Int. J. Nonlinear Anal. Appl. 13 (2022) 1, 3257-3280
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.6080

Controllability of impulsive fractional nonlinear control system with Mittag-Leffler kernel in Banach space

Fadhil Abbas Naji[a,*], Iftichar Al-Sharaa[a]

[a]*Department of mathematics, College of Education for Pure Sciences, University of Babylon, Babel, Iraq*

(Communicated by Madjid Rshaghi Gordji)

Abstract

In this paper, we study the controllability of a nonlinear impulsive fractional control system with Mittag-Leffler kernel in Banach space. Firstly, we present the mild solution of the control system using fractional calculus and semigroup theory. We set sufficient conditions to prove the controllability of the control system using the Nussbaum fixed point theorem. Finally, to illustrate our results, an example is given.

Keywords: Controllability, Fractional calculus, Mittag-Leffler, semigroup theory.

1. Introduction

Fractional calculus has received significant interest from researchers because it describes many scientific phenomena with great accuracy. This concept was originally described in 1695 by Leibniz and L'Hospital as a generalization of the integer-order derivative. However, fractional calculus was used in the 1960s. There are many critical applications of fractional calculus in many fields such as physics, engineering, biology, medicine, etc. Several authors discussed fractional calculus in [1], [2] and [3].

Impulsive equation theory explains processes whose states change rapidly at specific points. As a result, it has a wide range of applications in medicine, biology, physics, electrical engineering, and other fields. This has drawn the attention of many researchers. For more details for this topic see [4]

*Corresponding author

Email addresses: fadel.naji@student.uobabylon.edu.iq (Fadhil Abbas Naji), pure.iftichar.talb@uobabylon.edu.iq (Iftichar Al-sharaa)

Received: June 2021 *Accepted:* October 2021

[5] [6] . Solutions of impulsive fractional systems have been investigated in numerous publications [7],[8] and [9].

Controllability is an essential qualitative characteristic of a dynamical system that affects object behavior to achieve the desired objective. Many authors have studied the controllability problem of impulsive fractional control systems in finite and infinite dimensional spaces, [10], [11] and [12].

In 2020, Hattaf [13] introduced a new definition of the fractional derivative with a non-singular kernel which we will refer to by Hattaf-fractional derivative. we will go over this definition and some of its features in the next section.

This paper investigates the controllability of the fractional impulsive nonlinear system

$$\begin{cases} {}^{C}D^{\rho,\omega}\left[y\left(t\right)-h\left(t,y\left(t\right)\right)\right]=Ay\left(t\right)+Bu\left(t\right)+f\left(t,y\left(t\right)\right) \qquad t\in J=\left[0,\Upsilon\right],t\neq t_{\gamma}, \\ \Delta y\left(t_{\gamma}\right)=q_{\gamma}\left(y\left(t_{\gamma}^{-}\right)\right),\ \gamma=1,2,\ldots.p, \\ y\left(0\right)=y_{0,} \end{cases} \tag{1.1}$$

where $\rho,\omega\in(0,1)$, ${}^{C}D^{\rho,\omega}$ is Hattaf-fractional derivative in the sense of Caputo of order ρ,ω, $y(.)$ takes values in Banach space X, $A:D(A)\subset X\rightarrow X$ is linear operator, the control $u\in L^{2}\left(\mathcal{J},U\right)$ with U as a Banach space, $B:L^{2}\left(\mathcal{J},U\right)\rightarrow X$ is bounded linear operator, the functions $f:\mathcal{J}\times X\rightarrow X$ and $h:\mathcal{J}\times X\rightarrow D(A)$ are continuous, $0=t_0<t_1<t_2<\cdots<t_{\gamma}<t_{\gamma+1}=\Upsilon$, $y\left(t_{\gamma}^{+}\right)$ and $y\left(t_{\gamma}^{-}\right)$ indicate to the right and left limits of $y(t)$ at $t=t_{\gamma}$ respectively and $\Delta y\left(t_{\gamma}\right)=y\left(t_{\gamma}^{+}\right)-y\left(t_{\gamma}^{-}\right)$.

The following is the organization of the article. In the next section we review the basic concepts and properties of Hattaf-fractional derivative, Hattaf-fractional integral, Nussbaum fixed point theorem, and provide the mild solution of system (1.1). In the third section, we prove the system (1.1) is controllable under appropriate conditions. We provide an application that explains our main results in section four.

2. Preliminaries

In this part, we present the fundamental ideas that we use in our work.
Assume the linear operator $A:D(A)\subset X\rightarrow X$ is the generator of C_0-semigroup $\{\mathcal{G}\left(t\right),t\geq 0\}$ on a Banach space X, where $sup_{t\geq 0}\left\|\mathcal{G}\left(t\right)\right\|=\mathcal{S},\ \mathcal{S}\geq 1$. We consider the bounded linear operator $E:=\rho A_{\rho}A$. Clearly E is the generator of uniformly continuous semigroup $\{T\left(t\right),t\geq 0\}$ and $sup_{t\geq 0}\left\|T\left(t\right)\right\|=\mathcal{S}$ [14]. All through this paper, we assume $\{T\left(t\right),t\geq 0\}$ is a compact semigroup.

We introduce the family of functions

$$\begin{aligned} PC\left(\mathcal{J},X\right)=&\{y:\mathcal{J}\rightarrow X: y\ is\ continuous\ at\ t\in\mathcal{J}\backslash\left\{t_1,t_2,\ldots,t_{\gamma}\right\}, \\ &and\ \ there\ exist\ y\left(t_r^{+}\right)\ and\ y\left(t_r^{-}\right)\ with\ y\left(t_r^{-}\right)=\ \ y\left(t\right)\ for\ t=1,2,\ldots,\gamma\}. \end{aligned}$$

It is clear that $\left(PC\left(\mathcal{J},X\right),\left\|\cdot\right\|_{PC}\right)$ is a Banach space with the norm $\left\|y\right\|_{PC}=sup_{t\in I}\left\|y(t)\right\|$.

Let us introduce the definition of Hattaf-fractional derivative.

Definition 2.1 ([13]). *Let $\rho\in[0,1)$, $\omega,\lambda>0$ and $f\in H^{1}(c,d)$. Then*

$$ {}^{C}D_{\eta}^{\rho,\omega,\lambda}f\left(t\right)=\frac{N(\rho)}{1-\rho}\frac{1}{\eta(t)}\int_{c}^{t}E_{\omega}\left[-\frac{\rho}{1-\rho}(t-\zeta)^{\lambda}\right]\frac{d}{d\zeta}\left(\eta f\right)\left(\zeta\right)d\zeta, \tag{2.1}$$

is Hattaf-fractional derivative of order ρ in sense of Caputo of the function f with respect to the weight function $\eta \in C^1(c,d)$, $\eta, \acute{\eta} > 0$ on $[c,d]$. $N(\rho)$ is normalization function with $N(0) = N(1) = 1$ and $E_\omega(t) = \sum_{\gamma=1}^{\infty} \frac{t^\gamma}{\Gamma(\omega\gamma+1)}$ is Mittag-Leffler function of one parameter ω.
When $\lambda = \omega$ and $N(\rho) = \eta(t) = 1$, then the fractional derivative (2.1) will be in the form

$$ {}^{C}D^{\rho,\omega} f(t) = \frac{1}{1-\rho} \int_c^t E_\omega \left[-\frac{\rho}{1-\rho} (t-\zeta)^\omega \right] \frac{d}{d\zeta} f(\zeta)\, d\zeta. \tag{2.2} $$

Throughout this paper, we consider the fractional derivative (2.2) with $0 < \omega < 1$.

Definition 2.2 ([1]). *Riemann-Liouville fractional integral of order ω on $[c,d]$ is*

$$ \mathcal{I}^\omega f(t) = \frac{1}{\Gamma(\omega)} \int_c^t (t-\zeta)^{\omega-1} f(\xi)\, d\xi $$

where $\Gamma(\omega)$ is Gamma function.

Lemma 2.3 ([1]).
- *The Laplace transform of Riemann-Liouville fractional integral is*

$$ \mathcal{L}\{\mathcal{I}^\omega f(t)\} = \frac{1}{\lambda^\omega} \mathcal{L}\{f(t)\}(\lambda). $$

- *Riemann-Liouville fractional integral satisfies the following:*

$$ \mathcal{I}^\omega (y_1(t) + y_2(t)) = \mathcal{I}^\omega y_1(t) + \mathcal{I}^\omega y_2(t), $$

and

$$ \mathcal{I}^\alpha \mathcal{I}^\beta y(t) = \mathcal{I}^{\alpha+\beta} y(t), 0 < \alpha, \beta < \infty. $$

Definition 2.4 ([13]). *The fractional integral corresponding to the fractional derivative (2.2) on $[c,d]$ is*

$$ \mathcal{I}^{\rho,\omega} f(t) = (1-\rho) f(t) + \rho \mathcal{I}^\omega f(t) \tag{2.3} $$

where $\mathcal{I}^\omega f(t)$ is Riemann-Liouville integral of order ω.

Lemma 2.5 ([13]). *The Laplace transform of the fractional derivative (2.2) is*

$$ \mathcal{L}\left\{ {}^{C}D^{\rho,\omega} f(t) \right\}(\lambda) = \frac{1}{1-\rho} \frac{\lambda^\omega \mathcal{L}\{f(t)\}(\lambda) - \lambda^{\omega-1} f(0)}{\lambda^\omega + \frac{\rho}{1-\rho}}. $$

From Lemma 2.3, we can give the following remark.

Remark 2.6. *(i) The Laplace transform of the fractional integral (2.3) is*

$$ \mathcal{L}\left\{ {}^{C}\mathcal{I}^{\rho,\omega} f(t) \right\}(\lambda) = (1-\rho) \mathcal{L}\{f(t)\}(\lambda) + \frac{\rho}{\lambda^\omega} \mathcal{L}\{f(t)\}(\lambda), $$

(ii) $\mathcal{I}^{\rho,\omega}(y_1(t) + y_2(t)) = \mathcal{I}^{\rho,\omega} y_1(t) + \mathcal{I}^{\rho,\omega} y_2(t)$, where $0 < \rho, \omega < 1$ and $t \in [c,d]$

Lemma 2.7. *The fractional derivative (2.2) can be written as*

$$ {}^{C}D^{\rho,\omega}y\left(t\right)=\frac{1}{1-\rho}\sum_{n=0}^{\infty}\left(-\frac{\rho}{1-\rho}\right)^{n}\mathcal{I}^{\rho n+1}\dot{y}\left(t\right). $$

where $\mathcal{I}^{\rho n+1}$ *is Riemann-Leoville fractional integral of order* $\rho n+1,\ 0<n<\infty$.

Proof .

$$ \begin{aligned} {}^{C}D^{\rho,\omega}y\left(t\right)&=\frac{1}{1-\rho}\int_{c}^{t}\dot{y}\left(t\right)E_{\rho}\left(\frac{-\rho}{1-\rho}(t-s)^{\omega}\right)ds\\ &=\frac{1}{1-\rho}\int_{c}^{t}\dot{y}\left(t\right)\sum_{n=0}^{\infty}\left(-\frac{\rho}{1-\rho}\right)^{n}\frac{(t-\rho)^{\omega n}}{\Gamma\left(\omega n+1\right)}ds\\ &=\frac{1}{1-\rho}\sum_{n=0}^{\infty}\left(\frac{-\rho}{1-\rho}\right)^{n}\frac{1}{\Gamma\left(\omega n+1\right)}\int_{c}^{t}\dot{y}\left(t\right)\left(t-s\right)^{\omega n}ds\\ &=\frac{1}{1-\rho}\sum_{n=0}^{\infty}\left(-\frac{\rho}{1-\rho}\right)^{n}\mathcal{I}^{\rho n+1}\dot{y}\left(t\right). \end{aligned} $$

□

Lemma 2.8. *Let* $0<\rho,\omega<1,\ c<d$ *and* $y\in C\left(\left[c,d\right],X\right)$, *then*

$$ \mathcal{I}^{\rho,\omega C}D^{\rho,\omega}y\left(t\right)=y\left(t\right)-y\left(c\right). $$

Proof .

$$ \mathcal{I}^{\rho,\omega C}D^{\rho,\omega}y\left(t\right)=\left(1-\rho\right){}^{C}D^{\rho,\omega}y\left(t\right)+\rho\mathcal{I}^{\omega C}D^{\rho,\omega}y\left(t\right). $$

From Lemma 2.7 we have,

$$ \mathcal{I}^{\rho,\omega C}D^{\rho,\omega}y\left(t\right)=\sum_{k=0}^{\infty}\mu_{\rho}^{k}\mathcal{I}^{\omega k+1}\dot{y}(t)+\mathcal{I}^{\omega}\frac{\rho}{1-\rho}\sum_{k=0}^{\infty}\mu_{\rho}^{k}\mathcal{I}^{\omega k+1}\dot{y}(t) $$

where $\mu_{\rho}=\frac{-\rho}{1-\rho}$. By using Lemma 2.3, it follows

$$ \begin{aligned} \mathcal{I}^{\rho,\omega C}D^{\rho,\omega}y\left(t\right)&=\sum_{k=0}^{\infty}\mu_{\rho}^{k}\mathcal{I}^{\omega k+1}\dot{y}(t)+\frac{\rho}{1-\rho}\sum_{k=0}^{\infty}\mu_{\rho}^{k}\mathcal{I}^{(1+k)\omega+1}\dot{y}(t)\\ &=\sum_{k=0}^{\infty}\mu_{\rho}^{k}\mathcal{I}^{\omega k+1}\dot{y}(t)-\sum_{k=0}^{\infty}\mu_{\rho}^{k+1}\mathcal{I}^{(1+k)\omega+1}\dot{y}\left(t\right)\\ &=\mathcal{I}\dot{y}\left(t\right)=\int_{c}^{t}\dot{y}\left(s\right)ds=y\left(t\right)-y\left(c\right). \end{aligned} $$

□

Lemma 2.9. *Let* $0<\rho,\omega<1,\ c<d$ *and* $y\in PC\left(\left[c,d\right],X\right)$, *then*

$$ \mathcal{I}^{\rho,\omega C}D^{\rho,\omega}y\left(t\right)=y\left(t\right)-y\left(c\right)-\sum_{\gamma=1}^{p}\Delta y(t_{\gamma}) $$

for $\gamma=1,2,\ldots,p$ *and* $t\in[c,d]$.

Proof . Using the same technique as in Lemma 2.8, we get

$$\mathcal{I}^{\rho,\omega C}D^{\rho,\omega}y\left(t\right)=\mathcal{I}\dot{y}\left(t\right)=\int_{c}^{t}\dot{y}\left(s\right)ds.$$

Using integral by part for picewise continuous functions, we get

$$\mathcal{I}^{\rho,\omega C}D^{\rho,\omega}y\left(t\right)=y\left(t\right)-y\left(c\right)-\sum_{\gamma=1}^{p}\Delta y(t_{\gamma})$$

for $\gamma=1,2,\dots,p$ and $t\in[c,d]$. □

Assume the one-sided stable probability density [15]

$$\psi_{\omega}\left(\delta\right)=\frac{1}{\pi}\sum_{i=1}^{\infty}\left(-1\right)^{i-1}\delta^{-\omega i-1}\frac{\Gamma\left(i\omega+1\right)}{i!}sin\left(i\pi\omega\right)$$

then the Laplace transform of $\psi_{\omega}\left(\delta\right)$ is

$$\mathcal{L}\left\{\psi_{\omega}\left(\delta\right)\right\}\left(\lambda\right)=e^{-\lambda^{\omega}} \tag{2.4}$$

where $0<\delta<\infty,\ 0<\omega<1$ and $\lambda>0$.
Additionally, assume the probability density function

$$\varphi_{\omega}\left(\delta\right)=\frac{1}{\omega}\delta^{-1-\frac{1}{\omega}}\psi_{\omega}\left(\delta^{-\frac{1}{\omega}}\right),0<\delta<\infty,\ 0<\omega<1, \tag{2.5}$$

then, for $0\leq\xi\leq1$

$$\int_{0}^{\infty}\delta^{\xi}\varphi_{\omega}\left(\delta\right)d\delta=\int_{0}^{\infty}\frac{1}{\delta^{\omega\xi}}\psi_{\omega}\left(\delta\right)d\delta=\frac{\Gamma(1+\xi)}{\Gamma(1+\omega\xi)},\ 0<\delta<\infty,\ 0<\omega<1.$$

To define the mild solution of system (1.1),
We begin by proving the following lemma before defining a mild solution of (1.1).

Lemma 2.10. *If $y\in PC(\mathcal{J},X)$ and y is a solution of system (1.1), then y satisfy the following*

$$y\left(t\right)=\begin{cases}A_{\rho}h\left(t,y\left(t\right)\right)+A_{\rho}\int_{0}^{t}\left(t-\zeta\right)^{\omega-1}EL_{\omega}\left(t-\zeta\right)h\left(\zeta,y\left(\zeta\right)\right)d\zeta\\+A_{\rho}T_{\omega}\left(t\right)\left(y_{0}-h\left(0,y_{0}\right)\right)+\left(1-\rho\right)A_{\rho}\left[Bu\left(t\right)+f\left(t,y\left(t\right)\right)\right] & t\in[0,t_{1}]\\+\rho A_{\rho}^{2}\int_{0}^{t}\left(t-\zeta\right)^{\omega-1}L_{\omega}\left(t-\zeta\right)\left[Bu\left(\zeta\right)+f\left(\zeta,y\left(\zeta\right)\right)\right]d\zeta\\A_{\rho}h\left(t,y\left(t\right)\right)+A_{\rho}\int_{0}^{t}\left(t-\zeta\right)^{\omega-1}EL_{\omega}\left(t-\zeta\right)h\left(\zeta,y\left(\zeta\right)\right)d\zeta\\+A_{\rho}T_{\omega}\left(t\right)\left(y_{0}-h\left(0,y_{0}\right)\right)+\left(1-\rho\right)A_{\rho}\left[Bu\left(t\right)+f\left(t,y\left(t\right)\right)\right]\\+\rho A_{\rho}^{2}\int_{0}^{t}\left(t-\zeta\right)^{\omega-1}L_{\omega}\left(t-\zeta\right)\left[Bu\left(\zeta\right)+f\left(\zeta,y\left(\zeta\right)\right)\right]d\zeta & t\in(t_{\gamma},t_{\gamma+1}]\\+A_{\rho}\sum_{\gamma=1}^{p}\Delta y\left(t_{\gamma}\right)T_{\omega}\left(t-t_{\gamma}\right),\end{cases} \tag{2.6}$$

where $L_{\omega}\left(t\right)=\omega\int_{0}^{\infty}\theta\varphi_{\omega}\left(\theta\right)T\left(\theta t^{\omega}\right)d\theta,\ T_{\omega}\left(t\right)=\int_{0}^{\infty}\varphi_{\omega}(\theta)T\left(\theta t^{\omega}\right)d\theta,\ \varphi_{\omega}\left(\theta\right)$ *given in identically* (2.5) *and* $A_{\rho}=\left[(1-\rho)\left(\frac{1}{1-\rho}I-A\right)\right]^{-1}$, $0<\rho<1$.

Proof . We see that $y(\cdot)$ is decomposable into $m(\cdot)+n(\cdot)$, where m is a continuous mild solution of the system

$$
{}^{c}D^{\rho,\omega}m(t) = \begin{cases} {}^{c}D^{\rho,\omega}h\left(t,y\left(t\right)\right) + Am\left(t\right) + Bu\left(t\right) + f\left(t,y\left(t\right)\right) & t\in\mathcal{J} \\ m\left(0\right) = y_0 \end{cases} \tag{2.7}
$$

and n is picewise continuous solution of the system

$$
{}^{c}D^{\rho,\omega}n\left(t\right) = \begin{cases} An\left(t\right) & t\in J,\ t\neq t_\gamma \\ \Delta y\left(t_\gamma\right) = q_\gamma\left(n\left(t_\gamma^-\right)\right) & \gamma = 1,2,\ldots,p \\ n\left(0\right) = 0. \end{cases} \tag{2.8}
$$

To find the mild solution $m(t)$, we applying fractional integral (2.3) for both sides of the system (2.7),

$$
\mathcal{I}^{\rho,\omega c}D^{\rho,\omega}m\left(t\right) = \mathcal{I}^{\rho,\omega c}D^{\rho,\omega}h\left(t,y(t)\right) + \mathcal{I}^{\rho,\omega}\left[Am\left(t\right) + Bu\left(t\right) + f\left(t,y\left(t\right)\right)\right].
$$

using lemma 2.8, we have

$$
m\left(t\right) = h\left(t,y\left(t\right)\right) - h\left(0,y_0\right) + y_0 + \mathcal{I}^{\rho,\omega}\left[Am\left(t\right) + Bu\left(t\right) + f\left(t,y\left(t\right)\right)\right], \tag{2.9}
$$

By taking Laplace transform for both sides of (2.9), we have

$$
\begin{aligned}
M\left(\lambda\right) =& H\left(\lambda\right) - \frac{h\left(0,y_0\right)}{\lambda} + \frac{y_0}{\lambda} + \left(1-\rho\right)\mathcal{L}\left\{Am\left(t\right) + Bu\left(t\right) + f\left(t,y\left(t\right)\right)\right\}\left(\lambda\right) \\
&+ \frac{\rho}{\lambda^\omega}\mathcal{L}\left\{Am\left(t\right) + Bu\left(t\right) + f\left(t,y\left(t\right)\right)\right\}\left(\lambda\right)
\end{aligned}
$$

where $M\left(\lambda\right) = \mathcal{L}\left\{m(t)\right\}\left(\lambda\right)$ and $H\left(\lambda\right) = \mathcal{L}\left\{h(t,y(t))\right\}\left(\lambda\right)$.
It follows

$$
M\left(\lambda\right) - \left(1-\rho\right)AM\left(\lambda\right) - \frac{\rho}{\lambda^\omega}AM\left(\lambda\right) = H\left(\lambda\right) - \frac{h\left(0,y_0\right)}{\lambda} + \frac{y_0}{\lambda} + \left[\left(1-\rho\right) + \frac{\rho}{\lambda^\omega}\right]F\left(\lambda\right)
$$

where $F\left(\lambda\right) = \mathcal{L}\left\{Bu\left(t\right) + f(t,y(t)\right\}\left(\lambda\right)$, then

$$
\left[\left(I-(1-\rho)A\right) - \frac{\rho}{\lambda^\omega}A\right]M\left(\lambda\right) = H\left(\lambda\right) + \frac{y_0 - h\left(0,y_0\right)}{\lambda} + \left[\left(1-\rho\right) + \frac{\rho}{\lambda^\omega}\right]F(\lambda)
$$

$$
\left(A_\rho^{-1} - \frac{\rho}{\lambda^\omega}A\right)M\left(\lambda\right) = H\left(\lambda\right) + \frac{y_0 - h\left(0,y_0\right)}{\lambda} + \left[\left(1-\rho\right) + \frac{\rho}{\lambda^\omega}\right]F(\lambda)
$$

$$
\left(\lambda^\omega A_\rho^{-1} - \rho A\right)M\left(\lambda\right) = \lambda^\omega H\left(\lambda\right) + \lambda^{\omega-1}\left(y_0 - h\left(0,y_0\right)\right) + \left[\lambda^\omega\left(1-\rho\right) + \rho\right]F(\lambda)
$$

$$
\left[\left(\lambda^\omega I - \rho AA_\rho\right)A_\rho^{-1}\right]M\left(\lambda\right) = \lambda^\omega H\left(\lambda\right) + \lambda^{\omega-1}\left[y_0 - h\left(0,y_0\right)\right] + \left[\lambda^\omega\left(1-\rho\right) + \rho\right]F(\lambda)
$$

$$
\begin{aligned}
M\left(\lambda\right) =& A_\rho(\lambda^\omega I - \rho AA_\rho)^{-1}\lambda^\omega H\left(\lambda\right) + A_\rho(\lambda^\omega I - \rho AA_\rho)^{-1}\lambda^{\omega-1}\left[y_0 - h\left(0,y_0\right)\right] \\
&+ A_\rho(\lambda^\omega I - \rho AA_\rho)^{-1}\lambda^\omega\left(1-\rho\right)F\left(\lambda\right) + A_\rho(\lambda^\omega I - \rho AA_\rho)^{-1}\rho F\left(\lambda\right).
\end{aligned}
$$

$$\begin{aligned}\lambda^{\omega}(\lambda^{\omega}I-\rho AA_{\rho})^{-1}&=\left(\lambda^{\omega}I-\rho AA_{\rho}+\rho AA_{\rho}\right)\left(\lambda^{\omega}I-\rho AA_{\rho}\right)^{-1}\\&=\left(\lambda^{\omega}I-\rho AA_{\rho}\right)\left(\lambda^{\omega}I-\rho AA_{\rho}\right)^{-1}+\rho AA_{\rho}(\lambda^{\omega}I-\rho AA_{\rho})^{-1}\\&=I+\rho AA_{\rho}(\lambda^{\omega}I-\rho AA_{\rho})^{-1}\end{aligned}$$

then

$$\begin{aligned}M\left(\lambda\right)=&A_{\rho}(\lambda^{\omega}I-\rho AA_{\rho})^{-1}\lambda^{\omega}H\left(\lambda\right)+A_{\rho}(\lambda^{\omega}I-\rho AA_{\rho})^{-1}\lambda^{\omega-1}\left(y_{0}-h\left(0,y_{0}\right)\right)\\&+\left(1-\rho\right)A_{\rho}\left[I+\rho AA_{\rho}(\lambda^{\omega}I-\rho AA_{\rho})^{-1}\right]F\left(\lambda\right)+\rho A_{\rho}(\lambda^{\omega}I-\rho AA_{\rho})^{-1}F\left(\lambda\right)\\=&A_{\rho}(\lambda^{\omega}I-\rho AA_{\rho})^{-1}\lambda^{\omega}H\left(\lambda\right)+A_{\rho}(\lambda^{\omega}I-\rho AA_{\rho})^{-1}\lambda^{\omega-1}\left(y_{0}-h\left(0,y_{0}\right)\right)\\&+\left(1-\rho\right)A_{\rho}F\left(\lambda\right)+\left(1-\rho\right)\rho AA_{\rho}^{2}(\lambda^{\omega}I-\rho AA_{\rho})^{-1}F\left(\lambda\right)+\rho A_{\rho}(\lambda^{\omega}I-\rho AA_{\rho})^{-1}F\left(\lambda\right)\\=&A_{\rho}(\lambda^{\omega}I-\rho AA_{\rho})^{-1}\lambda^{\omega}H\left(\lambda\right)+A_{\rho}(\lambda^{\omega}I-\rho AA_{\rho})^{-1}\lambda^{\omega-1}\left(y_{0}-h\left(0,y_{0}\right)\right)+\left(1-\rho\right)A_{\rho}F\left(\lambda\right)\\&+\left[\left(1-\rho\right)A+A_{\rho}^{-1}\right]\rho A_{\rho}^{2}(\lambda^{\omega}I-\rho AA_{\rho})^{-1}\ F\left(\lambda\right).\end{aligned}$$

Since

$$A_{\rho}=\left(I-\left(1-\rho\right)A\right)^{\,-1}$$

then

$$I=\left(1-\rho\right)A+A_{\rho}^{-1}.$$

Therefore

$$\begin{aligned}M\left(\lambda\right)=&A_{\rho}(\lambda^{\omega}I-E)^{-1}\lambda^{\omega}H\left(\lambda\right)+A_{\rho}(\lambda^{\omega}I-E)^{-1}\lambda^{\omega-1}\left(y_{0}-h\left(0,y_{0}\right)\right)\\&+\left(1-\rho\right)A_{\rho}F\left(\lambda\right)+\rho A_{\rho}^{2}(\lambda^{\omega}I-E)^{-1}\ F\left(\lambda\right).\end{aligned}\tag{2.10}$$

Now,

$$\begin{aligned}A_{\rho}(\lambda^{\omega}I-E)^{-1}\lambda^{\omega}H\left(\lambda\right)&=A_{\rho}\lambda^{\omega}\int_{0}^{\infty}e^{-\lambda^{\omega}s}T\left(s\right)H\left(\lambda\right)ds\\&=A_{\rho}\lambda^{\omega}\int_{0}^{\infty}e^{-(\lambda u)^{\omega}}T\left(u^{\omega}\right)H\left(\lambda\right)\omega u^{\omega-1}du\\&=A_{\rho}\int_{0}^{\infty}\lambda\omega(\lambda u)^{\omega-1}e^{-(\lambda u)^{\omega}}T\left(u^{\omega}\right)H\left(\lambda\right)du\\&=A_{\rho}\int_{0}^{\infty}-\frac{de^{-(\lambda u)^{\omega}}}{du}T\left(u^{\omega}\right)H\left(\lambda\right)du,\end{aligned}$$

using integration by part, we obtain

$$\begin{aligned}A_{\rho}(\lambda^{\omega}I-E)^{-1}\lambda^{\omega}H\left(\lambda\right)&=A_{\rho}\left[\left[-T(u^{\omega})H\left(\lambda\right)e^{-(\lambda u)^{\omega}}\right]_{0}^{\infty}-\int_{0}^{\infty}-e^{-(\lambda u)^{\omega}}ET(u^{\omega})\omega u^{\omega-1}H\left(\lambda\right)du\right]\\&=A_{\rho}H\left(\lambda\right)+A_{\rho}\int_{0}^{\infty}e^{-(\lambda u)^{\omega}}ET(u^{\omega})\omega u^{\omega-1}H\left(\lambda\right)du,\end{aligned}$$

using (2.4), we have

$$\begin{aligned}A_{\rho}(\lambda^{\omega}I-E)^{-1}\lambda^{\omega}H\left(\lambda\right)&=A_{\rho}H\left(\lambda\right)+A_{\rho}\int_{0}^{\infty}\int_{0}^{\infty}e^{-\lambda u\delta}\psi_{\omega}(\delta)ET(u^{\omega})H\left(\lambda\right)\omega u^{\omega-1}d\delta du\\&=A_{\rho}H\left(\lambda\right)+A_{\rho}\int_{0}^{\infty}\int_{0}^{\infty}\omega e^{-\lambda v}\psi_{\omega}(\delta)ET(\frac{v}{\delta})^{\omega}\frac{v^{\omega-1}}{\delta^{\omega}}H\left(\lambda\right)d\delta dv\\&=A_{\rho}H\left(\lambda\right)+A_{\rho}\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}\omega e^{-\lambda(v+\zeta)^{\text{`}}}\psi_{\omega}\left(\delta\right)ET\Big(\frac{v}{\delta}\Big)^{\omega}\frac{v^{\omega-1}}{\delta^{\omega}}h\left(\zeta,y\left(\zeta\right)\right)d\zeta d\delta dv\end{aligned}$$

$$= A_\rho H(\lambda) + A_\rho \int_0^\infty \int_0^\infty \int_\zeta^\infty \omega e^{-\lambda t}\psi_\omega(\delta) ET\left(\frac{t-\zeta}{\delta}\right)^\omega \frac{(t-\zeta)^{\omega-1}}{\delta^\omega} h(\zeta, y(\zeta))\, dt d\zeta d\delta$$

$$= A_\rho H(\lambda) + A_\rho \int_0^\infty e^{-\lambda t}\left[\omega \int_0^t \int_0^\infty \psi_\omega(\delta) ET\left(\frac{t-\zeta}{\delta}\right)^\omega \frac{(t-\zeta)^{\omega-1}}{\delta^\omega} h(\zeta, y(\zeta))\, d\delta d\zeta\right] dt$$

$$= A_\rho H(\lambda) + A_\rho \mathcal{L}\left\{\omega \int_0^t \int_0^\infty \psi_\omega(\delta) ET\left(\frac{t-\zeta}{\delta}\right)^\omega \frac{(t-\zeta)^{\omega-1}}{\delta^\omega} h(\zeta, y(\zeta))\, d\delta d\zeta\right\}(\lambda)$$

using (2.5), we have

$$A_\rho(\lambda^\omega I - E)^{-1}\lambda^\omega H(\lambda) =$$

$$A_\rho H(\lambda) + A_\rho \mathcal{L}\left\{\omega \int_0^t \int_0^\infty \varphi_\omega(\theta) ET(\theta(t-\zeta)^\omega)\theta(t-\zeta)^{\omega-1} h(\zeta, y(\zeta))\, d\theta d\zeta\right\}(\lambda).$$

Therefore

$$A_\rho(\lambda^\omega I - E)^{-1}\lambda^\omega H(\lambda) = A_\rho H(\lambda) + A_\rho \mathcal{L}\left\{\int_0^t (t-\zeta)^{\omega-1} E L_\omega(t-\zeta) h(\zeta, y(\zeta))\, d\zeta\right\}(\lambda). \quad (2.11)$$

$$\begin{aligned} A_\rho\lambda^{\omega-1}(\lambda^\omega I - E)^{-1}[y_0 - h(0,y_0)] &= A_\rho\lambda^{\omega-1}\int_0^\infty e^{-\lambda^\omega s} T(s)(y_0 - h(0,y_0))\, ds \\ &= A_\rho\lambda^{\omega-1}\int_0^\infty e^{-(\lambda u)^\omega} T(u^\omega)[y_0 - h(0,y_0)]\,\omega u^{\omega-1} du \\ &= A_\rho \int_0^\infty \omega(\lambda u)^{\omega-1} e^{-(\lambda u)^\omega} T(u^\omega)[y_0 - h(0,y_0)]\, du \\ &= A_\rho \int_0^\infty -\frac{1}{\lambda}\frac{de^{-(\lambda u)^\omega}}{du} T(u^\omega)[y_0 - h(0,y_0)]\, du, \end{aligned}$$

from (2.4), we have

$$e^{-(\lambda u)^\omega} = \int_0^\infty e^{-\lambda u \delta}\psi_\omega(\delta)\, d\delta$$

then

$$\frac{de^{-(\lambda u)^\omega}}{du} = \int_0^\infty -\lambda\delta e^{-\lambda u\delta}\psi_\omega(\delta)\, d\delta.$$

Therefore

$$\begin{aligned} A_\rho\lambda^{\omega-1}(\lambda^\omega I - E)^{-1}[y_0 - h(0,y_0)] &= A_\rho \int_0^\infty \int_0^\infty \delta e^{-\lambda u\delta}\psi_\omega(\delta) T(u^\omega)[y_0 - h(0,y_0)]\, d\delta du \\ &= A_\rho \int_0^\infty \int_0^\infty e^{-\lambda t}\psi_\omega(\delta) T\left(\left(\frac{t}{\delta}\right)^\omega\right)[y_0 - h(0,y_0)]\, d\delta dt \\ &= A_\rho \int_0^\infty \int_0^\infty -\frac{1}{\omega}\theta^{-\frac{1}{\omega}-1} e^{-\lambda t}\psi_\omega\left(\theta^{-\frac{1}{\omega}}\right) T(\theta t^\omega)[y_0 - h(0,y_0)]\, d\theta dt \end{aligned}$$

using (2.5), we have

$$\begin{aligned} A_\rho\lambda^{\omega-1}(\lambda^\omega I - E)^{-1}[y_0 - h(0,y_0)] &= A_\rho \int_0^\infty \int_0^\infty e^{-\lambda t}\varphi_\omega(\theta) T(\theta t^\omega)[y_0 - h(0,y_0)]\, d\theta dt \\ &= A_\rho \mathcal{L}\left\{\int_0^\infty \varphi_\omega(\theta) T(\theta t^\omega)(y_0 - h(0,y_0))\, d\theta\right\}(\lambda). \end{aligned}$$

Therefore,

$$A_{\rho}\lambda^{\omega-1}\left(\lambda^{\omega}I-E\right)^{-1}\left[y_0-h\left(0,y_0\right)\right]=A_{\rho}\mathcal{L}\left\{T_{\omega}\left(t\right)\left[y_0-h\left(0,y_0\right)\right]\right\}\left(\lambda\right). \tag{2.12}$$

$$\rho A_{\rho}^{2}(\lambda^{\omega}I-E)^{-1}\ F\left(\lambda\right)=\rho A_{\rho}^{2}\int_{0}^{\infty}e^{-\lambda^{\omega}s}T\left(s\right)F\left(\lambda\right)ds=\rho A_{\rho}^{2}\int_{0}^{\infty}e^{-(\lambda u)^{\omega}}T\left(u^{\omega}\right)F\left(\lambda\right)\omega u^{\omega-1}du$$

using (2.4), we have

$$\begin{aligned}
\rho A_{\rho}^{2}(\lambda^{\omega}I-E)^{-1}\ F\left(\lambda\right)&=\rho A_{\rho}^{2}\int_{0}^{\infty}\int_{0}^{\infty}\omega e^{-\lambda u\delta}\psi_{\omega}(\delta)T\left(u^{\omega}\right)F\left(\lambda\right)u^{\omega-1}d\delta du\\
&=\rho A_{\rho}^{2}\int_{0}^{\infty}\int_{0}^{\infty}\omega e^{-\lambda v}\psi_{\omega}(\delta)T\left(\left(\frac{v}{\delta}\right)^{\omega}\right)\frac{v^{\omega-1}}{\delta^{\omega}}F\left(\lambda\right)d\delta dv\\
&=\rho A_{\rho}^{2}\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}\omega e^{-\lambda(v+\zeta)}\psi_{\omega}(\delta)T\left(\left(\frac{v}{\delta}\right)^{\omega}\right)\frac{v^{\omega-1}}{\delta^{\omega}}\left[Bu\left(\zeta\right)+f(\zeta,y\left(\zeta\right)\right]\ d\zeta d\delta dv\\
&=\rho A_{\rho}^{2}\int_{0}^{\infty}\int_{0}^{\infty}\int_{\zeta}^{\infty}\omega e^{-\lambda t}\psi_{\omega}(\delta)T\left(\left(\frac{(t-\zeta)}{\delta}\right)^{\omega}\right)\frac{(t-\zeta)^{\omega-1}}{\delta^{\omega}}\left[Bu\left(\zeta\right)+f(\zeta,y\left(\zeta\right)\right]\ dt d\zeta d\delta\\
&=\rho A_{\rho}^{2}\int_{0}^{\infty}e^{-\lambda t}\left[\omega\int_{0}^{t}\int_{0}^{\infty}\psi_{\omega}(\delta)T\left(\left(\frac{(t-\zeta)}{\delta}\right)^{\omega}\right)\frac{(t-\zeta)^{\omega-1}}{\delta^{\omega}}\left[Bu\left(\zeta\right)+f(\zeta,y\left(\zeta\right)\right]d\delta d\zeta\right]dt\\
&=\rho A_{\rho}^{2}\mathcal{L}\left\{\omega\int_{0}^{t}\int_{0}^{\infty}\psi_{\omega}(\delta)T\left(\left(\frac{(t-\zeta)}{\delta}\right)^{\omega}\right)\frac{(t-\zeta)^{\omega-1}}{\delta^{\omega}}\left[Bu\left(\zeta\right)+f(\zeta,y\left(\zeta\right)\right]d\delta d\zeta\right\}\left(\lambda\right)
\end{aligned}$$

using (2.5), we have

$$\rho A_{\rho}^{2}(\lambda^{\omega}I-E)^{-1}F\left(\lambda\right)=\rho A_{\rho}^{2}\mathcal{L}\left\{\omega\int_{0}^{t}\int_{0}^{\infty}\theta\varphi_{\omega}(\theta)T\left(\theta(t-\zeta)^{\omega}\right)(t-\zeta)^{\omega-1}\left[Bu\left(\zeta\right)+f(\zeta,y\left(\zeta\right)\right]d\theta d\zeta\right\}\left(\lambda\right),$$

$$\rho A_{\rho}^{2}(\lambda^{\omega}I-E)^{-1}F\left(\lambda\right)=\rho A_{\rho}^{2}\mathcal{L}\left\{\int_{0}^{t}L_{\omega}(t-\zeta)(t-\zeta)^{\omega-1}\left[Bu\left(\zeta\right)+f(\zeta,y\left(\zeta\right)\right]d\zeta\right\}\left(\lambda\right). \tag{2.13}$$

By substitute (2.11), (2.12) and (2.13) in (2.10), we get

$$\begin{aligned}
M\left(\lambda\right)=&A_{\rho}H(\lambda)+A_{\rho}\mathcal{L}\left\{\int_{0}^{t}(t-\zeta)^{\omega-1}EL_{\omega}\left(t-\zeta\right)h\left(\zeta,y\left(\zeta\right)\right)d\zeta\right\}\left(\lambda\right)+A_{\rho}\mathcal{L}\left\{\mathcal{T}_{\omega}\left(t\right)\left(y_0-h\left(0,y_0\right)\right)\right\}\left(\lambda\right)\\
&+\left(1-\rho\right)A_{\rho}F\left(\lambda\right)+\rho A_{\rho}^{2}\mathcal{L}\left\{\int_{0}^{t}L_{\omega}(t-\zeta)(t-\zeta)^{\omega-1}\left[Bu\left(\zeta\right)+f(\zeta,y\left(\zeta\right)\right])d\zeta\right\}\left(\lambda\right).
\end{aligned} \tag{2.14}$$

Taking the inverse of Laplace transform for both sides of (2.14), we get

$$\begin{aligned}
m\left(t\right)=&A_{\rho}g\left(t,y\left(t\right)\right)+A_{\rho}\int_{0}^{t}(t-\zeta)^{\omega-1}EL_{\omega}\left(t-\zeta\right)h\left(\zeta,y\left(\zeta\right)\right)d\zeta\\
&+A_{\rho}T_{\omega}\left(t\right)\left[y_0-h\left(0,y_0\right)\right]+\left(1-\rho\right)A_{\rho}[Bu\left(t\right)+f(t,y(t))]\\
&+\rho A_{\rho}^{2}\int_{0}^{t}L_{\omega}\left(t-\zeta\right)\left(t-\zeta\right)^{\omega-1}\left[Bu\left(\zeta\right)+f\left(\zeta,y\left(\zeta\right)\right)\right]d\zeta.
\end{aligned} \tag{2.15}$$

Now, to find the mild solution $n(t)$, we applying fractional integral (2.3) for both sides of the system (2.8),

$$\mathcal{I}^{\rho,\omega} D^{\rho,\omega} n(t) = \mathcal{I}^{\rho,\omega} \left[An(t)\right].$$

By using Lemma 2.9 we have,

$$n\left(t\right) = \sum_{\gamma=1}^{p} \Delta y\left(t_{\gamma}\right) \sigma_{\gamma}\left(t\right) + \mathcal{I}^{\rho,\omega}\left[An(t)\right] \tag{2.16}$$

where

$$\sigma_{\gamma}\left(t\right) = \begin{cases} 0 & t \in [0, t_1] \\ 1 & t \in (t_{\gamma}, t_{\gamma+1}] \qquad \gamma = 1, 2, \ldots, p \end{cases} \tag{2.17}$$

and

$$\mathcal{L}\left\{\sigma_{\gamma}\left(t\right)\right\}\left(\lambda\right) = \frac{e^{-\lambda t_{\gamma}}}{\lambda}.$$

Taking Laplace transform for both sides of (2.16),

$$N\left(\lambda\right) = \sum_{\gamma=1}^{p} \Delta y\left(t_{\gamma}\right) \lambda^{-1} e^{-\lambda t_{\gamma}} + \left(1-\rho\right) AN\left(\lambda\right) + \frac{\rho}{\lambda^{\omega}} AN\left(\lambda\right)$$

where $N\left(\lambda\right)$ denotes the Laplace transform of $n(t)$.
It follows,

$$N\left(\lambda\right) - \left(1-\rho\right) AN\left(\lambda\right) - \frac{\rho}{\lambda^{\omega}} AN\left(\lambda\right) = \sum_{\gamma=1}^{p} \Delta y\left(t_{\gamma}\right) \lambda^{-1} e^{-\lambda t_{\gamma}}$$

$$\left[I - \left(1-\rho\right) A - \frac{\rho}{\lambda^{\omega}} A\right] N\left(\lambda\right) = \sum_{\gamma=1}^{p} \Delta y\left(t_{\gamma}\right) \lambda^{-1} e^{-\lambda t_{\gamma}}$$

$$\left[A_{\rho}^{-1} - \frac{\rho}{\lambda^{\omega}} A\right] N\left(\lambda\right) = \sum_{\gamma=1}^{p} \Delta y\left(t_{\gamma}\right) \lambda^{-1} e^{-\lambda t_{\gamma}}$$

$$\left[\lambda^{\omega} A_{\rho}^{-1} - \rho A\right] N\left(\lambda\right) = \sum_{\gamma=1}^{p} \Delta y\left(t_{\gamma}\right) \lambda^{\omega-1} e^{-\lambda t_{\gamma}}$$

$$\left[(\lambda^{\omega} I - \rho A A_{\rho}) A_{\rho}^{-1}\right] N\left(\lambda\right) = \sum_{\gamma=1}^{p} \Delta y\left(t_{\gamma}\right) \lambda^{\omega-1} e^{-\lambda t_{\gamma}}$$

$$N\left(\lambda\right) = A_{\rho}(\lambda^{\omega} I - \rho A A_{\rho})^{-1} \lambda^{\omega-1} \sum_{\gamma=1}^{p} \Delta y\left(t_{\gamma}\right) e^{-\lambda t_{\gamma}}$$

$$= A_{\rho}(\lambda^{\omega} I - E)^{-1} \lambda^{\omega-1} \sum_{\gamma=1}^{p} \Delta y\left(t_{\gamma}\right) e^{-\lambda t_{\gamma}}$$

$$= A_{\rho} \lambda^{\omega-1} \sum_{\gamma=1}^{p} \int_{0}^{\infty} \Delta y\left(t_{\gamma}\right) e^{-\lambda^{\omega} s} T\left(s\right) e^{-\lambda t_{\gamma}} ds.$$

$$= A_\rho \lambda^{\omega-1} \sum_{\gamma=1}^{p} \int_0^\infty \Delta y\left(t_\gamma\right) e^{-(\lambda u)^\omega} T\left(u^\omega\right) e^{-\lambda t_\gamma} \omega u^{\omega-1} du$$

$$= A_\rho \sum_{\gamma=1}^{p} \int_0^\infty \Delta y\left(t_\gamma\right) \omega (\lambda u)^{\omega-1} e^{-(\lambda u)^\omega} T\left(u^\omega\right) e^{-\lambda t_\gamma} du$$

$$= A_\rho \sum_{\gamma=1}^{p} \int_0^\infty -\Delta y\left(t_\gamma\right) \frac{1}{\lambda} \frac{de^{-(\lambda u)^\omega}}{du} T\left(u^\omega\right) e^{-\lambda t_\gamma} du$$

using (2.4), we have

$$N\left(\lambda\right) = A_\rho \sum_{\gamma=1}^{p} \int_0^\infty \int_0^\infty \Delta y\left(t_\gamma\right) \delta e^{-\lambda u \delta} \psi_\omega\left(\delta\right) T\left(u^\omega\right) e^{-\lambda t_\gamma} d\delta du.$$

$$= A_\rho \sum_{\gamma=1}^{p} \int_0^\infty \int_0^\infty \Delta y\left(t_\gamma\right) e^{-\lambda v} \psi_\omega(\delta) T\left(\frac{v}{\delta^\omega}^\omega\right) e^{-\lambda t_\gamma} d\delta dv$$

$$= A_\rho \sum_{\gamma=1}^{p} \int_0^\infty \int_0^\infty \Delta y\left(t_\gamma\right) e^{-\lambda(v+t_\gamma)} \psi_\omega(\delta) T\left(\frac{v}{\delta^\omega}^\omega\right) d\delta dv.$$

$$= A_\rho \sum_{\gamma=1}^{p} \int_0^\infty \int_0^\infty \Delta y\left(t_\gamma\right) e^{-\lambda t} \psi_\omega(\delta) T\left(\frac{\left(t-t_\gamma\right)^\omega}{\delta^\omega}\right) d\delta dt.$$

using (2.5), we have

$$N\left(\lambda\right) = A_\rho \sum_{\gamma=1}^{p} \int_0^\infty \int_0^\infty \Delta y\left(t_\gamma\right) e^{-\lambda t} \varphi_\omega(\theta) T\left(\theta (t-t_\gamma)^\omega\right) d\theta dt$$

$$= A_\rho \mathcal{L}\left\{\sum_{\gamma=1}^{p} \int_0^\infty \Delta y\left(t_\gamma\right) \varphi_\omega\left(\theta\right) T\left(\theta (t-t_\gamma)^\omega\right) d\theta\right\}\left(\lambda\right)$$

$$N\left(\lambda\right) = A_\rho \mathcal{L}\left\{\sum_{\gamma=1}^{p} \Delta y\left(t_\gamma\right) T_\omega (t-t_\gamma)^\omega\right\}\left(\lambda\right) \tag{2.18}$$

taking the inverse of Laplace transform for both sides of (2.18), we get

$$n\left(t\right) = A_\rho \sum_{\gamma=1}^{p} \Delta y\left(t_\gamma\right) T_\omega (t-t_\gamma)^\omega. \tag{2.19}$$

From (2.15) and (2.19), we have

$$y(t)=\begin{cases} A_\rho h(t,y(t))+A_\rho\int_0^t (t-\zeta)^{\omega-1}EL_\omega(t-\zeta)h(\zeta,y(\zeta))d\zeta \\ +A_\rho T_\omega(t)[y_0-h(0,y_0)]+(1-\rho)A_\rho[Bu(t)+f(t,y(t))] & t\in[0,t_1] \\ +\rho A_\rho^2\int_0^t (t-\zeta)^{\omega-1}L_\omega(t-\zeta)[Bu(\zeta)+f(\zeta,y(\zeta))]d\zeta \\ A_\rho g(t,y(t))+A_\rho\int_0^t (t-\zeta)^{\omega-1}EL_\omega(t-\zeta)h(\zeta,y(\zeta))d\zeta \\ +A_\rho T_\omega(t)(y_0-h(0,y_0))+(1-\rho)A_\rho[Bu(t)+f(t,y(t))] \\ +\rho A_\rho^2\int_0^t (t-\zeta)^{\omega-1}L_\omega(t-\zeta)[Bu(\zeta)+f(\zeta,y(\zeta))]d\zeta & t\in(t_\gamma,t_{\gamma+1}] \\ +A_\rho\sum_{\gamma=1}^{p}\Delta y(t_\gamma)T_\omega(t-t_\gamma) \end{cases}$$

□

Lemma 2.11 ([16]). *The operators L_ω and T_ω have the following properties.*

i. For any fixed $t\geq 0$, $L_\omega(t)$ and $T_\omega(t)$ are linear and bounded operators, i.e. for any $y\in X$,

$$\|L_\omega(t)y\|=\left\|\omega\int_0^\infty \theta\varphi_\omega(\theta)T(\theta t^\omega)yd\theta\right\|\leq\frac{\omega\mathcal{S}}{\Gamma(1+\omega)}\|y\|=\frac{\mathcal{S}}{\Gamma(\omega)}\|y\|$$

$$\|T_\omega(t)y\|=\left\|\int_0^\infty \varphi_\omega(\theta)T(\theta t^\omega)yd\theta\right\|\leq\mathcal{S}\|y\|$$

where $\mathcal{S}=sup_{t\geq 0}\|T(t)\|$.

ii. The operators $\{L_\omega(t)\}_{t\geq 0}$ and $\{T_\omega(t)\}_{t\geq 0}$ are strongly continuous.

iii. The operators $\{L_\omega(t)\}_{t\geq 0}$ and $\{T_\omega(t)\}_{t\geq 0}$ are compact.

Note that, for $y\in D(A)$ we have [14]

$$\frac{d}{dt}T(t)y=ET(t)y=T(t)Ey.$$

Definition 2.12. *The mild solution $y(t)$ of system (1.1) is*

$$y(t)=\begin{cases} A_\rho h(t,y(t))+A_\rho\int_0^t (t-\zeta)^{\omega-1}L_\omega(t-\zeta)Eh(\zeta,y(\zeta))d\zeta \\ +A_\rho T_\omega(t)(y_0-h(0,y_0))+(1-\rho)A_\rho[Bu(t)+f(t,y(t))] & t\in[0,t_1] \\ +\rho A_\rho^2\int_0^t (t-\zeta)^{\omega-1}L_\omega(t-\zeta)[Bu(\zeta)+f(\zeta,y(\zeta))]d\zeta \\ A_\rho h(t,y(t))+A_\rho\int_0^t (t-\zeta)^{\omega-1}L_\omega(t-\zeta)Eh(\zeta,y(\zeta))d\zeta \\ +A_\rho T_\omega(t)(y_0-h(0,y_0))+(1-\rho)A_\rho[Bu(t)+f(t,y(t))] \\ +\rho A_\rho^2\int_0^t (t-\zeta)^{\omega-1}L_\omega(t-\zeta)[Bu(\zeta)+f(\zeta,y(\zeta))]d\zeta & t\in(t_\gamma,t_{\gamma+1}] \\ +A_\rho\sum_{\gamma=1}^{p}\Delta y(t_\gamma)T_\omega(t-t_\gamma), \end{cases}$$

Theorem 2.13 (Nussbaum fixed point theorem[17]). *Assume G is closed, bounded and convex subset of a Banach space X. If the continuous mappings ϕ_1, ϕ_2 from G to X satisfies the following:*

1. $(\phi_1 + \phi_2)\, G \subset G$,
2. $\|\phi_1 x - \phi_1 y\| \leq C\, \|x - y\|$ *for all* $x, y \in G$ *where* $0 < C < 1$,
3. ϕ_2 *is completely continuous,*

then the operator $\phi_1 + \phi_2$ has a fixed point in G.

3. Controllability results

In this section, we prove the system (1.1) is controllable. The following conditions are assumed.

H1 The linear operator $W_\rho : L_2(\mathcal{J}, U) \to X$ defined as

$$W_\rho u = (1-\rho)\, A_\rho B u + \rho A_\rho^2 W u$$

has an inverse operator ${W_\rho}^{-1}$ on and $\left\|{W_\rho}^{-1}\right\| \leq K, \quad K > 0$, $\|A_\rho\| \leq \eta$, $\eta > 0$.
where W is linear operator from $L_2(\mathcal{J}, U)$ into X such that

$$Wu = \int_0^\Upsilon (\Upsilon - s)^{\omega - 1} L_\omega (\Upsilon - s)\, Bu(s)\, ds.$$

H2 There exist constants $\mathcal{M}_h, \widehat{\mathcal{M}_h} > 0$ such that

$$\|Eh(t, y_1(t)) - Eh(t, y_2(t))\| \leq \mathcal{M}_h \|y_1(t) - y_2(t)\|$$

and

$$\widehat{\mathcal{M}_h} = sup_{t \in \mathcal{J}} \|Eh(t, 0)\|.$$

H3 The continuous function $f : \mathcal{J} \times X \to X$ satisfies Lipchitz condition i.e. there exist a constant $\mathcal{M}_f > 0$ such that

$$\|f(t, y_1(t)) - f(t, y_2(t))\| \leq \mathcal{M}_f \|y_1(t) - y_2(t)\|$$

and

$$\widehat{\mathcal{M}_f} = sup_{t \in \mathcal{J}} \|f(t, 0)\|.$$

where $\widehat{\mathcal{M}_f} > 0$.

H4 The function $q_\gamma : X \to X$, $\gamma = 1, 2, \ldots, p$ is continuous and satisfies Lipchitz condition, i.e. there exists a constant $\mathcal{M}_\gamma > 0$ such that

$$\|q_\gamma(y_1) - q_\gamma(y_2)\| \leq \mathcal{M}_\gamma \|y_1(t) - y_2(t)\|$$

and

$$\sum_{\gamma=1}^{p} \mathcal{M}_\gamma = \mathcal{M}$$

where $\mathcal{M} > 0$.

Definition 3.1. *System* (1.1) *is controllable on* $\mathcal{J} = [0, \Upsilon]$ *if for each initial state* $y_0 \in X$ *and each* $y_1 \in X$*, there is suitable control* $u \in L^2(\mathcal{J}, U)$ *such that the PC-mild solution* $y(t)$ *of system* (1.1) *satisfies* $y(\Upsilon) = y_1$.

Theorem 3.2. *The system* (1.1) *is controllable on* $\mathcal{J} = [0, \Upsilon]$ *if it satisfies the conditions H1-H4, and*

$$0 < D + \eta K \|B\| D \left[(1-\rho) + \frac{\rho\eta\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\right] < 1 \tag{3.1}$$

where

$$D = \eta\left[\left\|E^{-1}\right\|\mathcal{M}_h + \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_h + (1-\rho)\mathcal{M}_f + \eta\rho\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_f + \mathcal{S}\mathcal{M}\right].$$

Proof . Using condition H1, define the control $u_y(t)$ for an arbitrary function $y(\cdot) \in PC(\mathcal{J}, X)$ as

$$u_y(t) = \begin{cases} W_{\rho}{}^{-1}\Big[y_1 - A_{\rho}h(\Upsilon, y(\Upsilon)) - A_{\rho}\int_0^{\Upsilon}(\Upsilon-\zeta)^{\omega-1}L_{\omega}(\Upsilon-\zeta)Eh(\zeta, y(\zeta))\,d\zeta \\ -A_{\rho}T_{\omega}(\Upsilon)[y_0 - h(0, y_0)] - (1-\rho)A_{\rho}f(\Upsilon, y(\Upsilon) \\ -\rho A_{\rho}^2\int_0^{\Upsilon}(\Upsilon-\zeta)^{\omega-1}L_{\omega}(\Upsilon-\zeta)f(\zeta, y(\zeta))\,d\zeta\Big], & t \in [0, t_1] \\ W_{\rho}{}^{-1}\Big[y_1 - A_{\rho}h(\Upsilon, y(\Upsilon)) - A_{\rho}\int_0^{\Upsilon}(\Upsilon-\zeta)^{\omega-1}L_{\omega}(\Upsilon-\zeta)Eh(\zeta, y(\zeta))\,d\zeta \\ -A_{\rho}T_{\omega}(\Upsilon)(y_0 - h(0, y_0)) - (1-\rho)A_{\rho}f(\Upsilon, y(\Upsilon)) \\ -\rho A_{\rho}^2\int_0^{\Upsilon}(\Upsilon-\zeta)^{\omega-1}L_{\omega}(\Upsilon-\zeta)f(\zeta, y(\zeta))\,d\zeta - A_{\rho}\sum_{\gamma=1}^{p}\Delta y(t_{\gamma})T_{\omega}(\Upsilon-t_{\gamma})\Big], & t \in (t_{\gamma}, t_{\gamma+1}] \end{cases}$$

By using (2.17), we can write $u_y(t)$ in the form

$$u_y(t) = W_{\rho}{}^{-1}\Big[y_1 - A_{\rho}h(\Upsilon, y(\Upsilon)) - A_{\rho}\int_0^{\Upsilon}(\Upsilon-\zeta)^{\omega-1}L_{\omega}(\Upsilon-\zeta)Eh(\zeta, y(\zeta))\,d\zeta$$
$$-A_{\rho}\mathcal{T}_{\omega}(\Upsilon)(y_0 - h(0, y_0)) - (1-\rho)A_{\rho}f(\Upsilon, y(\Upsilon))$$
$$-\rho A_{\rho}^2\int_0^{\Upsilon}(\Upsilon-\zeta)^{\omega-1}L_{\omega}(\Upsilon-\zeta)f(\zeta, y(\zeta))\,d\zeta - A_{\rho}\sum_{\gamma=1}^{p}\Delta y(t_{\gamma})\sigma_{\gamma}(t)\mathcal{T}_{\omega}(\Upsilon-t_{\gamma})\Big], \quad t \in [0, \Upsilon]$$

We have to show that the operator $\Phi : PC(\mathcal{J}, X) \to PC(\mathcal{J}, X))$ has a fixed point when applying this control, where

$$(\Phi y)(t) = \begin{cases} A_{\rho}h(t, y(t)) + A_{\rho}\int_0^{t}(t-\zeta)^{\omega-1}L_{\omega}(t-\zeta)Eh(\zeta, y(\zeta))\,d\zeta \\ +A_{\rho}T_{\omega}(t)[y_0 - h(0, y_0)] + (1-\rho)A_{\rho}[Bu_y(t) + f(t, y(t))] \\ +\rho A_{\rho}{}^2\int_0^{t}(t-\zeta)^{\omega-1}L_{\omega}(t-\zeta)[Bu_y(\zeta) + f(\zeta, y(\zeta))]\,d\zeta & t \in [0, t_1] \\ A_{\rho}h(t, y(t)) + A_{\rho}\int_0^{t}(t-\zeta)^{\omega-1}L_{\omega}(t-\zeta)Eh(\zeta, y(\zeta))\,d\zeta \\ +A_{\rho}T_{\omega}(t)[y_0 - h(0, y_0)] + (1-\rho)A_{\rho}[Bu_y(t) + f(t, y(t))] \\ +\rho A_{\rho}^2\int_0^{t}(t-\zeta)^{\omega-1}L_{\omega}(t-\zeta)[Bu_y(\zeta) + f(\zeta, y(\zeta))]\,d\zeta \\ +A_{\rho}\sum_{\gamma=1}^{p}\Delta y(t_{\gamma})T_{\omega}(t-t_{\gamma}) & t \in (t_{\gamma}, t_{\gamma+1}]. \end{cases}$$

By using (2.17), we can write $(\Phi y)(t)$ in the form

$$\begin{aligned}(\Phi y)(t) =& A_\rho h(t, y(t)) + A_\rho \int_0^t (t-\zeta)^{\omega-1} L_\omega (t-\zeta) Eh(\zeta, y(\zeta)) d\zeta \\ &+ A_\rho T_\omega (t) [y_0 - h(0, y_0)] + (1-\rho) A_\rho [Bu_y(t) + f(t, y(t))] \\ &+ \rho A_\rho^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega (t-\zeta) [Bu_y(\zeta) + f(\zeta, y(\zeta))] d\zeta \\ &+ A_\rho \sum_{\gamma=1}^{p} \Delta y(t_\gamma) \sigma_\gamma (t) T_\omega (t - t_\gamma), \qquad t \in \mathcal{J}.\end{aligned}$$

By using the control $u_y(t)$, we have $(\Phi y)(\Upsilon) = y_1$, indeed,

$$\begin{aligned}(\Phi y)(\Upsilon) =& A_\rho h(\Upsilon, y(\Upsilon)) + A_\rho \int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1} L_\omega (\Upsilon-\zeta) Eh(\zeta, y(\zeta)) d\zeta \\ &+ A_\rho T_\omega (\Upsilon) [y_0 - h(0, y_0)] + (1-\rho) A_\rho B W_\rho^{-1} \Lambda(\Upsilon) + (1-\rho) A_\rho f(\Upsilon, y(\Upsilon)) \\ &+ \rho A_\rho^2 \int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1} L_\omega (\Upsilon-\zeta) B W_\rho^{-1} \Lambda(\Upsilon) d\zeta \\ &+ \rho A_\rho^2 \int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1} L_\omega (\Upsilon-\zeta) f(\zeta, y(\zeta)) d\zeta + A_\rho \sum_{\gamma=1}^{p} \Delta y(t_\gamma) \sigma_\gamma (t) T_\omega (\Upsilon - t_\gamma),\end{aligned}$$

where

$$\begin{aligned}\Lambda(t) =& y_1 - A_\rho h(\Upsilon, y(\Upsilon)) - A_\rho \int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1} L_\omega (\Upsilon-\zeta) Eh(\zeta, y(\zeta)) d\zeta \\ &- A_\rho T_\omega (\Upsilon) [y_0 - h(0, y_0)] - (1-\rho) A_\rho f(\Upsilon, y(\Upsilon)) \\ &- \rho A_\rho^2 \int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1} L_\omega (\Upsilon-\zeta) f(\zeta, y(\zeta)) d\zeta - A_\rho \sum_{\gamma=1}^{p} \Delta y(t_\gamma) \sigma_\gamma (t) T_\omega (\Upsilon - t_\gamma).\end{aligned}$$

It follows,

$$\begin{aligned}(\Phi y)(\Upsilon) =& A_\rho h(\Upsilon, y(\Upsilon)) + A_\rho \int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1} L_\omega (\Upsilon-\zeta) Eh(\zeta, y(\zeta)) d\zeta + A_\rho T_\omega (\Upsilon) [y_0 - h(0, y_0)] \\ &+ \left[(1-\rho) A_\rho B + \rho A_\rho^2 W\right] W_\rho^{-1} \Lambda(\Upsilon) + (1-\rho) A_\rho f(\Upsilon, y(\Upsilon)) \\ &+ \rho A_\rho^2 \int_0^\Upsilon (\Upsilon-\zeta)^{\omega-1} L_\omega (\Upsilon-\zeta) f(\zeta, y(\zeta)) d\zeta + A_\rho \sum_{\gamma=1}^{p} \Delta y(t_\gamma) \sigma_\gamma (t) T_\omega(\Upsilon - t_\gamma)\end{aligned}$$

$$
\begin{aligned}
=&A_\rho h\left(\Upsilon,y\left(\Upsilon\right)\right)+A_\rho\int_0^\Upsilon\left(\Upsilon-\zeta\right)^{\omega-1}L_\omega\left(\Upsilon-\zeta\right)Eh\left(\zeta,y\left(\zeta\right)\right)d\zeta+A_\rho T_\omega\left(\Upsilon\right)\left[y_0-h\left(0,y_0\right)\right]\\
&+W_\rho W_\rho^{-1}\Big[y_1-A_\rho h\left(\Upsilon,y\left(\Upsilon\right)\right)-A_\rho\int_0^\Upsilon\left(\Upsilon-\zeta\right)^{\omega-1}L_\omega\left(\Upsilon-\zeta\right)Eh\left(\zeta,y\left(\zeta\right)\right)d\zeta\\
&\quad-A_\rho T_\omega\left(\Upsilon\right)\left[y_0-h\left(0,y_0\right)\right]-\left(1-\rho\right)A_\rho f\left(\Upsilon,y\left(\Upsilon\right)\right)\\
&\quad-\rho A_\rho^2\int_0^\Upsilon\left(\Upsilon-\zeta\right)^{\omega-1}L_\omega\left(\Upsilon-\zeta\right)f\left(\zeta,y\left(\zeta\right)\right)d\zeta-A_\rho\sum_{\gamma=1}^{p}\Delta y(t_\gamma)\sigma_\gamma\left(t\right)T_\omega(\Upsilon-t_\gamma)\Big]\\
&+\left(1-\rho\right)A_\rho f\left(\Upsilon,y\left(\Upsilon\right)\right)+\rho A_\rho^2\int_0^\Upsilon\left(\Upsilon-\zeta\right)^{\omega-1}L_\omega\left(\Upsilon-\zeta\right)f\left(\zeta,y\left(\zeta\right)\right)d\zeta\\
&+A_\rho\sum_{\gamma=1}^{p}\Delta y(t_\gamma)\sigma_\gamma\left(t\right)T_\omega(\Upsilon-t_\gamma).\\
&=y_1.
\end{aligned}
$$

Now, for $t\in\mathcal{J}$,

$$
\begin{aligned}
\|u(t)\|\le&K\Big[\|y_1\|+\|A_\rho\|\left\|h\left(\Upsilon,y\left(\Upsilon\right)\right)\right\|+\|A_\rho\|\frac{\mathcal{S}}{\Gamma\left(\omega\right)}\int_0^\Upsilon\left(\Upsilon-\zeta\right)^{\omega-1}\left\|Eh\left(\zeta,y\left(\zeta\right)\right)\right\|d\zeta\\
&+\|A_\rho\|\mathcal{S}\left\|y_0-h\left(0,y_0\right)\right\|+\left(1-\rho\right)\|A_\rho\|\left\|f\left(\Upsilon,y\left(\Upsilon\right)\right)\right\|\\
&+\rho\left\|A_\rho^2\right\|\frac{\mathcal{S}}{\Gamma\left(\omega\right)}\int_0^\Upsilon\left(\Upsilon-\zeta\right)^{\omega-1}\left\|f\left(\zeta,y\left(\zeta\right)\right)\right\|d\zeta+\|A_\rho\|\left\|\sum_{\gamma=1}^{p}\Delta y(t_\gamma)\sigma_\gamma\left(t\right)T_\omega(\Upsilon-t_\gamma)\right\|\Big]\\
\le&K\Big[\|y_1\|+\eta\Big[\left\|E^{-1}\right\|\mathcal{M}_h\|y\|+\left\|E^{-1}\right\|\widehat{\mathcal{M}_h}+\frac{\mathcal{S}\Upsilon^\omega}{\omega\Gamma\left(\omega\right)}\left(\mathcal{M}_h\|y\|+\widehat{\mathcal{M}_h}\right)+\mathcal{S}\left(\|y_0\|+\left\|E^{-1}\right\|\widehat{\mathcal{M}_h}\right)\\
&+\left(1-\rho\right)\left(\mathcal{M}_f\|y\|+\widehat{\mathcal{M}_f}\right)+\eta\rho\frac{\mathcal{S}\Upsilon^\omega}{\omega\Gamma\left(\omega\right)}\left(\mathcal{M}_f\|y\|+\widehat{\mathcal{M}_f}\right)+\mathcal{S}\mathcal{M}\|y\|+\mathcal{S}\sum_{\gamma=1}^{p}\|q_\gamma(0)\|\Big]\Big]\\
=&K\Big[\|y_1\|+\eta\|y\|\Big[\left\|E^{-1}\right\|\mathcal{M}_h+\frac{\mathcal{S}\Upsilon^\omega}{\Gamma\left(\omega+1\right)}\mathcal{M}_h+\left(1-\rho\right)\mathcal{M}_f+\eta\rho\frac{\mathcal{S}\Upsilon^\omega}{\Gamma\left(\omega+1\right)}\mathcal{M}_f+\mathcal{S}\mathcal{M}\Big]\\
&+\eta\Big[\left\|E^{-1}\right\|\widehat{\mathcal{M}_h}+\frac{\mathcal{S}\Upsilon^\omega}{\Gamma\left(\omega+1\right)}\widehat{\mathcal{M}_h}+\mathcal{S}\left(\|y_0\|+\left\|E^{-1}\right\|\widehat{\mathcal{M}_h}\right)+\left(1-\rho\right)\widehat{\mathcal{M}_f}\\
&\quad+\eta\rho\frac{\mathcal{S}\Upsilon^\omega}{\Gamma\left(\omega+1\right)}\widehat{\mathcal{M}_f}+\mathcal{S}\sum_{\gamma=1}^{p}\|q_\gamma(0)\|\Big]\Big]\\
=&K\left[D\|y\|+\|y_1\|+\hat{D}\right],
\end{aligned}
$$

where D is given in assumption and

$$
\begin{aligned}
\hat{D}=\eta\Big[&\left\|E^{-1}\right\|\widehat{\mathcal{M}_h}+\frac{\mathcal{S}\Upsilon^\omega}{\Gamma\left(\omega+1\right)}\widehat{\mathcal{M}_h}+\mathcal{S}\left(\|y_0\|+\left\|E^{-1}\right\|\widehat{\mathcal{M}_h}\right)+\left(1-\rho\right)\widehat{\mathcal{M}_f}\\
&+\eta\rho\frac{\mathcal{S}\Upsilon^\omega}{\Gamma\left(\omega+1\right)}\widehat{\mathcal{M}_f}+\mathcal{S}\sum_{\gamma=1}^{p}\|q_\gamma(0)\|\Big].
\end{aligned}
$$

Also, for $y, \hat{y} \in PC(\mathcal{J}, X)$, we have

$$\begin{aligned}\|u_y - u_{\hat{y}}\| \leq & \left\|W_\rho^{-1}\right\| \|A_\rho\| \left[\int_0^{\Upsilon} (\Upsilon-\zeta)^{\omega-1} \|L_\omega\| \|Eh(\zeta, y(\zeta)) - Eh(\zeta, \hat{y}(\zeta))\| d\zeta \right. \\ & \left. +\rho \|A_\rho\| \int_0^{\Upsilon} (\Upsilon-\zeta)^{\omega-1} \|L_\omega\| \|f(\zeta, y(\zeta)) - f(\zeta, \hat{y}(\zeta))\| d\zeta + \sum_{\gamma=1}^{p} \|T_\omega\| \|q_\gamma y(t_\gamma) - q_\gamma \hat{y}(t_\gamma)\| \right] \\ \leq & K\eta \left[\frac{\mathcal{S}\Upsilon^\omega}{\omega\Gamma(\omega)} \mathcal{M}_h \|y-\hat{y}\| + \rho\eta \frac{\mathcal{S}\Upsilon^\omega}{\omega\Gamma(\omega)} \mathcal{M}_f \|y-\hat{y}\| + \mathcal{S}\mathcal{M} \|y-\hat{y}\|\right] \\ = & K\eta\mathcal{S} \left[\frac{\Upsilon^\omega}{\Gamma(\omega+1)} \mathcal{M}_h + \rho\eta \frac{\Upsilon^\omega}{\Gamma(\omega+1)} \mathcal{M}_f + \mathcal{M}\right] \|y-\hat{y}\|.\end{aligned}$$

Define the set

$$C_\epsilon = \{y : y \in PC(\mathcal{J}, X), \|y\| \leq \epsilon \ for\ each\ t \in \mathcal{J}\},$$

then C_ϵ is closed, convex and bounded subset of $PC(\mathcal{J}, X)$ for each ϵ.
Define the operators

$$\begin{aligned}(\Phi_1)y(t) = & A_\rho h(t, y(t)) + A_\rho \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) Eh(\zeta, y(\zeta)) d\zeta \\ & + A_\rho T_\omega(t)(y_0 - h(0, y_0)) + (1-\rho) A_\rho [Bu_y(t) + f(t, y(t))] \\ & + \rho A_\rho^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) Bu_y(\zeta) d\zeta + A_\rho \sum_{\gamma=1}^{p} \Delta y(t_\gamma) T_\omega(t-t_\gamma), \qquad t \in \mathcal{J}.\end{aligned}$$

$$(\Phi_2)y(t) = \rho A_\rho^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) f(\zeta, y(\zeta)) d\zeta, \qquad t \in \mathcal{J}.$$

It is clear that

$$(\Phi_1 + \Phi_2)y = \Phi y.$$

To show operator Φ has a fixed point on C_ϵ, we need to choose $\epsilon_0 > 0$, such that $(\Phi_1 + \Phi_2)y$ has a fixed point on C_{ϵ_0}.
Taking

$$\epsilon_0 = \frac{\eta K \|B\| \left[(1-\rho) + \frac{\rho\eta\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\right] \left(\|y_1\| + \hat{D}\right) + \hat{D}}{1 - \left[D + \eta K \|B\| D \left[(1-\rho) + \frac{\rho\eta\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\right]\right]}.$$

We will show the operator $(\Phi_1 + \Phi_2)y$ has a fixed point on C_{ϵ_0}. There are three steps to our proof:

Step I . In this part of our proof, we will show $\Phi C_{\epsilon_0} \subset C_{\epsilon_0}$.
Let $y \in C_{\epsilon_0}$, then

$$\begin{aligned}\|(\Phi y)(t)\| \leq \|A_\rho\| & \left[\|h(t, y(t))\| + \frac{\mathcal{S}}{\Gamma(\omega)} \int_0^t (t-\zeta)^{\omega-1} \|Eh(\zeta, y(\zeta)\| d\zeta + \mathcal{S} \|(y_0 - h(0, y_0))\| \right. \\ & + (1-\rho)[\|Bu_y(t)\| + \|f(t, y(t))\|] + \rho \|A_\rho\| \frac{\mathcal{S}}{\Gamma(\omega)} \int_0^t (t-\zeta)^{\omega-1} \|Bu_y(\zeta)\| d\zeta \\ & \left. + \rho \|A_\rho\| \frac{\mathcal{S}}{\Gamma(\omega)} \int_0^t (t-\zeta)^{\omega-1} \|f(\zeta, y(\zeta))\| d\zeta + \mathcal{S} \sum_{\gamma=1}^{p} \|\Delta y(t_\gamma)\|\right]\end{aligned}$$

$$
\begin{aligned}
\leq & \eta\Bigg[\left\|E^{-1}\right\|\mathcal{M}_h\epsilon_0+\left\|E^{-1}\right\|\widehat{\mathcal{M}_h}+\frac{\mathcal{S}\Upsilon^\omega}{\omega\Gamma(\omega)}\left(\mathcal{M}_h\epsilon_0+\widehat{\mathcal{M}_h}\right)+\mathcal{S}\left(\|y_0\|+\left\|E^{-1}\right\|\widehat{\mathcal{M}_h}\right) \\
& \quad +(1-\rho)\|B\|\|u_y(t)\|+(1-\rho)\left(\mathcal{M}_f\epsilon_0+\widehat{\mathcal{M}_f}\right)+\rho\eta\frac{\mathcal{S}\Upsilon^\omega}{\omega\Gamma(\omega)}\|B\|\|u_y(t)\| \\
& \quad +\rho\eta\frac{\mathcal{S}\Upsilon^\omega}{\omega\Gamma(\omega)}\left(\mathcal{M}_f\epsilon_0+\widehat{\mathcal{M}_f}\right)+\mathcal{S}\mathcal{M}\epsilon_0+\mathcal{S}\sum_{\gamma=1}^{p}\|q_\gamma(0)\|\Bigg] \\
= & \eta\Bigg[\epsilon_0\left[\left\|E^{-1}\right\|\mathcal{M}_h+\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\mathcal{M}_h+(1-\rho)\mathcal{M}_f+\rho\eta\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\mathcal{M}_f+\mathcal{S}\mathcal{M}\right] \\
& \quad +\left\|E^{-1}\right\|\widehat{\mathcal{M}_h}+\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\widehat{\mathcal{M}_h}+\mathcal{S}\left(\|y_0\|+\left\|E^{-1}\right\|\widehat{\mathcal{M}_h}\right)+(1-\rho)\widehat{\mathcal{M}_f} \\
& \quad +\rho\eta\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\widehat{\mathcal{M}_f}+\mathcal{S}\sum_{\gamma=1}^{p}\|q_\gamma(0)\|+\left[(1-\rho)+\rho\eta\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\right]\|B\|\|u_y(t)\|\Bigg] \\
= & D\epsilon_0+\hat{D}+\left[(1-\rho)+\rho\eta\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\right]\eta\|B\|\|u_y(t)\| \\
\leq & D\epsilon_0+\hat{D}+\left[(1-\rho)+\rho\eta\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\right]\eta K\|B\|\left(D\epsilon_0+\|y_1\|+\hat{D}\right) \\
= & \left[D+\eta K\|B\|D\left[(1-\rho)+\rho\eta\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\right]\right]\epsilon_0 \\
& +\eta K\|B\|\left[(1-\rho)+\rho\eta\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\right]\left(\|y_1\|+\hat{D}\right)+\hat{D} \\
= & \epsilon_0.
\end{aligned}
$$

Therefore $(\Phi_1+\Phi_2)C_{\epsilon_0}=\Phi C_{\epsilon_0}\subset C_{\epsilon_0}$.

Step II . We prove the operator Φ_1 is contraction on C_{ϵ_0}.
Let $y,\hat{y}\in C_{\epsilon_0}$, then

$$
\begin{aligned}
\|\Phi_1 y-\Phi_1\hat{y}\|\leq\|A_\rho\|\Bigg[& \|h(t,y(t))-h(t,\hat{y}(t))\| \\
& +\int_0^t(t-\zeta)^{\omega-1}\|L_\omega(t-\zeta)\|\|Eh(\zeta,y(\zeta)-Eh(\zeta,\hat{y}(\zeta)\|d\zeta \\
& +(1-\rho)\|B\|\|u_y(t)-u_{\hat{y}}(t)\|+(1-\rho)\|f(t,y(t))-f(t,\hat{y}(t))\| \\
& +\rho\|A_\rho\|\int_0^t(t-\zeta)^{\omega-1}\|L_\omega(t-\zeta)\|\|B\|\|u_y(\zeta)-u_{\hat{y}}(\zeta)\|d\zeta \\
& +\sum_{\gamma=1}^{p}\|T_\omega\|\|q_\gamma y(t_\gamma)-q_\gamma\hat{y}(t_\gamma)\|\Bigg]
\end{aligned}
$$

$$\leq\eta\Bigg[\left\|E^{-1}\right\|\mathcal{M}_h+\frac{\mathcal{S}\Upsilon^{\omega}}{\omega\Gamma(\omega)}\mathcal{M}_h+(1-\rho)\|B\|K\eta\mathcal{S}\left(\frac{\Upsilon^{\omega}}{\omega\Gamma(\omega)}\mathcal{M}_h+\rho\eta\frac{\Upsilon^{\omega}}{\omega\Gamma(\omega)}\mathcal{M}_f+\mathcal{M}\right)$$
$$+(1-\rho)\mathcal{M}_f+\rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\omega\Gamma(\omega)}\|B\|K\eta\mathcal{S}\left(\frac{\Upsilon^{\omega}}{\omega\Gamma(\omega)}\mathcal{M}_h+\rho\eta\frac{\Upsilon^{\omega}}{\omega\Gamma(\omega)}\mathcal{M}_f+\mathcal{M}\right)$$
$$+\mathcal{S}\mathcal{M}\Bigg]\|y-\hat{y}\|$$
$$=\eta\Bigg[\left\|E^{-1}\right\|\mathcal{M}_h+\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_h+(1-\rho)\mathcal{M}_f+\mathcal{S}\mathcal{M}$$
$$+\|B\|K\eta\mathcal{S}\left((1-\rho)+\rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\right)\left(\frac{\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_h+\rho\eta\frac{\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_f\right.$$
$$\left.+\mathcal{M}\right)\Bigg]\|y-\hat{y}\|.$$

Let

$$N=\eta\Bigg[\left\|E^{-1}\right\|\mathcal{M}_h+\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_h+(1-\rho)\mathcal{M}_f+\mathcal{S}\mathcal{M}$$
$$+\|B\|K\eta\mathcal{S}\left((1-\rho)+\rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\right)\left(\frac{\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_h+\rho\eta\frac{\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_f+\mathcal{M}\right)\Bigg].$$

From (3.1), we can observe that $0<N<1$ which mean Φ_1 is a contraction.

Step III . We prove the operator Φ_2 is completely continuous.
Firstly, we prove Φ_2 is continuous.
Let y_n be a sequence in $PC(\mathcal{J},X)$ which converge to y. Since f is continuous function, then

$$\|\Phi_2y_n-\Phi_2y\|\leq\rho\left\|A_{\rho}^{2}\right\|\int_0^t(t-\zeta)^{\omega-1}\|L_{\omega}(t-\zeta)\|\|f(\zeta,y_n(\zeta))-f(\zeta,y(\zeta))\|d\zeta$$
$$\leq\rho\lambda^2\frac{\mathcal{S}}{\Gamma(\omega)}\int_0^t(t-\zeta)^{\omega-1}\|f(\zeta,y_n(\zeta))-f(\zeta,y(\zeta))\|d\zeta$$

which converge to zero as $n\to\infty$. Therefore $\Phi_2y_n\to\Phi_2y$ in $PC(\mathcal{J},X)$.
Next, we prove the family $\{\Phi_2y:y\in C_{\epsilon_0}\}$ is relatively compact. According to Arzela – Ascoli Theorem it is suffices to prove:

- $\{\Phi_2y:y\in C_{\epsilon_0}\}$ is uniformly bounded.
- $\{\Phi_2y:y\in C_{\epsilon_0}\}$ is equicontinuous.
- For each $t\in I$ then $\{(\Phi_2y)(t):y\in C_{\epsilon_0}\}$ is relatively compact in X.

By definition of C_{ϵ_0} we have $\|\Phi_2y\|\leq\epsilon_0$ for any $y\in C_{\epsilon_0}$, therefore $\{\Phi_2y:y\in C_{\epsilon_0}\}$ is uniformly bounded.

To prove $\{\Phi_2 y : y \in C_{\epsilon_0}\}$ is equicontinuous, let $t_1, t_2 \in \mathcal{J}$, $t_1 < t_2$ then

$$
\begin{aligned}
&\|\Phi_2 y(t_2) - \Phi_2 y(t_1)\| \\
&= \left\| \int_0^{t_2} (t_2-\zeta)^{\omega-1} L_\omega (t_2-\zeta) f(\zeta, y(\zeta))\, d\zeta - \int_0^{t_1} (t_1-\zeta)^{\omega-1} L_\omega (t_1-\zeta) f(\zeta, y(\zeta))\, d\zeta \right\| \\
&= \left\| \int_{t_1}^{t_2} (t_2-\zeta)^{\omega-1} L_\omega (t_2-\zeta) f(\zeta, y(\zeta))\, d\zeta + \int_0^{t_1} (t_2-\zeta)^{\omega-1} L_\omega (t_2-\zeta) f(\zeta, y(\zeta))\, d\zeta \right. \\
&\quad + \int_0^{t_1} (t_1-\zeta)^{\omega-1} L_\omega (t_2-\zeta) f(\zeta, y(\zeta))\, d\zeta - \int_0^{t_1} (t_1-\zeta)^{\omega-1} L_\omega (t_2-\zeta) f(\zeta, y(\zeta))\, d\zeta \\
&\qquad \left. - \int_0^{t_1} (t_1-\zeta)^{\omega-1} L_\omega (t_1-\zeta) f(\zeta, y(\zeta))\, d\zeta \right\| \\
&\le \left\| \int_{t_1}^{t_2} (t_2-\zeta)^{\omega-1} L_\omega (t_2-\zeta) f(\zeta, y(\zeta))\, d\zeta \right\| + \left\| \int_0^{t_1} (t_2-\zeta)^{\omega-1} L_\omega (t_2-\zeta) f(\zeta, y(\zeta))\, d\zeta \right. \\
&\qquad \left. - \int_0^{t_1} (t_1-\zeta)^{\omega-1} L_\omega (t_2-\zeta) f(\zeta, y(\zeta))\, d\zeta \right\| \\
&\quad + \left\| \int_0^{t_1} (t_1-\zeta)^{\omega-1} L_\omega (t_2-\zeta) f(\zeta, y(\zeta))\, d\zeta - \int_0^{t_1} (t_1-\zeta)^{\omega-1} L_\omega (t_1-\zeta) f(\zeta, y(\zeta))\, d\zeta \right\| \\
&= \left\| \int_{t_1}^{t_2} (t_2-\zeta)^{\omega-1} L_\omega (t_2-\zeta) f(\zeta, y(\zeta))\, d\zeta \right\| \\
&\quad + \left\| \int_0^{t_1} \left[(t_2-\zeta)^{\omega-1} - (t_1-\zeta)^{\omega-1}\right] L_\omega (t_2-\zeta) f(\zeta, y(\zeta))\, d\zeta \right\| \\
&\quad + \left\| \int_0^{t_1} (t_1-\zeta)^{\omega-1} \left[L_\omega (t_2-\zeta) - L_\omega (t_1-\zeta)\right] f(\zeta, y(\zeta))\, d\zeta \right\|.
\end{aligned}
$$

Let

$$
\begin{aligned}
O_1 &= \left\| \int_{t_1}^{t_2} (t_2-\zeta)^{\omega-1} L_\omega (t_2-\zeta) f(\zeta, y(\zeta))\, d\zeta \right\|, \\
O_2 &= \left\| \int_0^{t_1} \left[(t_2-\zeta)^{\omega-1} - (t_1-\zeta)^{\omega-1}\right] L_\omega (t_2-\zeta) f(\zeta, y(\zeta))\, d\zeta \right\|, \\
O_3 &= \left\| \int_0^{t_1} (t_1-\zeta)^{\omega-1} \left[L_\omega (t_2-\zeta) - L_\omega (t_1-\zeta)\right] f(\zeta, y(\zeta))\, d\zeta \right\|.
\end{aligned}
$$

We have to prove O_1, O_2 and O_3 tend to zero when $t_2 \to t_1$.
By using Lemma 2.11 and condition H4, we have

$$
\begin{aligned}
O_1 &\le \frac{\mathcal{S}}{\Gamma(\omega)} \int_{t_1}^{t_2} (t_2-\zeta)^{\omega-1} \|f(\zeta, y(\zeta))\|\, d\zeta \\
&\le \frac{\mathcal{S}}{\Gamma(\omega+1)} (t_2-t_1)^{\omega} \left(\mathcal{M}_f \epsilon_0 + \widehat{\mathcal{M}_f}\right)
\end{aligned}
$$

which tend to zero as $t_2 \to t_1$.
Also,

$$
O_2 \le \frac{\mathcal{S}}{\Gamma(\omega)} \frac{(t_2-t_1)^{\omega}}{\omega} \left(\mathcal{M}_f \epsilon_0 + \widehat{\mathcal{M}_f}\right)
$$

which is tend to zero as $t_2 \to t_1$.
Now, for O_3, if $t_1 = 0$ then $O_3 = 0$.
If $t_1 > 0$ and s is a small enough, then

$$\begin{aligned}O_3 \leq& \int_0^{t_1-s} (t_1-\zeta)^{\omega-1} \|L_\omega(t_2-\zeta) - L_\omega(t_1-\zeta)\| \|f(\zeta, y(\zeta))\| d\zeta \\ &+ \int_{t_1-s}^{t_1} (t_1-\zeta)^{\omega-1} \|L_\omega(t_2-\zeta) - L_\omega(t_1-\zeta)\| \|f(\zeta, y(\zeta))\| d\zeta \\ \leq& \, sup_{\zeta\in[0,t_1-s]} \|L_\omega(t_2-\zeta) - L_\omega(t_1-\zeta)\| \frac{t_1^\omega - s^\omega}{\omega} \left(\mathcal{M}_f\epsilon_0 + \widehat{\mathcal{M}_f}\right) \\ &+ sup_{\zeta\in[t_1-s,t_1]} \|L_\omega(t_2-\zeta) - L_\omega(t_1-\zeta)\| \frac{s^\omega}{\omega} \left(\mathcal{M}_f\epsilon_0 + \widehat{\mathcal{M}_f}\right).\end{aligned}$$

Since $T(t), t > 0$ is compact operator and by Lemma 2.11 we have L_ω is continuous in the uniform operator topology, then O_3 tend to zero as $t_2 \to t_1,\ s \to 0$.
Therefore $\|\Phi_2 y(t_2) - \Phi_2 y(t_1)\|$ tend to zero independently of $y \in C_{\epsilon_0}$ as $t_2 \to t_1$ which mean the family $\{\Phi_2 y : y \in C_{\sigma_0}\}$ is equicontinuous.
Finally, we prove the set $R(t) = \{(\Phi_2 y)(t), y \in C_{\epsilon_0}\}$ for any $t \in \mathcal{J}$ is relatively compact in X.
If $t = 0$, then $R(0) = \{(\Phi_2 y)(0)\} = \{0\}$ which is compact set.
If $t > 0$, we choose $\upsilon \in (0, t)$ and a real number $r > 0$ to define the operator

$$\begin{aligned}(\Phi_r^\upsilon y)(t) =& \omega\rho A_\rho^2 \int_0^{t-\upsilon} \int_r^\infty \theta(t-\zeta)^{\omega-1} \varphi_\omega(\theta) T(\theta(t-\zeta)^\omega) f(\zeta, y(\zeta)) d\theta d\zeta \\ =& \omega\rho A_\rho^2 \int_0^{t-\upsilon} \int_r^\infty \theta(t-\zeta)^{\omega-1} \varphi_\omega(\theta) T(\theta(t-\zeta)^\omega - \upsilon^\omega r + \upsilon^\omega r) f(\zeta, y(\zeta)) d\theta d\zeta \\ =& \omega\rho A_\rho^2 T(\upsilon^\omega r) \int_0^{t-\upsilon} \int_r^\infty \theta(t-\zeta)^{\omega-1} \varphi_\omega(\theta) T(\theta(t-\zeta)^\omega - \upsilon^\omega r) f(\zeta, y(\zeta)) d\theta d\zeta.\end{aligned}$$

Since $T(\upsilon^\omega r)$ is compact operator then the set $\{(\Phi_r^\upsilon y)(t), y \in C_{\epsilon_0}\}$ for any $t \in \mathcal{J}$ is relatively compact in X. Moreover, for any $y \in C_{\epsilon_0}$ we have

$$\begin{aligned}\|(\Phi_2 y)(t) - (\Phi_r^\upsilon y)(t)\| =& \Big\|\omega\rho A_\rho^2 \int_0^t \int_0^\infty \theta(t-\zeta)^{\omega-1} \varphi_\omega(\theta) T(\theta(t-\zeta)^\omega) f(\zeta, y(\zeta)) d\theta d\zeta \\ &- \omega\rho A_\rho^2 \int_0^{t-\upsilon} \int_r^\infty \theta(t-\zeta)^{\omega-1} \varphi_\omega(\theta) T(\theta(t-\zeta)^\omega) f(\zeta, y(\zeta)) d\theta d\zeta\Big\| \\ =& \Big\|\omega\rho A_\rho^2 \int_0^t \int_0^r \theta(t-\zeta)^{\omega-1} \varphi_\omega(\theta) T(\theta(t-\zeta)^\omega) f(\zeta, y(\zeta)) d\theta d\zeta \\ &+ \omega\rho A_\rho^2 \int_0^t \int_r^\infty \theta(t-\zeta)^{\omega-1} \varphi_\omega(\theta) T(\theta(t-\zeta)^\omega) f(\zeta, y(\zeta)) d\theta d\zeta \\ &- \omega\rho A_\rho^2 \int_0^{t-\upsilon} \int_r^\infty \theta(t-\zeta)^{\omega-1} \varphi_\omega(\theta) T(\theta(t-\zeta)^\omega) f(\zeta, y(\zeta)) d\theta d\zeta\Big\| \\ \leq& \Big\|\omega\rho A_\rho^2 \int_0^t \int_0^r \theta(t-\zeta)^{\omega-1} \varphi_\omega(\theta) T(\theta(t-\zeta)^\omega) f(\zeta, y(\zeta)) d\theta d\zeta\Big\| \\ &+ \Big\|\omega\rho A_\rho^2 \int_{t-\upsilon}^t \int_r^\infty \theta(t-\zeta)^{\omega-1} \varphi_\omega(\theta) T(\theta(t-\zeta)^\omega) f(\zeta, y(\zeta)) d\theta d\zeta\Big\|\end{aligned}$$

$$\leq \omega\rho\lambda^2\mathcal{S}\int_0^t (t-\zeta)^{\omega-1}\|f(\zeta,y(\zeta)\|\,d\zeta\int_0^r \theta\varphi_\omega(\theta)\,d\theta$$
$$+\,\omega\rho\lambda^2\mathcal{S}\int_{t-v}^t (t-\zeta)^{\omega-1}\|f(\zeta,y(\zeta)\|\,d\zeta\int_r^\infty \theta\varphi_\omega(\theta)\,d\theta$$
$$\leq \rho\lambda^2\mathcal{S}\Upsilon^\omega\left(\mathcal{M}_f\epsilon_0+\widehat{\mathcal{M}_f}\right)\int_0^r \theta\varphi_\omega(\theta)\,d\theta+\rho\lambda^2\mathcal{S}v^\omega\left(\mathcal{M}_f\epsilon_0+\widehat{\mathcal{M}_f}\right)\int_r^\infty \theta\varphi_\omega(\theta)\,d\theta.$$

Therefore, $\|(\Phi_2 y)(t)-(\Phi_r^v y)(t)\|$ tends to zero as $v,r\to 0$. Consequently, there are relatively compact sets arbitrary close to the set $R(t)$. Thus, $R(t)$ is relatively compact in X. Based on the above, the operator Φ_2 is completely continuous. According to Nussbaum fixed point theorem, operator Φ has a fixed point in C_{ϵ_0}. Therefore, the system (1.1) is controllable on $\mathcal{J}$. □

4. Application

Consider

$$\begin{cases} {}^C D^{\rho,\omega}\left[y(t,\gamma)-h(t,y(t,\gamma))\right]=Ay(t,\gamma)+Bu(t)+f(t,y(t,\gamma)), \\ \rho,\omega\in(0,1),\gamma\in[0,\pi],\ t\in[0,t_1)\cup(t_1,1]\ , \\ y(t,0)=y(t,\pi)=0,\ t\in[0,1], \\ \Delta y\left(t_1^+\right)=q_1\left(y\left(t_1^-\right)\right),\quad t_1=\frac{1}{2}, \end{cases}$$

and $X=L^2([0.\pi],R)$. Define

$$Ay(t,\gamma)=\frac{\partial^2 y}{\partial\gamma^2}(t,\gamma).$$

where

$$D(A)=\left\{y\in X:\ \frac{\partial y}{\partial\gamma},\frac{\partial^2 y}{\partial\gamma^2}\in X\ and\ \gamma(0)=\gamma(\pi)=0\right\}.$$

For $y\in D(A)$ then A can be written as the following

$$Ay=\sum_{s=1}^{\infty}-s^2\langle y,y_s\rangle y_s,$$

where $y_s(\gamma)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}sin(s\gamma)$, s=1,2,3.... Therefore A is the generator of C_0- semigroup $\{T(t),t\geq 0\}$ in $L^2[0,\pi]$ such that $T(t)y=\sum_{s=1}^{\infty}e^{-s^2t}\langle y,y_s\rangle y_s$, $y\in D(A)$ [18]. The functions h,f and q_1 defined as follows:

- $h:[0,1]\times X\to D(A)$ such that

$$h(t,y(t,\gamma))=\int_0^\gamma siny(t,\zeta)\,d\zeta,\ t\in[0,1],\gamma\in[0,\pi],\ y\in X.$$

- $f:[0,1]\times X\to X$ such that

$$f(t,y(t,\gamma))=\frac{t^2e^{-t}|y(t,\gamma)|}{b},\quad t\in[0,1],\gamma\in[0,\pi],\ y\in X,\ b>0.$$

- $q_1 : X \to X$ such that

$$q_1\left(y(t_1,\gamma)\right) = \frac{\left|y\left(\frac{1}{2}^-,\gamma\right)\right|}{2\left(1+\left|y(\frac{1}{2}^-,\gamma)\right|\right)},\ t\in[0,1],\gamma\in[0,\pi],\ y\in X.$$

For $y_1, y_2 \in X$,

$$\|f(t,y_1(t,\gamma)) - f(t,y_2(t,\gamma))\| = \left\|\frac{t^2e^{-t}|y_1(t,\gamma)|}{b} - \frac{t^2e^{-t}|y_2(t,\gamma)|}{b}\right\| \le \frac{1}{b}\|y_1-y_2\|$$

and

$$\|Eh(t,y_1(t,\gamma)) - Eh(t,y_2(t,\gamma))\| = \left\|\frac{\partial^2}{\partial\gamma^2}\int_0^\gamma siny_1(t,\zeta)\,d\zeta - \frac{\partial^2}{\partial\gamma^2}\int_0^\gamma siny_2(t,\zeta)\,d\zeta\right\|$$

$$= \left\|\frac{\partial}{\partial\gamma}siny_1(t,\zeta) - \frac{\partial}{\partial\gamma}siny_2(t,\zeta)\right\| \le L_h\|y_1-y_2\|$$

where

$$L_h = sup\left\{\left\|\frac{\partial^2}{\partial\gamma^2}sinu(t,\gamma)\right\| : u\in X, t\in[0,1],\gamma\in[0,\pi]\right\}.$$

Also,

$$\|q_1(y_1(t_1,\gamma)) - q_1(y_2(t_1,\gamma))\| \le \frac{1}{2}\|y_1-y_2\|.$$

Subsequently, the hypothesis of Theorem 3.2 are fulfilled. Therefore, the system is controllable.

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