})) < \kappa(\mathcal{B}),$$

for every bounded and nonprecompact subset $\mathcal{B}$ *of* $\mathcal{A}$.
In other words,

$$\kappa(\mathcal{T}(\mathcal{B})) \geq \kappa(\mathcal{B}), \textit{ implies } \kappa(\mathcal{B}) = 0.$$

Further we have $\mathcal{T} : \mathcal{A} \longrightarrow \mathcal{U}$ *is Lipschitz if we can find* $\ell > 0$ *such that*

$$\|\mathcal{T}(u) - \mathcal{T}(v)\| \leq \ell\|u - v\|, \textit{ for all } u, v \in \mathcal{A},$$

if $\ell < 1$, $\mathcal{T}$ *is said to be strict contraction.*

For the following results, we refer to [27].

Proposition 2.15. *If* $\mathcal{T}, \mathcal{S} : \mathcal{A} \longrightarrow \mathcal{U}$ *are* κ*-Lipschitz mapping with constants* ℓ_1 *and* ℓ_2 *respectively, then* $\mathcal{T} + \mathcal{S} : \mathcal{A} \longrightarrow \mathcal{U}$ *are* κ*-Lipschitz with constants* $\ell_1 + \ell_2$.

Proposition 2.16. *If* $\mathcal{T} : \mathcal{A} \longrightarrow \mathcal{U}$ *is compact, then* $\mathcal{T}$ *is* κ*-Lipschitz with constant* $\ell = 0$.

Proposition 2.17. *If $\mathcal{T}:\mathcal{A}\longrightarrow\mathcal{U}$ is Lipschitz with constant ℓ, then $\mathcal{T}$ is κ-Lipschitz with the same constant ℓ.*

Isaia [27] present the following results using topological degree theory.

Theorem 2.18. *Let $\mathcal{K}:\mathcal{A}\longrightarrow\mathcal{U}$ be κ-condensing and*

$$\Theta=\{u\in\mathcal{U}:\ \text{there exist}\,\xi\in[0,1]\ \text{such that}\ x=\xi\mathcal{K}u\}\,.$$

If Θ is a bounded set in $\mathcal{U}$, so there exists $r>0$ such that $\Theta\subset B_r(0)$, then the degree

$$\deg(I-\xi\mathcal{K},B_r(0),0)=1,\ \text{for all}\ \ \xi\in[0,1].$$

Consequently, $\mathcal{K}$ has at least one fixed point and the set of the fixed points of $\mathcal{K}$ lies in $B_r(0)$.

3. Main results

For the existence of solutions for the problem (1.1)-(4.1), we need the following auxiliary lemmas.

Lemma 3.1. *Let $\mathcal{F}_i:\mathrm{J}\times\mathbb{R}^2\to\mathbb{R}$ be a continuous function for each $i=1,2$. Then problem* (1.1)-(4.1) *is equivalent to the problem of obtaining the solutions of the integral equation*

$$u_i(\tau)=\mathcal{I}_q^{q_i}\mathcal{F}_{u_i}(\tau)+(\Lambda_{1,i}-\Lambda_{4,i}\tau)\,\mathcal{I}_q^{q_i+\beta_i}\mathcal{F}_{u_i}(\eta_i)+(\Lambda_{2,i}+\Lambda_{3i,}\tau)\left(b_i\mathcal{I}_q^{q_i+\alpha_i}\mathcal{F}_{u_i}(\sigma_i)-\mathcal{I}_q^{q_i}\mathcal{F}_{u_i}(1)\right)\quad(3.1)$$

if and only if u_i, $i=1,2$ is a solution of the fractional boundary-value problem

$$\begin{cases}\mathcal{D}_q^{q_1}u_1(\tau)=\mathcal{F}_{u_1},\\[2ex]\mathcal{D}_q^{q_2}u_2(\tau)=\mathcal{F}_{u_2},\end{cases}\qquad,\tau\in\mathrm{J}:=[1,1],\qquad(3.2)$$

$$\begin{cases}u_1(0)=a_1\mathcal{I}_q^{\beta_1}u(\eta_1),\ 0<\eta_1<1,\ \beta_1>0,\\u_1(1)=b_1\mathcal{I}_q^{\alpha_2}u(\sigma_1),\ 0<\sigma_1<1,\ \alpha_1>0,\\u_2(0)=a_2\mathcal{I}_q^{\beta_2}u(\eta_2),\ 0<\eta_2<1,\ \beta_2>0,\\u_2(1)=b_2\mathcal{I}_q^{\alpha_2}u(\sigma_2),\ 0<\sigma_2<1,\ \alpha_2>0,\end{cases}\qquad(3.3)$$

where

$$\begin{aligned}\Lambda_{1,i}&=\frac{a_i}{\Lambda_i}\left(1-\frac{b_i\sigma_i^{\alpha_i+1}}{\Gamma(\alpha_i+2)}\right),\quad\Lambda_{2,i}=\frac{a_i\eta_i^{\beta_i+1}}{\Lambda_i\Gamma(\beta_i+2)},\\\Lambda_{3,i}&=\frac{1}{\Lambda_i}\left(1-\frac{a_i\eta_i^{\beta_i}}{\Gamma(\beta_i+1)}\right),\quad\Lambda_{4,i}=\frac{a_i}{\Lambda_i}\left(1-\frac{b_i\sigma_i^{\alpha_i}}{\Gamma(\alpha_i+1)}\right),\\\Lambda_i&=\left(1-\frac{a_i\eta_i^{\beta_i}}{\Gamma(\beta_i+1)}\right)\left(1-\frac{b_i\sigma_i^{\alpha_i+1}}{\Gamma(\alpha_i+2)}\right)+\frac{a_i\eta_i^{\beta_i+1}}{\Gamma(\beta_i+2)}\left(1-\frac{b_i\sigma_i^{\alpha_i}}{\Gamma(\alpha_i+1)}\right).\end{aligned}\qquad(3.4)$$

Proof . For some constants $c_{0,i}$, $c_{1,i}\in\mathbb{R}$ and $1<q_i\leq 2$, the general solution of $\mathcal{D}_q^{q_i}u_i(\tau)=\mathcal{F}_{u_i}(\tau)$ can be written as

$$u_i(\tau)=\mathcal{I}_q^{q_i}\mathcal{F}_{u_i}(\tau)+c_{0,i}+c_{1,i}\ \tau.\qquad(3.5)$$

Using the boundary conditions (3.3) in (3.5) we may obtain

$$\left(1-\frac{a_i\eta_i^{\beta_i}}{\Gamma(\beta_i+1)}\right)c_{0,i}-\frac{a_i\eta_i^{\beta_i+1}}{\Gamma(\beta_i+2)}c_{1,i}=a_i\mathcal{I}_q^{q_i+\beta_i}\mathcal{F}_{u_i}(\eta_i),$$
$$\left(1-\frac{b_i\sigma_i^{\alpha_i}}{\Gamma(\alpha_i+1)}\right)c_{0,i}+\left(1-\frac{b_i\sigma_i^{\alpha_i+1}}{\Gamma(\alpha_i+2)}\right)c_{1,i}=b_i\mathcal{I}_q^{q_i+\alpha_i}\mathcal{F}_{u_i}(\sigma_i)-\mathcal{I}_q^{q_i}\mathcal{F}_{u_i}(1). \tag{3.6}$$

which, on solving, yields

$$c_{0,i}=\frac{1}{\Lambda_i}\left\{a_i\left(1-\frac{b_i\sigma_i^{\alpha_i+1}}{\Gamma(\alpha_i+2)}\right)\mathcal{I}_q^{q_i+\beta_i}\mathcal{F}_{u_i}(\eta_i)+\frac{a_i\eta_i^{\beta_i+1}}{\Gamma(\beta_i+2)}\left(b_i\mathcal{I}_q^{q_i+\alpha_i}\mathcal{F}_{u_i}(\sigma_i)-\mathcal{I}_q^{q_i}\mathcal{F}_{u_i}(1)\right)\right\},$$

and

$$c_{1,i}=\frac{1}{\Lambda_i}\left\{a_i\left(\frac{b_i\sigma_i^{\alpha_i}}{\Gamma(\alpha_i+1)}-1\right)\mathcal{I}_q^{q_i+\beta_i}\mathcal{F}_{u_i}(\eta_i)+\left(1-\frac{a_i\eta_i^{\beta_i}}{\Gamma(\beta_i+1)}\right)\left(b_i\mathcal{I}_q^{q_i+\alpha_i}\mathcal{F}_{u_i}(\sigma_i)-\mathcal{I}_q^{q_i}\mathcal{F}_{u_i}(1)\right)\right\}.$$

Substituting the value of $c_{0,i}, c_{1,i}$ in (3.5) we get (3.1), which completes the proof. □ We use the following sufficient assumptions in the proofs of our main results.

(H1) There exist constants $\mathcal{L}_i>0$, $i=1,2$ such that for $\tau\in$ J and each $u_i, v_i\in\mathcal{C}$, $i=1,2$.

$$\|\mathcal{F}_1(\tau,u_1,u_2)-\mathcal{F}(\tau,v_1,v_2)\|\le\mathcal{L}_1\sum_{i=1}^{2}\left(\|u_i-v_i\|\right),$$
$$\|\mathcal{F}_2(\tau,u_1,u_2)-\mathcal{F}(\tau,v_1,v_2)\|\le\mathcal{L}_2\sum_{i=1}^{2}\left(\|u_i-v_i\|\right). \tag{3.7}$$

(H2) For arbitrary $\tau\in$ J and each $u_1,u_2\in\mathcal{C}$ there exist constants $K_i, M_i, N_i>0, i=1,2$, and $p\in(0,1)$ such that

$$\|\mathcal{F}_1(\tau,u_1(s),u_2(s))\|\le K_1\|u_1\|^p+M_1\|u_2\|^p+N_1,$$
$$\|\mathcal{F}_2(\tau,u_1(s),u_2(s))\|\le K_2\|u_1\|^p+M_2\|u_2\|^p+N_2. \tag{3.8}$$

In the following, we set an abbreviated notation for the fractional q-integral of the Caputo type of order $q_i>0$, for a function with two variables as

$$\mathcal{I}_q^{q_i}\mathcal{F}_{u_i}(\tau)=\frac{1}{\Gamma(q_i)}\int_0^\tau(\tau-qs)^{\alpha-1}\mathcal{F}(s,u_1(s),u_2(s))ds.$$

Moreover, for computational convenience we put

$$\omega_i=\left\{(|\Lambda_{1,i}|+|\Lambda_{4,i}|)\frac{\eta_i^{q_i+\beta_i}}{\Gamma(q_i+\beta_i+1)}+(|\Lambda_{2,i}|+|\Lambda_{3,i}|)\left(\frac{|b_i|\sigma_i^{q_i+\alpha_i}}{\Gamma(q_i+\alpha_i+1)}+\frac{1}{\Gamma(q_i+1)}\right)\right\}, \tag{3.9}$$

and

$$\bar{\omega}_i=\left\{|\Lambda_{4,i}|\frac{\eta_i^{q_i+\beta_i}}{\Gamma(q_i+\beta_i+1)}+|\Lambda_{3,i}|\left(\frac{|b_i|\sigma_i^{q_i+\alpha_i}}{\Gamma(q_i+\alpha_i+1)}+\frac{1}{\Gamma(q_i+1)}\right)\right\}. \tag{3.10}$$

By Lemma 3.1, we consider two operators $\mathcal{T}, \mathcal{S} : \mathcal{C} \longrightarrow \mathcal{C}$ as follows:

$$\mathcal{T}u_i(\tau) = \mathcal{I}_q^{q_i}\mathcal{F}_{u_i}(\tau), \ \tau \in \mathrm{J},$$

and

$$\mathcal{S}u_i(\tau) = (\Lambda_{1,i} - \Lambda_{4,i}\ \tau)\, \mathcal{I}_q^{q_i+\beta_i}\mathcal{F}_{u_i}(\eta_i) + (\Lambda_{2,i} + \Lambda_{3,i}\ \tau)\left(b_i \mathcal{I}_q^{q_i+\alpha_i}\mathcal{F}_{u_i}(\sigma_i) - \mathcal{I}_q^{q_i}\mathcal{F}_{u_i}(1)\right), \ \tau \in \mathrm{J}.$$

Then the integral equation (3.1) in Lemma 3.1 can be written as an operator equation

$$\mathcal{K}u_i(\tau) = \mathcal{T}u_i(\tau) + \mathcal{S}u_i(\tau), \quad \tau \in \mathrm{J}.$$

The continuity of $\mathcal{F}_i$, i=1,2, shows that the operator $\mathcal{K} : \mathcal{C} \to \mathcal{C}$ is well define and fixed points of the operator equation are solutions of the integral equations (3.1) in Lemma 3.1.

Lemma 3.2. *The operator $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is Lipschitz with constant $\sum_{i=1}^{2} \ell_{\mathcal{F}_i} = \sum_{i=1}^{2} \frac{\mathcal{L}_i}{\Gamma(q_i+1)}$. Moreover, $\mathcal{T}$ satisfies the growth condition given below*

$$\|\mathcal{T}(u_1,u_2)\| \le \sum_{i=1}^{2} \frac{1}{\Gamma(\alpha+1)}(K_i\|u_1\|^p + M_i\|u_2\|^p + N_i),$$

for every $u_i \in \mathcal{C}$.

Proof .
To show that the operator $\mathcal{T}$ is Lipschitz. Let $u_i, v_i \in \mathcal{C}$,i=1,2, then we have

$$\begin{aligned}
|\mathcal{T}u_i(\tau) - \mathcal{T}v_i(\tau)| &= \left|\mathcal{I}_q^{q_i}\mathcal{F}_{i,u_i} - \mathcal{I}_q^{q_i}\mathcal{F}_{i,v_i}\right| \\
&\le \mathcal{I}_q^{q_i}|\mathcal{F}_{i,u_i} - \mathcal{F}_{i,v_i}|(\tau) \\
&\le \mathcal{I}_q^{q_i}(1)\ \mathcal{L}_i \sum_{i=1}^{2} (\|u_i - v_i\|) \\
&= \frac{\mathcal{L}_i}{\Gamma(q_i+1)} \sum_{i=1}^{2} (\|u_i - v_i\|)\,.
\end{aligned}$$

For all $\tau \in \mathrm{J}$, we obtain

$$\|\mathcal{T}u_i - \mathcal{T}v_i\| \le \frac{\mathcal{L}_i}{\Gamma(q_i+1)} \sum_{i=1}^{2} (\|u_i - v_i\|)\,.$$

Hence, $\mathcal{T} : \mathcal{C} \longrightarrow \mathcal{C}$ is a Lipschitzian on $\mathcal{C}$ with Lipschitz constant $\ell_{\mathcal{F}_i} = \frac{\mathcal{L}_i}{\Gamma(q_i+1)}$. By Proposition 2.17, $\mathcal{T}$ is κ–Lipschitz with constant $\ell_{\mathcal{F}_i}$. Moreover, for growth condition, we have

$$\begin{aligned}
|\mathcal{T}u_i(\tau)| &\le \mathcal{I}_q^{q_i}|\mathcal{F}_{u_i}|(\tau) \\
&\le (K_i\|u_1\|^p + M_i\|u_2\|^p + N_i)\, \mathcal{I}_q^{\alpha}(1) \\
&= \frac{1}{\Gamma(q_i+1)}(K_i\|u_1\|^p + M_i\|u_2\|^p + N_i).
\end{aligned}$$

Hence it follows that

$$\|\mathcal{T}u_i\| \le \frac{1}{\Gamma(q_i+1)}(K_i\|u_1\|^p + M_i\|u_2\|^p + N_i),$$

which implies that

$$\|\mathcal{T}(u_1,u_2)\| \leq \sum_{i=1}^{2} \frac{1}{\Gamma(\alpha+1)}(K_i\|u_1\|^p + M_i\|u_2\|^p + N_i).$$

□

Lemma 3.3. *$\mathcal{S}$ is continuous and satisfies the growth condition given as below,*

$$\|\mathcal{S}u_i\| \leq (K_i\|u_1\|^p + M_i\|u_2\|^p + N_i)\omega_i, \text{ for every } u_i \in \mathcal{C},$$

where ω_i is given by (3.9)*.*

Proof . Choose a bounded subset $D_r = \{(u_1,u_2) \in \mathcal{C} : \|(u_1,u_2)\| \leq r\} \subset \mathcal{C}$ and consider a sequence $\{z_n = (u_{1,n}, u_{2,n})\} \in D_r$ such that $z_n \to z = (u_1,u_2)$ as $n \to \infty$ in D_r. We need to show that $\|\mathcal{S}z_n - \mathcal{S}z| \to 0, n \to \infty$. From the continuity of $\mathcal{F}_{i,u}$, it follows that $\mathcal{F}_{i,u_n} \to \mathcal{F}_{i,u}$, as $n \to \infty$. In view of (H_2) , we obtain the following relations:

$$\begin{aligned}
&(\tau - sq)^{q_i-1}\|\mathcal{F}_{i,u_n} - \mathcal{F}_{i,u}\| \leq (K_i\|u_1\|^p + M_i\|u_2\|^p + N_i)(\tau - sq)^{q_i-1}, i = 1,2.\\
&(\eta_i - sq)^{q_i+\beta_i-1} \mapsto (K_i\|u_1\|^p + M_i\|u_2\|^p + N_i)(\eta_i - sq)^{q_i+\beta_i-1}, i = 1,2,\\
&(\sigma_i - sq)^{q_i+\alpha_i-1} \mapsto (K_i\|u_1\|^p + M_i\|u_2\|^p + N_i)(\sigma_i - sq)^{q_i+\alpha_i-1}, i = 1,2,\\
&(1 - sq)^{q_i-1} \mapsto (K_i\|u_1\|^p + M_i\|u_2\|^p + N_i)(1 - sq)^{q_i-1}, i = 1,2,
\end{aligned}$$

which implies that each term on the left is integrable. By Lebesgue Dominated convergent theorem, we obtain

$$\begin{aligned}
&\mathcal{I}_q^{q_i+\beta_i}|\mathcal{F}_{i,u_n} - \mathcal{F}_{i,u}|(\eta_i) \to 0 \text{ as } n \to +\infty,\\
&\mathcal{I}_q^{q_i+\alpha_i}|\mathcal{F}_{i,u_n} - \mathcal{F}_{i,u}|(\sigma_i) \to 0 \text{ as } n \to +\infty,\\
&\mathcal{I}_q^{q_i}|\mathcal{F}_{i,u_n} - \mathcal{F}_{i,u}|(1) \to 0 \text{ as } n \to +\infty.
\end{aligned}$$

It follows that $\|\mathcal{S}z_n - \mathcal{S}z\| \to 0$ as $n \to +\infty$. Which implies the continuity of the operator $\mathcal{S}$. For the growth condition, using the assumption (H2) we have

$$\begin{aligned}
|\mathcal{S}u_i(\tau)| &\leq (|\Lambda_{1,i}| + |\Lambda_{4_i}|)\,\mathcal{I}_q^{q_i+\beta_i}\mathcal{F}_{u_i}(\eta_i) + (|\Lambda_{2,i}| + |\Lambda_{3,i}|)\left(|b|\mathcal{I}_q^{q_i+\alpha_i}\mathcal{F}_{u_i}(\sigma_i) + \mathcal{I}_q^{q_i}\mathcal{F}_{u_i}(1)\right),\\
&\leq (K_i\|u_1\|^p + M_i\|u_2\|^p + N_i)(|\Lambda_{1,i}| + |\Lambda_{4_i}|)\,\mathcal{I}_q^{q_i+\beta_i}(1)(\eta_i)\\
&+ (K_i\|u_1\|^p + M_i\|u_2\|^p + N_i)(|\Lambda_{2,i}| + |\Lambda_{3,i}|)\left(|b_i|\mathcal{I}_q^{q_i+\alpha_i}(1)(\sigma_i) + \mathcal{I}_q^{q_i}(1)\right)\\
&\leq (K_i\|u_1\|^p + M_i\|u_2\|^p + N_i)\Bigg\{(|\Lambda_{1,i}| + |\Lambda_{4_i}|)\frac{\eta_i^{q_i+\beta_i}}{\Gamma(q_i+\beta_i+1)}\\
&+ (|\Lambda_{2,i}| + |\Lambda_{3,i}|)\left(|b_i|\frac{\sigma_i^{q_i+\alpha_i}}{\Gamma(q_i+\alpha_i+1)} + \frac{1}{\Gamma(q_i+1)}\right)\Bigg\}\\
&= (K_i\|u_1\|^p + M_i\|u_2\|^p + N_i)\omega_i.
\end{aligned}$$

Which implies that,

$$\|\mathcal{S}(u_1,u_2)\| \leq \sum_{i=1}^{2}(K_i\|u_1\|^p + M_i\|u_2\|^p + N_i)\omega_i, i = 1,2. \tag{3.11}$$

where ω_i, i=1,2 is given by (3.9). This completes the proof of Lemma 3.3. □

Lemma 3.4. *The operator $\mathcal{S} : \mathcal{C} \longrightarrow \mathcal{C}$ is compact. Consequently, $\mathcal{S}$ is κ-Lipschitz with zero constant.*

Proof . In order to show that $\mathcal{S}$ is compact. Let us take a bounded set $\Omega \subset \mathcal{B}_r$, i=1,2. We are required to show that $\mathcal{S}(\Omega)$ is relatively compact in $\mathcal{C}$. For arbitrary $u_i \in \Omega \subset \mathcal{B}_r$, then with the help of the estimates (3.11) we can obtain

$$\|\mathcal{S}u\| \leq (K_i r^p + M_i r^p + N_i)\,\omega_i,$$

where ω_i is given by (3.9), which shows that $\mathcal{S}(\Omega)$ is uniformly bounded.
Now, for equi-continuity of $\mathcal{S}$ take $\tau_1, \tau_2 \in$ J with $\tau_1 < \tau_2$, and let $u_i \in \Omega$. Thus, we get

$$\begin{aligned}|\mathcal{S}u_i(\tau_2) - \mathcal{S}u_i(\tau_1)| &\leq |\Lambda_{4,i}|\,(\tau_2-\tau_1)\,\mathcal{I}_q^{q_i+\beta_i}\mathcal{F}_{u_i}(\eta_i)\\ &\quad + |\Lambda_{3,i}|\,(\tau_2-\tau_1)\left(b_i\mathcal{I}_q^{q_i+\alpha_i}\mathcal{F}_{u_i}(\sigma_i) - \mathcal{I}_q^{q_i}\mathcal{F}_{u_i}(1)\right)\\ &\leq \bar{\omega}_i\,(K_i\|u_i\|^p + M_i\|v_i\|^p + N_i)\,(\tau_2-\tau_1).\end{aligned}$$

Which implies that,

$$|\mathcal{S}(u_1,u_2)(\tau_2) - \mathcal{S}(u_1,u_2)(\tau_1)| \leq \sum_{i=1}^{2}\bar{\omega}_i\,(K_i\|u_i\|^p + M_i\|v_i\|^p + N_i)\,(\tau_2-\tau_1).$$

where $\bar{\omega}_i$ is given by (3.10). From the last estimate, we deduce that $\|\mathcal{S}(u_1,u_2)(\tau_2) - \mathcal{S}(u_1,u_2)(\tau_1)\| \to 0$ when $\tau_2 \to \tau_1$. Therefore, $\mathcal{S}$ is equicontinuous. Thus, by Ascoli–Arzelà theorem, the operator $\mathcal{S}$ is compact and hence by Proposition 2.16. $\mathcal{S}$ is κ–Lipschitz with zero constant. □

Theorem 3.5. *Suppose that (H1)–(H2) are satisfied, then the BVP* (1.1) *has at least one solution $(u_1,u_2) \in \mathcal{C}$, provided that $\sum_{i=1}^{2}\ell_{\mathcal{F}_i} < 1$, $i = 1,2$, and the set of the solutions is bounded in $\mathcal{C}$.*

Proof . Let $\mathcal{T}, \mathcal{S}, \mathcal{K}$ are the operators defined in the start of this section. These operators are continuous and bounded. Moreover, by Lemma 3.2, $\mathcal{T}$ is κ–Lipschitz and by Lemma 3.4, $\mathcal{S}$ is κ–Lipschitz with constant 0. Thus, $\mathcal{K}$ is κ–Lipschitz with constant $\ell_{\mathcal{F}_i}$. Hence $\mathcal{K}$ is strict κ–contraction with constant $\ell_{\mathcal{F}_i}$. Since $\sum_{i=1}^{2}\ell_{\mathcal{F}_i} < 1$, so $\mathcal{K}$ is κ-condensing.

Now consider the following set

$$\Theta = \left\{(u_1,u_2) \in \mathcal{C} :\ \text{there exist } \xi \in [0,1] \text{ such that } u_i = \xi\mathcal{K}u_i, i = 1,2\right\}.$$

We will show that the set Θ is bounded. For $u_i \in \Theta$, we have $u_i = \xi\mathcal{K}u_i = \xi(\mathcal{T}(u_i) + S(u_i))$, which implies that

$$\begin{aligned}\|u_i\| &\leq \xi(\|\mathcal{T}u_i\| + \|\mathcal{S}u_i\|)\\ &\leq \left[\frac{1}{\Gamma(q_i+1)} + \omega_i\right](K_i\|u_1\|^p + M_i\|u_2\|^p + N_i),\end{aligned}$$

hence we get

$$\begin{aligned}\|(u_1,u_2)\| &\leq \xi(\|\mathcal{T}(u_1,u_2)\| + \|\mathcal{S}(u_1,u_2)\|)\\ &\leq \sum_{i=1}^{2}\left[\frac{1}{\Gamma(q_i+1)} + \omega_i\right](K_i\|u_1\|^p + M_i\|u_2\|^p + N_i),\end{aligned}$$

where ω_i is given by (3.9). From the above inequalities, we conclude that Θ is bounded in $\mathcal{C}$. If it is not bounded, then dividing the above inequality by $a := \|u_i\|$ and letting $a \to \infty$, we arrive at

$$1 \leq \sum_{i=1}^{2} \left[\frac{1}{\Gamma(q_i+1)} + \omega_i\right] \lim_{a\to\infty} \frac{K_i a^p + M_i a^p + N_i}{a} = 0,$$

which is a contradiction. Thus the set Θ is bounded in $\mathcal{C}$ and the operator $\mathcal{K}$ has at least one fixed point which represent the solution of BVP (1.1). □ To end this section, we give an existence and uniqueness result.

Theorem 3.6. *Under assumption (H1) the BVP* (1.1) *has a unique solution if*

$$\sum_{i=1}^{2} \left[\frac{1}{\Gamma(q_i+1)} + \omega_i\right] \mathcal{L}_i < 1. \tag{3.12}$$

Proof . Let $u_i, v_i \in \mathcal{C}$ and $\tau \in$ J, then we have

$$\begin{aligned}
|\mathcal{K}u_i(\tau) - \mathcal{K}v_i(\tau)| &\leq {}^{H}\mathcal{I}_q^{q_i}|\mathcal{F}_{u_i} - \mathcal{F}_{v_i}|(\tau) + (|\Lambda_{1,i}| + |\Lambda_{4,i}|)\, \mathcal{I}_q^{q_i+\beta_i}|\mathcal{F}_{u_i} - \mathcal{F}_{v_i}|(\eta_i) \\
&\quad + (|\Lambda_{2,i}| + |\Lambda_{3,i}|)\, |b_i| \mathcal{I}_q^{q_i+\alpha_i}|\mathcal{F}_{u_i} - \mathcal{F}_{v_i}|(\sigma_i) + (|\Lambda_{2,i}| + |\Lambda_{3,i}|T)\, \mathcal{I}_q^{q_i}|\mathcal{F}_{u_i} - \mathcal{F}_{v_i}|(1) \\
&\leq \mathcal{L}_i \sum_{i=1}^{2}(\|u_i - v_i\|) \left\{\mathcal{I}_q^{q_i}(1)(1) + (|\Lambda_{1,i}| + |\Lambda_{4,i}|)\, \mathcal{I}_q^{q_i+\beta_i}(1)(\eta_i)\right. \\
&\quad \left. + (|\Lambda_{2,i}| + |\Lambda_{3,i}|)\, |b_i| \mathcal{I}_q^{q_i+\alpha_i}(1)(\sigma_i) + (|\Lambda_{2,i}| + |\Lambda_{3,i}|T)\, \mathcal{I}_q^{q_i}(1)\right\} \\
&\leq \mathcal{L}_i \sum_{i=1}^{2}(\|u_i - v_i\|) \left(\frac{1}{\Gamma(q_i+1)} + \left\{(|\Lambda_{1,i}| + |\Lambda_{4,i}|) \frac{\eta_i^{q_i+\beta_i}}{\Gamma(q_i+\beta_i+1)}\right.\right. \\
&\quad \left.\left. + (|\Lambda_{2,i}| + |\Lambda_{3,i}|) \left(|b_i| \frac{\sigma_i^{q_i+\alpha_i}}{\Gamma(q_i+\alpha_i+1)} + \frac{1}{\Gamma(q_i+1)}\right)\right\}\right) \\
&= \left[\frac{1}{\Gamma(q_i+1)} + \omega_i\right] \mathcal{L}_i \sum_{i=1}^{2}(\|u_i - v_i\|).
\end{aligned}$$

Hence $\mathcal{K}$ is contraction as $\sum_{i=1}^{2} \left[\frac{1}{\Gamma(q_i+1)} + \omega_i\right] \mathcal{L}_i < 1$ and by Banach contraction principle $\mathcal{K}$ has a unique fixed point which is a unique solution of problem (1.1). This completes the proof. □

Remark 3.7. *If the growth condition (H2) is formulated for $p = 1$, then the conclusions of Theorem 3.5 remain valid provided that*

$$\sum_{i=1}^{2} \left[\frac{1}{\Gamma(q_i+1)} + \omega_i\right] (K_i + M_i) < 1.$$

4. Examples

In this section, in order to illustrate the main result, we consider two examples.

Example 4.1. *Consider the following boundary value problem of a fractional differential equation:*

$$\begin{cases} \mathcal{D}^{\frac{4}{3}}_{\frac{1}{4}} u_1(t) = \frac{1}{e^{(t)}+9}\left(\frac{|u_1(t)|}{1+|u_1(t)|}\right) + \frac{\sqrt{3+t^2}|u_1(t)|}{20} + t, \quad t \in \mathrm{J} := [0,1], \\ \mathcal{D}^{q_2}_{\frac{1}{4}} u_2(t) = \frac{\sin\left(\sqrt{|u_2(t)|}\right)}{16} + \left(\frac{e^{-\pi t}|u_2(t)|}{16+|u_2(t)|}\right) + (1+t^2), \\ u_1(0) = a_1 \mathcal{I}^{\frac{5}{2}}_{\frac{1}{4}} u(\frac{1}{4}), \quad u_1(1) = b_1 \mathcal{I}^{\frac{1}{4}}_{\frac{1}{4}} u(\frac{1}{5}), \\ u_2(0) = a_2 \mathcal{I}^{\frac{4}{5}}_{\frac{1}{4}} u(\frac{2}{5}), \quad u_2(1) = b_2 \mathcal{I}^{\frac{4}{5}}_{\frac{1}{4}} u(\frac{2}{5}). \end{cases} \tag{4.1}$$

Note that, this problem is a particular case of BVP (1.1)*, where*

$$\begin{aligned} &q_1 = \frac{4}{3}, q_2 = \frac{7}{5}, \boldsymbol{q} = \boldsymbol{\frac{1}{4}}, T = 1, \\ &a_1 = b_2 = \frac{1}{2}; a_2 = b_1 = \frac{1}{5}; \eta_2 = \sigma_2 = \frac{2}{5}, \beta_1 = \frac{5}{2}, \\ &\alpha_1 = \eta_1 = \frac{1}{4}, \beta_2 = \frac{1}{3}, \alpha_2 = \frac{4}{5}, \sigma_1 = \frac{1}{5}. \end{aligned} \tag{4.2}$$

Using the given values of the parameters in (3.4) and (3.9), by the Matlab program, we find that

$$\sum_{i=1}^{2} \left[\frac{T^{q_i}}{\Gamma(q_i+1)} + \omega_i\right] = 2.332,$$

In order to illustrate Theorem 3.5, we take

$$\begin{aligned} f_1(t, u_1(t), u_2(t)) &= \frac{1}{e^{(t-1)}+9}\left(\frac{|u_1(t)|}{1+|u_1(t)|}\right) + \frac{\sqrt{3+t^2}|u_2(t)|}{20} + t, \\ f_2(t, u_1(t), u_2(t)) &= \frac{\sin\left(\sqrt{|u_1(t)|}\right)}{16} + \left(\frac{e^{-\pi t}|u_2(t)|}{16+|u_2(t)|}\right) + (1+t^2). \end{aligned} \tag{4.3}$$

We can easily show that

$$\begin{aligned} \|f_1(t, u_1, u_2) - f(t, v_1, v_2)\| &\leq \frac{1}{10}\sum_{i=1}^{2}[\|u_i - v_i\|], \\ \|f_2(t, u_1, u_2) - g(t, v_1, v_2)\| &\leq \frac{1}{16}\sum_{i=1}^{2}[\|u_i - v_i\|]. \end{aligned} \tag{4.4}$$

Hence the condition (H1) holds with $L_1 = \frac{1}{10}$, $L_2 = \frac{1}{16}$. *Further from the above given data it is easy to calculate*

$$\sum_{i=1}^{2} \ell_{f_i} = \sum_{i=1}^{2}\left[\frac{1}{\Gamma(q_i+1)}\right] L_i = 1.8703,$$

On the other hand, for any $t \in \mathrm{J}, u \in \mathbb{R}$ *we have*

$$|f(t, u_1, u_2)| \leq \frac{1}{10}|u_1| + \frac{1}{10}|u_2| + 1,$$

$$|f(t, u_1, u_2)| \leq \frac{1}{16}|u_1| + \frac{1}{16}|u_2| + 2,$$

Condition (H2) holds with $M_1 = K_1 = \frac{1}{10}$, $M_2 = K_2 = \frac{1}{16}$, $p = N_1 = 1$ *and* $N_2 = 2$. *In view of Theorem 3.5*

$$\Theta = \{(u_1, u_2) \in \mathcal{C} : \text{ there exist } \xi \in [0,1] \text{ such that } u_i = \xi \mathcal{K} u_i, i = 1,2\},$$

is the solution set; then

$$\begin{aligned} \| (u_1, u_2) \| &\leq \xi(\|\mathcal{T}(u_1, u_2)\| + \|\mathcal{S}(u_1, u_2)\|) \\ &\leq \sum_{i=1}^{2} \left[\frac{1}{\Gamma(q_i+1)} + \omega_i\right] ((K_i + M_i)(\|u_1\| + \|u_2\|) + N_i). \end{aligned}$$

From which, we have

$$\| (u_1, u_2) \| \leq \frac{\sum_{i=1}^{2} \left[\frac{1}{\Gamma(q_i+1)} + \omega_i\right] N_i}{1 - \sum_{i=1}^{2} \left[\frac{1}{\Gamma(q_i+1)} + \omega_i\right] (M_i + K_i)} = 19.8124.$$

By Theorem 3.5, the BVP (1.1) with the data (4.5) and (4.3) has at least a solution u *in* $C(\mathrm{J} \times \mathbb{R}, \mathbb{R})$. *Furthermore* $\sum_{i=1}^{2} \left[\frac{1}{\Gamma(q_i+1)} + \omega_i\right] L_i = 0.1854, < 1$. *Hence by Theorem 3.6 the boundary value problem (1.1) with the data (4.5) and (4.3) has a unique solution.*

Example 4.2. *Let us consider coupled system* (1.1) *with specific data:*

$$\begin{aligned} & q_1 = \frac{3}{2}, q_2 = \frac{5}{4}, q = \frac{1}{2}, T = 1, \\ & a_1 = b_1 = a_2 = b_2 = 1; \beta_1 = \eta_2 = \sigma_2 = \frac{1}{2}, \\ & \alpha_1 = \eta_1 = \frac{3}{4}, \beta_2 = \frac{2}{3}, \alpha_2 = \frac{2}{5}, \sigma_1 = \frac{1}{3}. \end{aligned} \tag{4.5}$$

In order to illustrate Theorem 3.5, we take

$$\begin{aligned} f_1(t, u_1, u_2) &= \tfrac{1}{4} + \tfrac{e^{-\pi t}\sqrt{|u_1(t)|}}{16+\sqrt{|u_1(t)|}} + \tfrac{\cos\sqrt{|u_2(t)|}}{16}, \\ f_2(t, u_1, u_2) &= \tfrac{1}{8} + \tfrac{\sin\sqrt{|u_1(t)|}}{24} + \tfrac{\sqrt{|u_2(t)|}}{24}. \end{aligned} \tag{4.6}$$

One has

$$\begin{aligned} \|f_1(t, u_1, u_2) - f(t, v_1, v_2)\| &\leq \frac{1}{16} \sum_{i=1}^{2} [\|u_i - v_i\|], \\ \|f_2(t, u_1, u_2) - g(t, v_1, v_2)\| &\leq \frac{1}{24} \sum_{i=1}^{2} [\|u_i - v_i\|]. \end{aligned} \tag{4.7}$$

The condition (H1) holds with $L_1 = \frac{1}{16}$ *and* $L_1 = \frac{1}{24}$. *Further from the above given data it is easy to calculate*

$$\sum_{i=1}^{2} \ell_{f_i} = \sum_{i=1}^{2} \frac{L_i T^{\alpha}}{\Gamma(\alpha+1)} = 0.1446.$$

Using the given values of the parameters in (3.4) *and* (3.9)*, by the Matlab program, we find that*

$$\sum_{i=1}^{2} \frac{T^{q_i}}{\Gamma(q_i+1)} + \omega_i = 4.6588. \tag{4.8}$$

Hence condition (H1) holds with $L_1 = \frac{1}{16}$*,* $L_2 = \frac{1}{24}$*. We shall check that condition* (3.12) *is satisfied. Indeed using the Matlab program, we can find*

$$\sum_{i=1}^{2} \left[\frac{T^{q_i}}{\Gamma(q_i+1)} + \omega_i\right] L_i = 0.2332 < 1.$$

By Theorem 3.6 the boundary value problem (1.1)-(4.1) *has a unique solution.*

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