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The dynamics of a modified Holling-Tanner prey-predator model with wind effect

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Abstract

Wind flow is one of the biosphere components that could change the amount of predation. This paper suggests and analyses a prey-predator model including wind in the predation task. The Holling-Tanner functional response has been considered to illustrate the global dynamics of the proposed model, considering the change in wind intensity. The persistence conditions are provided to reveal a threshold that will allow the coexistence of all species. Numerical simulations are provided to back up the theoretical analysis. The system's coexistence can be achieved in abundance as long as the wind flow increases.

Keywords: dynamic, Holling-Tanner functional, Wind flow

1. Introduction

Environment researchers have been centred on biotic factors, which could affect species' growth and diffusion. That means the majority of the research captures the result of any living component on other creatures to construct the related ecology [13, 6]. Theoretical studies on predator-prey relationships have evolved from Lotka and Volterra's early work [10]. After that, different mechanisms have been proposed in studying mathematical ecology over a lengthy time [7, 15, 12]. In the 1940s, Leslie has presented a model where individuals' survival rates and reproduction depend upon their ages [5]. Robert has developed Leslie's model by adding the network joining the reproducing individuals. Further, he described how many features of graphs develop under his system [9]. The

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Holling-Tanner is a system of two differential equations and is assumed by Kolmogorov type [3] provided by

$$\begin{aligned}\frac{dx}{dt} &= rx(1-\frac{x}{k}) - qxy \\ \frac{dy}{dt} &= Sy(1-\frac{y}{nx})\end{aligned} \tag{1}$$

The parameters in system (1) are explained in detail in the next section. This complex model has been used when the favoured food for the predator is not available in abundance. Therefore, the growth of the predator population is limited. The Holling-Tanner model has been modified in [2, 1] by adding a positive constant c to the predator environment carrying capacity. This modification has been studied in the case the predator switches into alternative prey. Therefore, the term $(1-y/nx)$ in the predator equation, is exchanged to $(1-y/(nx+c))$, which is known as a modified Holling-Tanner model [1].

$$\begin{aligned}\frac{dx}{dt} &= rx(1-\frac{x}{k}) - qxy \\ \frac{dy}{dt} &= Sy(1-\frac{y}{nx+c})\end{aligned} \tag{2}$$

Recently, ecologists have studied the effect of wind on the ecosystem, which may have a diverse impact on the interaction of species. Winds may have different patterns, and speed varies accordingly [11]. In [8], it has been shown that aerial predators better detect the reed warbler nest exposed to the wind blowing. If the prey is less able to be conscious of the nearness of predators, then the rate of predation may increase [14].

This paper considers the effect of wind blowing and harvesting in the prey-predator interaction. The predator attacks the prey according to the Holling-Tanner functional response. The residual of this article is arranged as follows: Section two investigates the equilibrium points for the proposed model. In section three, the behaviour of the possible steady points has been analysed. Finally, in the last section, some numerical analyses have been provided to confirm our analytical results.

2. Assumptions of the model

Suppose a prey-predator model with wind flow contains the following species: prey and a predator, with the mathematics being based on the following assumptions. $n_1(t)$ is the density of the harvested prey, $n_2(t)$ is the density of the predator.

Under the above assumptions, the model can be presented by:

$$\begin{aligned}\frac{dn_1}{dt} &= rn_1(1-\frac{n_1}{k}) - \frac{p(n_1)n_2}{\phi(w)} - eqn_1 = n_1 f_1(n_1,n_2) \\ \frac{dn_2}{dt} &= sn_2(1-\frac{n_2}{\phi(w)+\beta n_1+1}) - \gamma n_2 = n_2 f_2(n_1,n_2)\end{aligned} \tag{3}$$

Here, model (3) has been analysed with the initial conditions $n_1(0) \geq 0$ and $n_2(0) \geq 0$. $p(n_1) = \alpha n_1$ is the Lotka-Volterra type of functional response. $\phi(w) = 1 + w$ is the wind efficiency, where w represents the wind flow satisfying the following assumptions:

1. $\phi(0) = 1$; i.e., the predator's search efficiency continue as before in the absence of wind.

2. $\phi'(w) > 0$; i.e., the predator's search efficiency increases with the rise of wind.

All parameters of the system (3) are expected to be positive and labelled as follows: k is the carrying capacity of the prey with intrinsic growth rate r; βn_1 is the prey's carrying capacity for the predator with intrinsic growth rate s; e, q are the effort, and the catchability rate applied on the prey, i.e., eq represents the harvest amount rate of the prey; l represents an additional food source for the predator which is added to the predator's carrying capacity; α is the attack rate of the prey due to the predator; γ represents the predator's natural death rate.

The functions in system (3) are continuous and differentiable on $R^2_+ = \{(n_1, n_2), n_1 \geq 0, n_2 \geq 0\}$. Therefore, there exists a unique solution for system (3).

The positive invariance of R^2_+ for system (1) is examined first, and then boundedness is shown.

3. Positivity and boundedness of the solution

Lemma 3.1. *System (3) is positively invariant.*

Proof . Let $N = (n_1, n_2)^T \in \mathrm{R}^2$ and, $F(N) = [f_1(N), f_2(N)]^T$ where, $F(N) : R^2_+ \to R^2$ and $f \in C^\infty_+(R^2_+)$. Then the system (3) becomes

$$N' = f(N) \tag{4}$$

with $N(0) = N_0$. It is clear for any $N(0) \in R^2_+$, such that $N_i = 0$, then $[f_i(N)]_{n_i=0} \geq 0 (for i = 1; 2)$. Now, any solution of the Eq. (4) with $N_0 \in R^2_+$, say $N(t) = N(t; N_0)$, is such that $N(t) \in IR^2_+$ for all $t > 0$. Thus, the system is positively invariant [7]. □

Theorem 3.2. *All solutions $n_1(t)$ and $n_2(t)$ of the system (3) with the initial conditions (n_1, n_2) are uniformly bounded.*

Proof . Let $(n_1(t), n_2(t))$ be any system solution of (3) if the initial condition is non-negative. Then for $H(t) = n_1(t) + n_2(t)$, we have $\frac{dH}{dt} = \frac{dn_1}{dt} + \frac{dn_2}{dt}$.

$$\frac{dH}{dt} = rn_1 - \frac{rn_1^2}{k} - \frac{\beta n_1 n_2}{1+w} - eqn_1 + sn_2(1 - \frac{sn_2}{1+w+\beta n_1 + l}) - \gamma n_2$$

Hence, $\frac{dH}{dt} + \mu H \leq 2rn_1 - \frac{rn_1^2}{k} - \frac{\beta n_1 n_2}{1+w} + 2sn_2(1 - \frac{sn_2}{1+w+\beta n_1 + l})$.

Where $\mu = min\{r + eq, s + \gamma\}$, Then $\frac{dH}{dt} + \mu H \leq 2rn_1 + 2sn_2 = \xi$, then

$0 \leq H(n_1(t), n_2(t)) \leq \frac{\xi}{\mu}(1 - e^{-\mu t}) + H(0)e^{-\mu t}$, hence $0 \leq \sup_{t\to\infty} H(t) \leq \frac{\xi}{\mu}$

Therefore, all the solutions of the system (3) that are initiated in R^2_+ are attracted to the region $\vartheta = \{(n_1, n_2) \in R^2_+ : H = n_1 + n_2 \leq \frac{\xi}{\mu}\}$ under the given conditions. Thus, these solutions are uniformly bounded. □

4. Existence of equilibria and their stability

In this section, the existence and the stability analysis of the steady points of system (3) are studied. The computation shows that system (3) has four equilibria, namely

1. The vanishing equilibrium point: $E_0 = (0, 0)$.
2. The prey equilibrium point: $E_1 = (0, n_2^{"})$, where $n_2^{"} = \dfrac{(s-\gamma)(1+w+l)}{s}$, exists when
$$s > \gamma. \tag{5}$$
3. The predator equilibrium point $E_2 = (\widetilde{n_1}, 0)$, where $\widetilde{n_1} = \dfrac{k(r-eq)}{r}$, exists when
$$r > eq. \tag{6}$$
4. The positive equilibrium point $E_3 = (n_1^*, n_2^*)$, where $n_1^* = \dfrac{k[s(r-eq)(1+w) - \alpha(s-\gamma)(1+w+l)]}{rs(1+w)+\alpha\beta(s-\gamma)}$, and $n_2^* = \dfrac{(s-\gamma)(1+w+\beta n_1^*+l)}{s}$, exists when $0 < \alpha(s-\gamma)(1+w+l) < s(r-eq)(1+w)$.

Now, the local behaviour around the above steady points is found. First, the Jacobian matrix of the system (3) at each point is computed, and then, the eigenvalues of the resulting matrix are calculated.

The Jacobian matrix of system (1) at the vanishing fixed point $I_0 = (0, 0)$ can be written as:

$$J(E_0) = \begin{bmatrix} r-eq & 0 \\ 0 & s-\gamma \end{bmatrix}$$

Then, the eigenvalues of $J(I_1)$ are given by $\lambda_{01} = r - eq$ and $\lambda_{02} = s - eq$. That means E_0 is a locally asymptotically stable point if and only if

$$r < eq \text{ and } s < \gamma \tag{7}$$

The Jacobian matrix of the system at $E_1 = (0, n_2^{"})$ can be written as:

$$J(E_1) = \begin{bmatrix} r - eq - \dfrac{an_2^{"}}{1+w} & 0 \\ \dfrac{\beta(s-\gamma)^2}{s(1+w+l)} & -(s-\gamma) \end{bmatrix}$$

Then, the eigenvalues of $J(E_1)$ are given by $\lambda_{11} = r - eq - \dfrac{\alpha n_2^{"}}{1+w}$ and $\lambda_{12} = -(s-\gamma) < 0$. That means E_1 is a locally asymptotically stable point if and only if

$$r - eq < \frac{an_2^{"}}{1+w} \tag{8}$$

The Jacobian matrix of the system at $E_2 = (\widetilde{n_1}, 0)$ can be written as:

$$J(E_2) = \begin{bmatrix} -(r-eq) & \dfrac{-\alpha\widetilde{n_1}}{1+w} \\ 0 & s-\gamma \end{bmatrix}$$

Then, the eigenvalues of $J(E_2)$ are given by $\lambda_{21} = -(r - eq) < 0$ and $\lambda_{22} = s - \gamma$. That means E_2 is a locally asymptotically stable point if and only if

$$s < \gamma \tag{9}$$

The Jacobian matrix of the system at $E_3 = (n_1^*, n_2^*)$, can be written as:

$$J(E_3) = \begin{bmatrix} \dfrac{-rn_1^*}{k} & \dfrac{-\alpha n_1^*}{1+w} \\ \dfrac{\beta(s-\gamma)^2}{s} & 0 \end{bmatrix} \tag{10}$$

Straightforward computations show that the eigenvalues of the Jacobian matrix $J(E_3)$ satisfy the following relations:

$$\lambda_{31} + \lambda_{32} = \frac{-rn_1^*}{k} < 0,$$
$$\lambda_{31}.\lambda_{32} = \frac{\alpha\beta n_1^*(s-\gamma)^2}{s(1+w)} > 0$$

Hence E_3 is locally asymptotically stable in the R_+^2.

It should also be noted that the formulae of λ_{31} and λ_{32} are given by the following equations: $\lambda_{31,32} = -(\frac{rn_1^*}{2k}) \pm \frac{1}{2}\sqrt{(\frac{rn_1^*}{k})^2 - \frac{4\alpha\beta n_1^*(s-\gamma)^2}{s(1+w)}}$. It can be seen that real or complex eigenvalues are possible. In particular, as the carrying capacity of the prey $k \to 0$, the eigenvalues are real, and for k large, the eigenvalues are complex. In the following, the global stability of E_3 is investigated.

Theorem 4.1. *E_3 is globally asymptotically stable in R_+^2, whenever it exists.*

Proof . For any initial value (n_1, n_2) in R_+^2, let $H(n_1, n_2) = \frac{1}{n_1 n_2}$, $h_1(n_1, n_2) = rn_1(1 - \frac{n_1}{k}) - \frac{\alpha n_1 n_2}{1+w} - eqn_1$ and $h_2(n_1, n_2) = sn_2(1 - \frac{n_2}{1+w+\beta n_1+l}) - \gamma n_2$.

Clearly, $H(n_1, n_2) > 0$ for all $(n_1, n_2) \in R_+^2$ and it is a C^1 function in R_+^2.

Now, since $Hh_1(n_1, n_2) = \frac{r}{n_2} - \frac{rn_1}{kn_2} - \frac{\alpha}{1+w} - \frac{eq}{n_2}$; $Hh_2(n_1, n_2) = \frac{s}{n_1} - \frac{sn_2}{n_1(1+w+\beta n_1+l)} - \frac{\gamma}{n_1}$

Hence, $\Delta(n_1, n_2) = \frac{\partial Hh_1}{\partial n_1} + \frac{\partial Hh_2}{\partial n_2} = -\frac{r}{kn_2} - \frac{s}{n_1(1+w+\beta n_1+l)} < 0$

Note that $\Delta(n_1, n_2)$ does not change of sign and is not identically zero in the R_+^2. Then according to Bendixson-Dulic criteria, there is no periodic solution. Since all the solutions of model (3) are bounded and E_3 is a unique positive equilibrium point, hence by using the Poincare-Bendixson theorem, E_3 is globally asymptotically stable. □

Remark

According to the Jacobian matrix $J(E_3)$ given by (10), all the eigenvalues of $J(E_3)$ have negative real parts at the equilibrium point E_3. Therefore, E_3 is a hyperbolic equilibrium point, and thus, the system (3) has no bifurcation at E_3.

5. Persistence analysis

The persistence of model (3) indicates the survival of all system species over the long term. Mathematically means the strictly positive trajectories of system (3) that initiate in R_+^2 have no omega-limit sets on the boundary planes.

Theorem 5.1. *Assume that the boundary equilibrium points conditions hold, then system (3) is uniformly persistent.*

Proof . Consider the function $\omega(n_1, n_2) = n_1^{\alpha} n_2^{b}$, where a and b are positive constants. Obviously $\omega(n_1, n_2) > 0$ for all $(n_1, n_2) \in R_+^2$ and $\omega(n_1, n_2) \to 0$ when $n_1 \to 0$ or $n_2 \to 0$. Consequently, $\varphi(n_1, n_2) = \frac{\dot{\omega}}{\omega} = b[s - \frac{sn_2}{1 + w + \beta n_1 + l} - \gamma] + a[r - \frac{rn_1}{k} - \frac{\alpha n_2}{1 + w} - eq]$.

Now, the only possible omega limit sets of the system (3) on the boundary of the $n_1 n_2 -$ plane are the equilibrium points E_0, E_1 and E_2. Thus according to the Gard method [4], the proof follows, and the system is uniformly persistent, provided that $\varphi(n_1, n_2) > 0$ at the boundary fixed points. Now, since

$$\begin{aligned} \varphi(E_0) &= b(s - \gamma) + a(r - eq); \\ \varphi(E_1) &= a(r - eq - \frac{\alpha n_2^{"}}{1 + w}); \\ \varphi(E_2) &= b(s - \gamma), \end{aligned}$$

It follows that, $\varphi(E_0) > 0, \varphi(E_1) > 0$ and $\varphi(E_2) > 0$ under the existing conditions of the boundary equilibrium points for all values of a and b. Then, system (3) is uniformly persistent. □

6. Numerical analysis

This section explores model (3) dynamics in a windy atmosphere by providing some numerical simulations using MATLAB 2015a. For this determination, the following set of parameters is considered for model (3) throughout the article:

$$r = 5, k = 20, \alpha = 0.8, e = 0.03, q = 0.02, s = 4, \beta = 0.3, l = 0.2, \gamma = 0.001. \tag{11}$$

Now, the effect of the wind flow is discussed on the behaviour of model (3). The system without wind flow, i.e., w=0 shows stable behaviour for the positive steady point $E_3 = (8.24, 3.67)$ with different starting points, as exhibited in Fig. 1. This behaviour confirms the result that has been proved in Theorem 4.1, which states that E_3 has global stability whenever it exists. Moreover, system (3) illustrates unstable behaviour at E_0, E_1 and E_2. This shows that at least one of the conditions in Eqs. 7, 8 or 9, are violated.

To check the dynamics of system (3) in the presence of wind flow strength, we have chosen w=5. The results show that for different values of initial points, the trajectories of system (3) approach asymptotically to the positive equilibrium point $E_3 = (14.02, 11.2)$. From Fig.1 and Fig. 2, it is observed that the density of prey and predator populations rise to a certain level in the presence of wind flow, while they decrease in the absence of wind. Ecologically, in the presence of blowing wind, the prey is less able to observe the proximity of predators. Therefore, the predator becomes more effective to detect the prey.

7. Conclusion

In the proposed model, it has been observed that the system has four different equilibrium points. The values of the first and third equilibria E_0 and E_2 are independent of the wind flow w, but E_1 and E_3 are dependent on w. The system's stability at E_0, E_1 and E_2 has been determined based on specific conditions, while the equilibrium E_3 is always stable if the conditions for its existence are satisfied. E_3 can only exist if E_1 and E_2 both exist. Assuming that $0 < \alpha(s - \gamma) < s(r - eq)$, then E_3 exists if the wind speed is large enough, and E_3 exists for lower wind speeds if the second inequality is satisfied by a large amount.

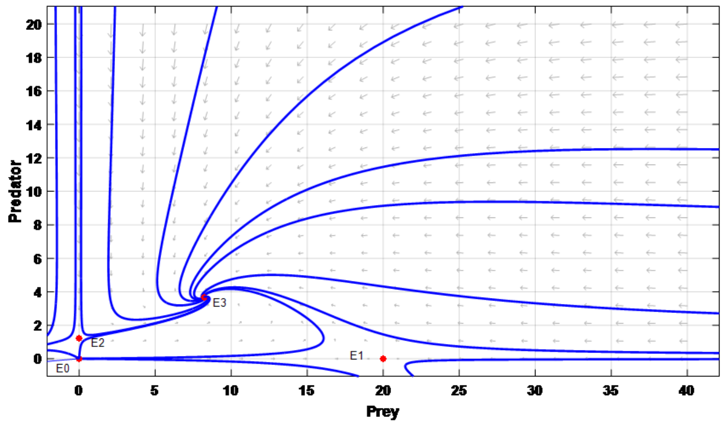

Figure 1: Phase plane analysis with the data given by Eq. (11) with $w = 0$.

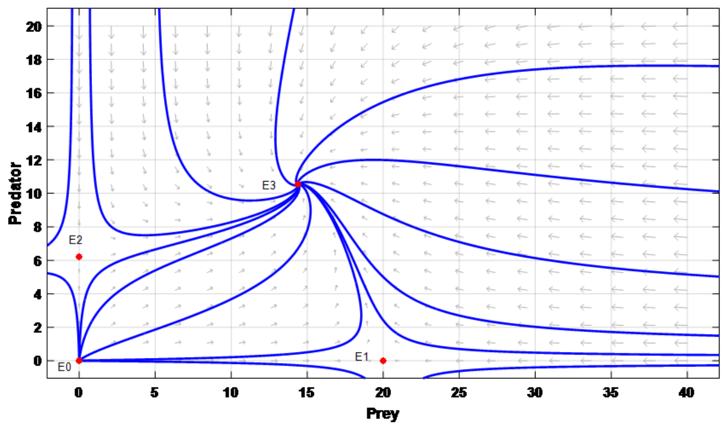

Figure 2: Phase plane analysis with the data given by Eq. (11) with $w = 5$.

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