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Ditzain-Totik modulus of smoothness for the fractional derivative of functions in L_p space of the partial neural network

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Abstract

Some scientists studied the weighted approximation of the partial neural network, but in this paper, we studied the weighted Ditzain-Totik modulus of smoothness for the fractional derivative of functions in L_p of the partial neural network and this approximation of the real-valued functions over a compressed period by the tangent sigmoid and quasi-interpolation operators. These approximations measurable left and right partial Caputo models of the committed function. Approximations are bitmap with respect to the standard base. Feed-forward neural networks with a single hidden layer. Our higher-order fractal approximation results in better convergence than normal approximation with some applications. All proved results are in $L_p[X]$ spaces, where $0<p<1$.

Keywords: approximation, Ditzain-Totik modulus, higher order fractal approximation, partial Caputo models, partial neural network, Sobolev space

1. Introduction

Here at the beginning of this work, we would like to know the space $W_p^k[a,b]$ is a Sobolev space, where $W_p^k[a,b]$ is the set of all functions from $L_p[a,b]$, $f^{(k)} \in L_p[a,b]$ and $L_p[a,b]$ the space of all measurable functions. We can define the norm of $f \in L_p[a,b]$ as follows, $\|f\|_p = \left(\int_a^b |f|^p \, dx\right)^{1/p}$.

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Moreover, we can define the k-th symmetric difference of f is given by

$$\Delta_h^k\left(f, x\left[a, b\right]\right) = \Delta_h^k\left(f, x\right) == \begin{cases} \sum_{i=0}^{r} \begin{pmatrix} r \\ i \end{pmatrix} (-1)^{r-i} f\left(x - \frac{rh}{2} + ih\right), & x \pm \frac{rh}{2} \in [a, b] \\ 0 & \text{o.w.} \end{cases}$$

Then the r-th usual modulus of smoothness of $f \in L_p\left[a, b\right]$ is defined by:

$$\omega_k(f, \delta, [a, b])_p = \sup_{0<|h|\leq\delta} \|\Delta_h^r\left(f, .\right)\|_{L_p[a,b]}, \qquad \delta \geq 0.$$

We can define Ditzain–Totik modulus of smoothness which defined for such an f as follows:

$$\omega_k^{\emptyset}(f, \delta, [a, b])_p = \sup_{0<|h|\leq\delta} \left\|\Delta_{h\emptyset(\cdot)}^r\left(f, .\right)\right\|_{L_p[a,b]} \quad \text{and} \quad \lim_{h\to 0} \omega_k^{\emptyset}\left(f, h\right) = 0.$$

In the applications the $\emptyset$ usually used $\emptyset\left(x\right) = \left(x\left(1-x\right)\right)^{1/2}$ and $\emptyset\left(x\right) = \sqrt{x}\left(1-x\right)$ for $J = [0.1]\,, \emptyset\left(x\right) = \left(1-x^2\right)^{\frac{1}{2}}$ for $J = [-1, 1]\,, \emptyset\left(x\right) = \sqrt{x}$ and $\emptyset\left(x\right) = \left(x\left(1+x\right)\right)^{\frac{1}{2}}$and $\Phi\left(x\right) = x$ for $J = [0, \infty]$. [2] is the first outhouse studied neural network approximations and introduce neural network operators, continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of the sigmoidal and hyperbolic tangent types which resulted in [4]. Working in this paper, neural network approximations are introduced at the fractal level resulting in higher approximations. We include the left and right Caputo derivatives of the function under approximate fractional calculus and neural networks, all of which are necessary to expose our work. Applications are presented at the end.

Feed-forward neural networks with a single hidden layer, are expressed mathematically as follows

$$N_n\left(x_1, \ldots, x_s\right) = \sum_{i=0}^{n} l_i \sigma\left(\sum_{k=1}^{s} a_{ik} x_i\right) + b_i$$

where $0 \leq i \leq n$, $a_i \in R^s, s \in N$ *are connection weights*, $l_i \in R$ is coefficients, a_i, x, σ activation function of the network. The network can be found in [4, 5].

2. Auxiliary Results

Definition 2.1. *[5] Let $f \in W_p\left[a, b\right]$ and K-modulus smoothness of f at t is given by*

$$\omega_k\left(f, h\right) = \sup_{0<|h|\leq\delta} \left\|\Delta_h^k \; f\right\|_p, \quad \textit{and} \quad \lim_{h\to 0} \omega_k\left(f, h\right) \; = 0,$$

and the Ditzian-Totik modulus of smoothness of f at t is given by

$$\omega_k^{\emptyset}\left(f, h\right) = \sup_{0<|h|\leq\delta} \left\|\Delta_{h\emptyset(\cdot)}^k \; f\right\|_p \textit{and} \quad \lim_{h\to o} \quad \omega_k^{\Phi}\left(f, h\right) = 0.$$

Definition 2.2. *[5, 6] Let $v \geq 0$, $n = \lceil v \rceil$ where $\lceil . \rceil$ (calling of number), $f \in W_p^n$ (Sobolve space) and $f^{(n-1)} \in W_p$. We can define left Caputo fractional derivative as follow.*

$$D_{*a}^v \; f\left(x\right) = \frac{1}{\Gamma\left(n-v\right)} \int_a^x \left(x-t\right)^{n-v-1} \; f^n\left(t\right) \; dt, \; \forall x \in \; [a, b]\,,$$

where Γ is the gamma function:

$$\Gamma\left(v\right) = \int_o^\infty e^{-t} \; t^{v-1} \; dt \; , \; v > 0.$$

Note that $D^{v}_{*a}\ f\ \in L_p\ [a,b]$ and D^{v}_{*a} exist a.e on $[a,b]$. We set $D^{o}_{*a}\ f\ (\ x) = f\ (x)\ ,\ \forall\ x\ \in\ [a,b]$.

Definition 2.3. *[3] Let $f\ \in W^{m}_{p}\,[a,b]\ ,\ m = \lceil\propto\rceil\ ,\propto > 0$, the right Caputo fractional derivative of order $\propto > o$ is given as follows*

$$D^{v}_{b-}\ f\ (x) = \frac{(-1)^m}{\varGamma\ (m-\propto)}\int_x^b (\varepsilon - x)^{m-\propto-1}\quad f^m\ (\varepsilon)\ \ d\varepsilon,\ \ \forall x\in\ \ [a,b]\ ,$$

we set $D^{o}_{b-}\ f\ (x) =\ \ f\ (x)\ \ ,\ \forall x\in\ \ [a,b]$.

Note that $D^{\propto}_{b-}\ f\in\ L_p\ \ [a,b]$ and $D^{\propto}_{b-}\ f$ exists a.e on $[a,b]$. We let that $D^{\propto}_{*x_0}\ f\ (x) = 0$ for $x < x_0$ and $D^{\propto}_{x_0-}\ f\ (x) = 0$ for $x > x_0$ for all $x,\ x_0\ \in [a,b]$.

Remark 2.4. *Let $f\in W^{n-1}_{p}\,[a,b]\,,f^n\in W_p\,[a,b]\,,n = \lceil v\rceil,\ v > 0, v\notin N$, then we have*

$$\|D^{v}_{*a}\ f\ (x)\|_{L_p[a,b]}\ \ \leq\ \ c\,(p,k)\,\frac{\|\ f^n\ (x)\|_{L_p[a,b]}}{\varGamma\ (n-v+1)}(x-a)^{n-v}$$

thus we observe that

$$\begin{aligned}\omega^{\emptyset}_{k}(D^{v}_{*a}\ f,\ \delta)_{L_p[a,b]} &= \sup_{0<|h|\leq\delta}\left\|\Delta^{k}_{h\emptyset}D^{v}_{*a}\ f\ (x)\right\|_{L_p[a,b]}\\ &= \sup_{0<|h|\leq\delta}\left\|\sum_{i=0}^{k}\left(\begin{array}{c}k\\ i\end{array}\right)(-1)^{k-i}D^{v}_{*a}f\left(x-\frac{kh\emptyset}{2}+ih\emptyset\right)\right\|_{L_p[a,b]}\end{aligned}$$

where $x\pm kh\emptyset\in [a,b]$

$$\begin{aligned}\omega^{\emptyset}_{k}(D^{v}_{*a}\ f,\ \delta)_{L_p[a,b]} &\leq c\,(p,k)\sup_{0<|h|\leq\delta}\sum_{i=0}^{k}\left(\begin{array}{c}k\\ i\end{array}\right)(-1)^{k-i}\|D^{v}_{*a}f\ (x)\|_{L_p[a,b]}\\ &\leq \|D^{v}_{*a}f\ (x)\|_{L_p[a,b]}\\ &\leq \left\|\frac{1}{\Gamma\,(n-v)}\int_a^b (b-t)^{n-v-1}f^n\ (x)\ \ dt\right\|_{L_p[a,b]}\\ &= \frac{\left\|f^{(n)}\right\|_{L_p[a,b]}}{\Gamma\,(n-v)}\left.\frac{(b-t)^{n-v}}{n-v}\right|_a^b\\ &= \frac{\|\ f^n\|_{L_p[a,b]}}{(n-v)\,\Gamma\,(n-v)}\left\|(b-b)^{n-v}(b-a)^{n-v}\right\|_{L_p[a.b]}\\ &\leq c\,(p,k)\,\frac{\left\|f^{(n)}\right\|_{L_p[a,b]}}{\Gamma\,(n-v+1)}(b-a)^{n-v}. \end{aligned}\tag{2.1}$$

Note that $(n-v)\,\varGamma\,(n-v) = \varGamma\,(n-v+1)$. Similarly, let $f\in W^{m-1}_{p}\,([a,b])\,,f^{(m)}\in W_p\,[a,b]$, $m =\ [\propto]\,,\propto > 0,\propto\notin N$, then

$$\begin{aligned}w^{\emptyset}_{k}\ \left(D^{\propto}_{b-}f,\delta\right) &= \sup_{o<|h|\leq\delta}\|\ \Delta^{k}_{h\emptyset}\ \ D^{\propto}_{b-}f(x)\ \|_{L_p[a,b]}\\ &= \sup_{o<|h|\leq\delta}\left\|\sum_{i=o}^{k}\left(\begin{array}{c}k\\ i\end{array}\right)(-1)^{k-i}\ D^{\propto}_{b-}f(x-\ \frac{kh\emptyset}{2}+ih\emptyset\right\|_{L_p\ \ [a,b]}.\end{aligned}$$

where $x \mp \dfrac{kh\emptyset}{2} \in [a,b]$.

$$\begin{aligned}
\omega_k^{\emptyset}\left(D_{b-}^{\propto} f,\delta\right)_{L_P\ [a,b]} &\le c(p,k) \sup_{o<|h|\le\delta} \sum_{i=o}^{k} \binom{k}{i} (-1)^{k-i} \left\| D_{b-}^{\propto} f(x)\right\|_{L_p\ [a,b]} \\
&\le \left\| D_{b-}^{\propto} f(x)\right\|_{L_{p[a,b]}} \\
&\le \left\| \frac{1}{\Gamma(m-\propto)} \int_a^b (\mathcal{E}-t)^{m-\propto-1} f^{(m)}(\mathcal{E})\, d\mathcal{E} \right\|_{L_{p[a,b]}} \\
&\le \frac{\| f^m \|_{L_{p[a,b]}}}{\Gamma(m-\propto)} \int_a^b (\mathcal{E}-t)^{m-\propto-1}\, d\mathcal{E} \\
&\le \frac{\| f^m \|_{L_{p[a,b]}}}{\Gamma(m-\propto)} \left. \frac{(\mathcal{E}-t)^{(m-\propto-1)+1}}{(m-\propto-1)+1} \right|_a^b
\end{aligned}$$

since $x \in [a,b]$, *we have*

$$\omega_k^{\emptyset}\left(D_{b-}^{\propto} f,\delta\right)_{L_p\ [a,b]} \le c(p,k) \frac{\| f^m \|_{L_{p[a,b]}}}{\Gamma(m-\propto)} \frac{(b-a)^{m-\propto} - (a-a)^{m-\propto}}{m-\propto}$$

$$\omega_k^{\emptyset}\left(D_{b-}^{\propto} f,\delta\right)_{L_{p[a,b]}} \le c(p,k) \frac{\| f^m \|_{L_{p[a,b]}}}{(m-\propto)\,\Gamma(m-\propto)} (b-a)^{m-\propto}. \tag{2.2}$$

Then from (2.1) *and* (3.2), *we find that*

$$\varsigma_1(\delta) = \omega_k^{\emptyset}\left(D_{*xo}^{\propto} f,\delta\right) \le c(p,k) \frac{\| f^m \|_{L_{p[a,b]}}}{\Gamma(m-\propto)} (b-a)^{m-a} \tag{2.3}$$

$$\varsigma_2(\delta) = \omega_k^{\emptyset}\left(D_{xo-}^{\propto} f,\delta\right) \le c(p,k) \frac{\| f^m \|_{L_{p[a,b]}}}{\Gamma(m-\propto)} (b-a)^{m-a}.$$

Thus, $D_{*xo}^{\propto} f \in L_p\ [x_0,b]$ *and* $D_{x_0-}^{\propto} f \in L_p\ ([a,x_o])$. *Clearly, we have* $\varsigma_1(\delta) \to 0$ *and* $\varsigma_2(\delta) \to 0$ *as* $\delta \to 0$.

Definition 2.5. *[1] We define here the sigmoidal function of logarithmic type*

$$\varsigma(x) = \frac{1}{1+e^{-x}}, \quad x \in R$$

and

$$\lim_{x\longrightarrow+\infty} \varsigma(x) = 1 \text{ and } \lim_{x\longrightarrow-\infty} \varsigma(x) = 0.$$

This function plays the role of activation function in the hidden layer of networks we consider that

Lemma 2.6. *Let* $x \in [a,b] \subset R$ *and* $n \in N$ *so that* $\lceil na \rceil \le \lfloor nb \rfloor$ *it holds*

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Psi(nx-k)} < 4.1488766 = \frac{1}{\Psi(1)}$$

$$\lim_{n\longrightarrow\infty} \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Psi(nx-k) \ne 1, \text{ for least some } x \in [a,b].$$

Definition 2.7. *Let $f \in L_p[a,b]$ and $n \in N$ such that $\lceil na \rceil \leq \lfloor nb \rfloor$. We introduced the positive linear neural network operator*

$$f_n\ (f,x) = \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f(x' + \frac{kh\emptyset}{2} + ih\emptyset)\, \Psi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi\,(nx-k)}\ ,\quad x, x' \in [a,b]\,.$$

We study F_n similarly to G_n

$$F^*{}_n\ (f,x) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f(x' + \frac{kh\emptyset}{2} + ih\emptyset)\, \Psi(nx-k)$$

That is $F_n\ (f,x) = \frac{F_n^*\ (f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k)}$

Definition 2.8. *Let $f \in L_p[a,b]$ and $n \in N$ such that $\lceil na \rceil \leq \lfloor nb \rfloor$, we further study the positive linear neural network operator*

$$F_n\ (f,x) = \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f(x' + \frac{kh\emptyset}{2} + ih\emptyset)\, \Psi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi\,(nx-k)}\ ,\quad x \in [a,b]\,.$$

Let $F_n^* = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \sum_{i=0}^{k} \begin{pmatrix} k \\ i \end{pmatrix} (-1)^{k-i}\ f\left(x' - \frac{kh\emptyset}{2} + ih\emptyset\right)\ \Psi(nx-k)$

That is $F_n\ (f,x) = \frac{F_n^*\ (f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k)}$. Note that by Lemma 2.1, $\frac{1}{\sum_{\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k)} = \frac{1}{\Psi(1)} = 4.14887$. Then

$$F_n\ (f,x) - f\,(x) = \frac{F_n^*\ (f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi\,(nx-k)} - f\,(x) = \frac{F_n^*\ (f,x) - f(x)\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi\,(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi\,(nx-k)}$$

By Lemma 2.1

$$\|F_n\ (f,x) - f\,(x)\|_{L_{p[a,b]}} = \frac{1}{\Psi\,(1)} \left\| F_n^*\,(f,x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi\,(nx-k) \right\|_{L_{p[a,b]}}$$

$$\leq 4.14887 \left\| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \sum_{i=0}^{\lfloor k \rfloor} \begin{pmatrix} k \\ i \end{pmatrix} (-1)^{k-i} f\left(x' - \frac{kh\emptyset}{2} + ih\emptyset\right) \Psi\,(nx-k) - f\,(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi\,(nx-k) \right\|_{L_{p[a,b]}}$$

$$\leq 2^k.4.148876 \left\| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(x - \frac{kh\emptyset}{2} + ih\emptyset\right)\ \Psi\,(nx-k) \right\|_{L_{p[a,b]}}$$

We will estimate the right hand side of above equation involving the right and left caputo fractional derivative of f.

3. Main Results

Theorem 3.1. *Let* $\propto > 0, N = \lceil \propto \rceil, \propto \notin N, f \in W_p[a,b]$ *with* $f^{(N)} \in Lp[a,b], 0 < B < 1, x \in [a,b], n \in N$, *then*

$$\text{(i)} \left\| F_n(f,x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} F_n((.,x)^j) - (x)f(x) \right\|_{Lp[a,b]}$$
$$\leq \frac{8.2\ c(p)}{\Gamma(\propto +1)} [\frac{1}{n^B}(w_k^{\emptyset}(D_{x-}^{\propto} f, \frac{1}{n^B})_{Lp[a,x]} + [w_k^{\emptyset}(D_{*x}^{\propto} f, \frac{1}{n^B})_{Lp[x,b]})$$
$$+ e4\ e^{-n(1-B)}(\left\|D_{x-}^{\propto} f\right\|_{Lp[a,x]}(x-a)^{\propto} + \|D_{*x}^{\propto} f\|_{Lp[x,b]}(b-x)^{\propto}].$$

When $\propto > 1$ *note here extremely high rate of convergence at* $n^{-(\propto+1)B}$.

(ii) *If* $f^{(j)}(x) = 0$, *for* $j = 1, \ldots, N-1$, *we have*

$$\|F_n(f,x) - f(x)\|_{Lp[a,b]} \leq \frac{8.2\ c(p)}{\Gamma(\propto +1)} [w_k^{\emptyset}(D_{x-}^{\propto} f, \frac{1}{n^B})_{Lp[a,x]} + [w_k^{\emptyset}(D_{*x}^{\propto} f, \frac{1}{n^B})_{Lp[x,b]}$$
$$+ 3.19\ e^{-n(1-B)}(\left\|D_{x-}^{\propto} f\right\|_{Lp[a,x]}(x-a)^{\propto} + \|D_{*x}^{\propto} f\|_{Lp[x,b]}(b-x)^{\propto}].$$

When $\propto > 1$ *note here extremely high rate of convergence at* $n^{-(\propto+1)B}$.

$$\text{(iii)} \|F_n(f,x) - f(x)\|_{Lp[a,b]}$$
$$\leq \frac{8.2\ c(p)}{\Gamma(\propto +1)} [\sum_{j=1}^{N-1} \frac{\left\|f^{(j)}\right\|}{j!} \{\frac{1}{n^B} + (b-a)^j 3.19\ e^{-n(1-B)}\}(\frac{8.2 c(p)}{\Gamma(\propto +1)} [(w_k^{\emptyset}(D_{x-}^{\propto} f, \frac{1}{n^B})_{Lp[a,x]}$$
$$+ [w_k^{\emptyset}(D_{*x}^{\propto} f, \frac{1}{n^B})_{Lp[x,b]} + 3.19\ e^{-n(1-B)}(\left\|D_{x-}^{\propto} f\right\|_{Lp[a,x]}(x-a)^{\propto} + \|D_{*x}^{\propto} f\|_{Lp[x,b]}(b-x)^{\propto})]$$

Proof . Let $x, x' \in [a,b]$ we have $D_{x-}^{\propto} f = D_{*x}^{\propto} f = 0$, we get by the left Caputo fractional weighted Taylor formula that

$$f\left(x' + \frac{kh\emptyset}{2} + ih\emptyset\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(x' + \frac{kh\emptyset}{2} + ih\emptyset\right)^j$$
$$+ \frac{1}{\Gamma(\propto)} \int_x^{x+\frac{kh\emptyset}{2}++ih\emptyset} (\left(x' + \frac{kh\emptyset}{2} + ih\emptyset) - j\right)^{\propto-1} . (D_{*x}^{\propto} f(j-x)\, dj$$

where $x \leq x' + \frac{kh\emptyset}{2} + ih\emptyset \leq b$. Using the right Caputo fractional weighted Taylor formula, we get

$$f\left(x' + \frac{kh\emptyset}{2} + ih\emptyset\right) = \sum_{j=0}^{N-1} \frac{f^j(x)}{j!} \left(x' + \frac{kh\emptyset}{2} + ih\emptyset - x\right)^j$$
$$+ \frac{1}{\Gamma(\propto)} \int_{x'+\frac{kh\emptyset}{2}+ih\emptyset}^{x} \left(j - x' + \frac{kh\emptyset}{2} + ih\emptyset\right) \left(D_{x-}^{\propto} f(j+x)\, dj\right)^{\propto-1},$$

where $a \le x' + \frac{kh\emptyset}{2} + ih\emptyset \le x$. Then

$$f\left(x' + \frac{kh\emptyset}{2} + ih\emptyset\right)\Psi(nx-k) = \sum_{j=0}^{N-1}\frac{f^{j}(x)}{j!}\Psi(nx-k) + \frac{\Psi(nx-k)}{\Gamma(\propto)}\int_{x}^{x'+\frac{kh\emptyset}{2}+ih\emptyset}\left(x' + \frac{kh\emptyset}{2} + ih\emptyset - j\right)^{\propto-1}(D_{*x}^{\propto} f(j+x))dj,$$

where for all $x \le x' + \frac{kh\emptyset}{2} + ih\emptyset \le b$ if and only if $\lceil nx\rceil \le k \le \lfloor nb \rfloor$ and

$$f\left(x' + \frac{kh\emptyset}{2} + ih\emptyset\right)\Psi(nx-k) = \sum_{j=0}^{N-1}\frac{f^{j}(x)}{j!}\Psi(nx-k)\left(x' + \frac{kh\emptyset}{2} + ih\emptyset - x\right)^{j} + \frac{\Psi(nx-k)}{\Gamma(\propto)}\int_{x'+\frac{kh\emptyset}{2}+ih\emptyset}^{x}\left(j-(x'+\frac{kh\emptyset}{2} + ih\emptyset\right)^{\propto-1}\left(D_{x-}^{\propto} f(j+x)\right)dj,$$

where $a \le x' + \frac{kh\emptyset}{2} + ih\emptyset \le x$ if and only if $\lceil na\rceil \le k \le \lfloor nx \rfloor$ we have $\lceil nx\rceil \le \lfloor nx \rfloor + 1$. Therefore

$$\sum_{k=\lceil nx\rceil+1}^{\lfloor nb\rfloor} f\left(x' + \frac{kh\emptyset}{2} + ih\emptyset\right)\Psi(nx-k) = \sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\Psi(nx-k)\left(x' + \frac{kh\emptyset}{2} + ih\emptyset\right)^{j} + \frac{1}{\Gamma(\propto)}\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\Psi(nx-k)\int_{x}^{x'+\frac{kh\emptyset}{2}+ih\emptyset}\left(\left(x' + \frac{kh\emptyset}{2} + ih\emptyset\right) - j\right)^{\propto-1} D_{*x}^{\propto} f(j+x)\,dj \tag{3.1}$$

and

$$\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor} f\left(x' + \frac{kh\emptyset}{2} + ih\emptyset\right)\Psi(nx-k) = \sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\Psi(nx-k)\left((x' + \frac{kh\emptyset}{2} + ih\emptyset - x\right)^{j} + \frac{1}{\Gamma(\propto)}\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\Psi(nx-k)\int_{x'+\frac{kh\emptyset}{2}+ih\emptyset}^{x}\left(J-(x' + \frac{kh\emptyset}{2} + ih\emptyset\right)^{\propto-1} D_{x-}^{\propto} f(j+x)\,dj. \tag{3.2}$$

By (3.1) and (3.2) we get

$$F_n^{*}(f,x) = \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(x' + \frac{kh\emptyset}{2} + ih\emptyset\right)\Psi(nx-k)$$

$$\frac{F_n^{*}(f,x)}{\sum_{i=0}^{k}\binom{k}{i}} = \sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\Psi(nx-k)\left((x' + \frac{kh\emptyset}{2} + ih\Phi - x\right)^{j} + \frac{1}{\Gamma(\propto)}\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\Psi(nx-k)\int_{x'+\frac{kh\emptyset}{2}+ih\emptyset}^{x}\left(J-(x' + \frac{kh\emptyset}{2} + ih\emptyset\right)^{\propto-1}$$

$$D_{x-}^{\propto} f(j+x)\,dj + \frac{1}{\Gamma(\propto)} \sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} \Psi(nx-k) \int_{x}^{x'+\frac{kh\emptyset}{2}+ih\emptyset} \left(\left(x'+\frac{kh\emptyset}{2}+ih\emptyset\right)-j\right)^{\propto-1} D_{*x}^{\propto} f(j+x)\,dj$$

$$F_n^*(f,x) = c(p)\,[\sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \Psi(nx-k))\left((x'+\frac{kh\emptyset}{2}+ih\Phi-x\right)^{j}$$
$$+\frac{1}{\Gamma(\propto)} \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \Psi(nx-k) \int_{x'+\frac{kh\emptyset}{2}+ih\emptyset}^{x} \left(J-(x'+\frac{kh\emptyset}{2}+ih\emptyset\right)^{\propto-1} D_{x-}^{\propto} f(j+x)\,dj$$
$$+\frac{1}{\Gamma(\propto)} \sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} \Psi(nx-k) \int_{x}^{x'+\frac{kh\emptyset}{2}+ih\emptyset} \left(\left(x'+\frac{kh\emptyset}{2}+ih\emptyset\right)-j\right)^{\propto-1} D_{*x}^{\propto} f(j+x)\,dj]$$

Where assume $\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i}$ estimation equal $c(p)$ and assume

$$U_n(x) = \frac{1}{\Gamma(\propto)} (\sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \Psi(nx-k) \int_{x'+\frac{kh\emptyset}{2}+ih\emptyset}^{x} \left(j-(x'+\frac{kh\emptyset}{2}+ih\emptyset\right)^{\propto-1} D_{x-}^{\propto} f(j+x)\,dj$$
$$+\sum_{k=\lceil nx \rceil+1}^{\lfloor nb \rfloor} \Psi(nx-k) \int_{x}^{x'+\frac{kh\emptyset}{2}+ih\emptyset} \left(x'+\frac{kh\emptyset}{2}+ih\emptyset)-j\right)^{\propto-1} D_{*x}^{\propto} f(j-x)\,dj).$$

Then

$$F_n^*(f,x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k) = c(p) \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} F_n^*(.-x)^j(x) + U_n(x). \tag{3.3}$$

We put

$$U_{1n}(x) = \frac{1}{\Gamma(\propto)} \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \Psi(nx-k) \int_{x'+\frac{kh\emptyset}{2}+ih\emptyset}^{x} \left(j-\left(x'+\frac{kh\emptyset}{2}+ih\emptyset\right)\right) D_{x-}^{\propto} f(j-x)\,dj$$

$$U_{2n}(x) = \frac{1}{\Gamma(\propto)} \sum_{k=\lceil nx \rceil+1}^{\lfloor nb \rfloor} \Psi(nx-k) \int_{x}^{x'+\frac{kh\Phi}{2}+ih\Phi} \left(x'+\frac{kh\emptyset}{2}+ih\emptyset-j\right)^{\propto-1} D_{*x}^{\propto} f(j+x)\,)dj$$

We mean $U_n(x) = U_{1n}(x) + U_{2n}(x)$. Assume that $b-a > \frac{1}{n^B}$, where $0 < B < 1$, which always large enough $n \in N$ that is when $n > \left[(b-a)^{\frac{-1}{B}}\right]$ it is for $k = \lceil na \rceil, \ldots, \lfloor nx \rfloor$. We consider

$$y_{1k} = \left| \int_{x'+\frac{kh\emptyset}{2}+ih\emptyset}^{x} (j-(x'+\frac{kh\emptyset}{2}+ih\emptyset)^{\propto-1} D_{x-}^{\propto} f(j+x)\,dj \right|,$$

$$y_{1k} = \left| \int_{x'+\frac{kh\emptyset}{2}+ih\emptyset}^{x} (j-(x'+\frac{kh\emptyset}{2}+ih\emptyset)^{\propto-1} D_{x-}^{\propto} f(j)\,dj \right|,$$

$$y_{1k} \leq \int_{x'+\frac{kh\emptyset}{2}+ih\emptyset}^{x} (j-(x'+\frac{kh\emptyset}{2}+ih\emptyset)^{\propto-1} \left|D_{x-}^{\propto} f\right| dj,$$

$y_{1k} \leq \left|D_{x-}^{\propto} f\right| \frac{(x-\ a)^{\propto}}{\propto}$. Where $a \leq x' + \frac{kh\emptyset}{2} + ih\emptyset \leq x$ if and only if $\lceil na \rceil \leq k \leq \lfloor nx \rfloor$ we have $\lceil nx \rceil \leq \lfloor nx \rfloor + 1$. Then

$$\|y_{1k}\|_{Lp[a,b]} \leq \left\|D_{x-}^{\propto} f\right\|_{Lp[a,x].}$$

Also we have in case of $\left|x' + \frac{kh\emptyset}{2} + ih\emptyset - x\right| \leq \frac{1}{n^B},\ \ k\ = \lceil na \rceil, \ldots, \lfloor nx \rfloor$.

$$\|y_{1k}\|_{Lp[a,x]} \leq \frac{w_k^{\emptyset}(D_{x-}^{\propto} f, \frac{1}{n^B}\ )_{Lp[a,x]}}{\propto n^{\propto} B},$$

$$\|U_{1n}(x)\|_{Lp[a,x]} = \left\| \frac{1}{\Gamma(\propto)} \sum_{\lceil na \rceil}^{\lfloor nx \rfloor} \psi(nx-k)\, y_{1k} \right\|_{Lp[a,x]}.$$

$$\|U_{1n}(x)\|_{Lp[a,x]} = \frac{1}{\Gamma(\propto)} \left\| \sum_{\substack{k=\lceil na \rceil \\ \left|x'+\frac{kh\Phi}{2}+ih\Phi-x\right| \leq \frac{1}{nB}}}^{\lfloor nx \rfloor} \Psi(nx-k) \frac{w_k^{\emptyset}\left(D_{x-}^{\propto} f, \frac{1}{n^B}\ \right)_{Lp[a,x]}}{\propto n^{\propto} B} + \sum_{\substack{k=\lceil na \rceil \\ \left|x'+\frac{kh\Phi}{2}+ih\Phi-x\right| > \frac{1}{nB}}}^{\lfloor nx \rfloor} \Psi(nx-k) \left\|D_{*x}^{\propto} f\right\|_{Lp[a,x]} \frac{(x-\ a)^{\propto}}{\propto} \right\|_{Lp[a,x]}.$$

Since Ψis increasing on bounded interval, we have

$$\|U_{1n}(x)\|_{Lp[a,x]} \leq \frac{c(p)}{\propto \Gamma(\propto)} \left\{ \frac{w_k^{\emptyset}\left(D_{x-}^{\propto} f, \frac{1}{n^B}\ \right)_{Lp[a,x]}}{n^{\propto} B} + \sum_{\substack{k=\lceil na \rceil \\ \left|x'+\frac{kh\Phi}{2}+ih\Phi-x\right| > \frac{1}{nB}}}^{\lfloor nx \rfloor} \Psi(nx-k) \left\|D_{x-}^{\propto} f\right\|_{Lp[a,x]} (x-a)^{\propto} \right\},$$

$$\leq \frac{c(p)}{\Gamma(\propto+1)}\left\{\frac{w_k^{\emptyset}\left(D_{x-}^{\propto}f,\frac{1}{n^B}\right)_{Lp[a,x]}}{n^{\propto}B}\right.$$

$$\left.+\sum_{\substack{k=-\infty\\ \left|x'+\frac{kh\Phi}{2}+ih\Phi-x\right|>\frac{1}{nB}}}^{\infty}\Psi(nx-k)\left\|D_{x-}^{\propto}f\right\|_{Lp[a,x]}(x-a)^{\propto}\right\}$$

$$\leq \frac{c(p)}{\Gamma(\propto+1)}\left\{\frac{w_k^{\emptyset}\left(D_{x-}^{\propto}f,\frac{1}{n^B}\right)_{Lp[a,x]}}{n^{\propto}B}+e4\ e^{-n(1-B)}\left\|D_{x-}^{\propto}f\right\|_{Lp[a,x]}(x-a)^{\propto}\right\}$$

where $k=\lfloor nx\rfloor+1,\ldots,\lfloor nb\rfloor$, we consider $\left|x'+\frac{kh\emptyset}{2}+ih\emptyset-x\right|>\frac{1}{n^B}$,

$$y_{2k}=\left|\int_x^{x'+\frac{kh\emptyset}{2}+ih\emptyset}\left(\left(x'+\frac{kh\emptyset}{2}+ih\emptyset\right)-j\right)^{\propto-1}D_{*x}^{\propto}f(j+x)\,dj\right|$$

$$y_{2k}\leq\int_x^{x'+\frac{kh\emptyset}{2}+ih\emptyset}\left(\left(x'+\frac{kh\emptyset}{2}+ih\emptyset\right)-j\right)^{\propto-1}\left|D_{*x}^{\propto}f(j+x)\right|dj,$$

$$y_{2k}\leq\left|D_{*x}^{\propto}f\right|\frac{(b-x)^{\propto}}{\propto}.$$

$$\|y_{2k}\|_{Lp[a,x]}\leq\|D_{*x}^{\propto}f\|_{Lp[a,x].}\frac{(b-x)^{\propto}}{\propto}$$

Also we have in case of $\left|x'+\frac{kh\emptyset}{2}+ih\emptyset-x\right|\leq\frac{1}{n^B},\ k=\lceil na\rceil,\ldots,\lfloor nx\rfloor$.

$$y_{2k}=\left|\int_{x''}^{x'+kh\emptyset/2+ih\emptyset}\left(\left(x+\frac{kh\emptyset}{2}+ih\emptyset-j\right)^{\propto-1}D_{*x}^{\propto}f(j+x)\,dj\right|$$

where $x''<x$, then

$$\|y_{2k}\|_{Lp[x,b]}=\int_{x''}^{x'+kh\emptyset/2+ih\emptyset}\left(\left(x+\frac{kh\emptyset}{2}+ih\emptyset-j\right)^{\propto-1}\|D_{*x}^{\propto}f(x)\|_{Lp[x,b]}$$
$$\leq\frac{w_k^{\emptyset}(D_{*x}^{\propto}f,\frac{1}{n^B})_{Lp[x,b]}}{\propto n^{\propto}B},\qquad\delta\leq\frac{1}{nB}$$

Similar to the previous cases we can find

$$\|U_{2n}(x)\|_{Lp[x,b]}=\left\|\frac{1}{\Gamma(\propto)}\sum_{\lceil na\rceil}^{\lfloor nx\rfloor}\Psi(nx-k)\,y_{2k}\right\|_{Lp[x,b]}$$

$$\|U_{2n}(x)\|_{Lp[x,b]} = \frac{1}{\Gamma(\propto)} \left\| \sum_{\substack{k=\lceil nx \rceil +1 \\ \left|x' + \frac{kh\Phi}{2} + ih\Phi - x\right| \le \frac{1}{nB}}}^{\lfloor nb \rfloor} \Psi(nx-k)\, y_{2k} + \sum_{\substack{k=\lceil nx \rceil +1 \\ \left|x' + \frac{kh\Phi}{2} + ih\Phi - x\right| > \frac{1}{nB}}}^{\lfloor nb \rfloor} \Psi(nx-k)\, y_{2k} \right\|_{Lp[x,b]}$$

Since $\Psi$is increasing on bounded interval, we obtain that

$$\le \frac{c(p)}{\propto \Gamma(\propto)} \{ (\frac{w_k^{\emptyset}\left(D_{*x}^{\propto} f, \frac{1}{n^B}\right)_{Lp[x,b]}}{n^{\propto} B} + e^4 e^{-2n^B} \|D_{*x}^{\propto} f\|_{Lp[x,b]} (b-x)^{\propto}$$

$$\le \frac{c(p)}{\propto \Gamma(\propto)} \left\{ \frac{w_k^{\emptyset}\left(D_{*x}^{\propto} f, \frac{1}{n^B}\right)_{Lp[x,b]}}{n^{\propto} B} + \sum_{\substack{k=-\infty \\ \left|x' + \frac{kh\Phi}{2} + ih\Phi - x\right| > \frac{1}{nB}}}^{\infty} \Psi(nx-k)) \left\|D_{x-}^{\propto} f\right\|_{Lp[x,b]} (b-x)^{\propto} \right\}$$

$$\le \frac{c(p)}{\Gamma(\propto +1)} \left\{ \frac{w_k^{\emptyset}\left(D_{*x}^{\propto} f, \frac{1}{n^B}\right)_{Lp[a,x]}}{n^{\propto} B} + e^4\ e^{-n(1-B)} \|D_{*x}^{\propto} f\|_{Lp[a,x]} (b-x)^{\propto} \right\}$$

$$\|U_n(x)\|_{Lp[a,x]} \le c(p)(\|U_{1n}(x)\|_{Lp[a,x]} + \|U_{2n}(x)\|_{Lp[x,b]})$$

$$\le \frac{2c(p)}{\Gamma(\propto +1)} \left\{ \frac{w_k^{\emptyset}\left(D_{x-}^{\propto} f, \frac{1}{n^B}\right)_{Lp[a,x]} + w_k^{\emptyset}\left(D_{*x}^{\propto} f, \frac{1}{n^B}\right)_{Lp[x,b]}}{n^{\propto} B} + e4\ e^{-n(1-B)} \left(\left\|D_{x-}^{\propto} f\right\|_{Lp[a,x]} (x-a)^{\propto} + \|D_{*x}^{\propto} f\|_{Lp[x,b]} (b-x)^{\propto} \right) \right\}$$

In [4], we have

$$\left|F_n^{*}((.-x)^j)(x)\right| \le \frac{1}{n^{B_j}} + (b-a)^j e4e^{-n^{(1-B)}} \tag{3.4}$$

For $j = 1, \ldots, N$, for all $x \in [a,b]$.

Using (3.3) in (3.4) to get

$$\left\| F_n^*(f,x) - f(x)(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \psi(nx-k)) \right\|_{Lp[a,b]} \leq c(p)\left[\sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right|}{j!}) \left[\frac{1}{n^{Bj}} + (b-a)^j (e4)\, e^{-n^{(1-B)}}\right]\right.$$

$$+ \frac{2c(p)}{\Gamma(\propto +1)}\left\{ \frac{w_k^{\emptyset}\left(D_{x-}^{\propto} f, \frac{1}{n^B}\right)_{Lp[a,x]} + w_k^{\emptyset}\left(D_{*x}^{\propto} f, \frac{1}{n^B}\right)_{Lp[x,b]}}{n^{\propto}B}\right.$$

$$\left.\left. + e4\ e^{-n(1-B)}\left(\left\|D_{x-}^{\propto} f\right\|_{Lp[a,x]}(x-a)^{\propto} + \|D_{*x}^{\propto} f\|_{Lp[x,b]}(b-x)^{\propto}\right)\right\}\right] = \overline{A}_n, \quad \forall x\in[a,b]. \tag{3.5}$$

$$\|F_n(f,x) - f(x)\|_{Lp[a,b]} \leq (4.1)\,\overline{A}_n(x),\ \forall x\in[a,b] \tag{3.6}$$

Now let as estimate

$$\left\|\overline{A}_n\right\|_{Lp[a,b]} \leq c(p)\left[(\sum_{J=1}^{N-1} \frac{\left\|f^{(j)}\right\| Lp_{[a,b]}}{j!}) \left[\frac{1}{n^{Bj}} + (b-a)^j (e4)\, e^{-n^{(1-B)}}\right]\right.$$

$$+ 2c(p)\left\{ \frac{w_k^{\emptyset}\left(D_{x-}^{\propto} f, \frac{1}{n^B}\right)_{Lp[a,x]} + w_k^{\emptyset}\left(D_{*x}^{\propto} f, \frac{1}{n^B}\right)_{Lp[x,b]}}{n^{\propto}B}\right.$$

$$\left.\left. + e4\ e^{-n(1-B)}\left(\left\|D_{x-}^{\propto} f\right\|_{Lp[a,x]}(x-a)^{\propto} + \|D_{*x}^{\propto} f\|_{Lp[x,b]}(b-x)^{\propto}\right)\right\}\right] = \overline{B}n \tag{3.7}$$

Hence it holds

$$\|F_n(f,x) - f(x)\|_{Lp[a,b]} \leq 4.1\ \overline{B}n.$$

This complete the prove of (iii) and $j=0$ implies (ii).

We finally note that

$$F_n(f,x) - \sum_{J=1}^{N-1} \frac{f^{(j)}(x)}{j!} F_n((.-x)^j(x) - f(x)$$

$$= \frac{F_n^*(f,x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)} - \frac{\left(\sum_{J=1}^{N-1}\frac{f^{(j)}}{j!}F_n^*((.-x)^j(x)\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)} - f(x)$$

$$= \frac{F_n^*(f,x) - \left(\sum_{J=1}^{N-1}\frac{f^{(j)}(x)}{j!}F_n^*((.-x)^j(x)\right) - \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)\, f(x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)}$$

$$= \frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)]}[F_n^*(f,x) - \left(\sum_{J=1}^{N-1}\frac{f^{(j)}(x)}{j!}F_n^*((.-x)^j(x)\right) - \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)\, f(x)]$$

Then we have

$$\left\|F_n(f,x) - \sum_{J=1}^{N-1}\frac{f^{(j)}}{j!}F_n((.-x)^j(x) - f(x)\right\|_{Lp[a,b]}$$

$$\leq 5.25\left\|F_n^*(f,x) - \left(\sum_{J=1}^{N-1}\frac{f^{(j)}(x)}{j!}F_n^*((.-x)^j(x)\right) - \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)\, f(x)\right\|_{Lp[a,b]}.$$

Moreover, by(3.5),(3.6),(3.7), for all $x\in[a,b]$ the proof of (i) holds. □

4. Applications

Corollary 4.1. Let $\propto > 0$, $0 < B < 1$, $f \in W_p^1[a,b]$, $\grave{f} \in Lp[a,b], n \in N$ then

$$\|F_n f - f\|_{Lp[a,b]} \leq \frac{8.2}{\Gamma(\propto +1)} \left[\left\{ w_k^{\emptyset} \left(D_{x-}^{\propto} f, \frac{1}{n^B} \right)_{Lp[a,x]} + w_k^{\emptyset} \left(D_{*x}^{\propto} f, \frac{1}{n^B} \right)_{Lp[x,b]} \right\} \right.$$

$$\left. + e^4 \; e^{-2n(1-B)} \left((b-a)^{\propto} (\left\| D_{x-}^{\propto} f \right\|_{Lp[a,x]} + \| D_{*x}^{\propto} f \|_{Lp[x,b]} \right) \right].$$

Proof . Like the proof of theorem 3.1 when N=1 the $\sum_{j=1}^{N-1} . = 0$. In the same way this theorem is proved. □

We can specialized $\propto = 1/2$ in Theorems 3.1 it becomes the following .

Corollary 4.2. Let $\propto > 0$, $0 < B < 1$, $f \in W_p^1[a,b]$, $\grave{f} \in Lp[a,b], n \in N$ then

$$\|F_n f - f\|_{Lp[a,b]} \leq \frac{4.148}{\Gamma(3/2)} \left\{ w_k^{\emptyset} \left(D_{x-}^{\propto} f, \frac{1}{n^B} \right)_{Lp[a,x]} + w_k^{\emptyset} \left(D_{*x}^{\propto} f, \frac{1}{n^B} \right)_{Lp[x,b]} \right\}$$
$$+ e^4 \; e^{-2n(1-B)} \left((b-a)^{\propto} (\left\| D_{x-}^{\propto} f \right\|_{Lp[a,x]} + \| D_{*x}^{\propto} f \|_{Lp[x,b]} \right)$$

We can write as follows when $\Gamma(3/2) = 1/2\sqrt{\pi}$

$$\|F_n f - f\|_{Lp[a,b]} \leq \frac{8.2}{\Gamma(\propto +1)} [\left\{ \frac{1}{n^{\frac{B}{2}}} w_k^{\emptyset} \left(D_{x-}^{\propto} f, \frac{1}{n^B} \right)_{Lp[a,x]} + w_k^{\emptyset} \left(D_{*x}^{\propto} f, \frac{1}{n^B} \right)_{Lp[x,b]} \right\}$$
$$+ e^4 \; e^{-2n(1-B)} \left(\sqrt{(b-a)} \right) (\left\| D_{x-}^{\propto} f \right\|_{Lp[a,x]} + \| D_{*x}^{\propto} f \|_{Lp[x,b]} \Big)].$$

5. Conclusion

We proved some important result for weighted approximation of the partial neural network in terms of k-th order weighted modulus of smoothness of fractional derivative for functions in L_p, $0<p<1$.

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