Int. J. Nonlinear Anal. Appl. 13 (2022) 1, 3333-3345
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.21475.2262

Hermite-Hadamard type inequality for ψ-Riemann-Liouville fractional integrals via preinvex functions

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(Communicated by Madjid Eshaghi Gordji)

Abstract

The main aim of the present paper is to establish a new form of Hermite-Hadamard inequalities using left and right-sided ψ-Riemann-Liouville fractional integrals for preinvex functions and present two basic results of ψ-Riemann-Liouville fractional integral identities including the first-order derivative of a preinvex function. We derive some fractional Hermite-Hadamard inequalities with the help of these results. Further, we pointed out some applications for special means.

Keywords: Invex sets, preinvex functions, Hermite-Hadamard inequalities, ψ-Riemann-Liouville fractional integrals, Hölder's inequality.
2010 MSC: 26A51, 26A48, 26D99.

1. Introduction

Hanson [4] introduced one of the most important generalizations of convex functions is the class of invex functions. Ben Israel and Mond [2] gave a simple characterization of invexity for both constrained and unconstrained problems, and showed that invexity can be substituted for convexity in the saddle point problem and in the Slater constraint qualification. It is well known that ξ is invex if and only if every stationary point is a global minimum [3]. Pini [14] established the relationship between invexity and generalized convexity and showed that $\xi(w) = w^3$ is quasi-convex

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Received: September 2020 *Accepted:* September 2021

but not invex, since $w = 0$ is a stationary point but not a minimum point and also given an example $\xi(w, z) = -w^2 + wz - e^z$ which is invex but not quasi-convex because it fails to satisfy the second-order necessary and sufficient condition for quasi-convexity.

Noor proved some Hermite-Hadamard (H-H) type inequalities for preinvex [13], log-preinvex [12] and the product of two preinvex functions [11]. Further, Işcan [5] obtained some H-H type inequalities using fractional integrals for preinvex functions. For more generalizations of the H-H inequality, see [1, 8, 11, 12, 13, 15].

Recently, Wang *et al.* [15] investigated fractional integral identities for a differentiable mapping involving Riemann-Liouville (R-L) fractional integrals and Hadamard fractional integrals and given some inequalities via standard convex, r-convex, s-convex, m-convex, (s,m)-convex, (β, m)-convex functions, etc. Further, the H-H type inequality for fractional integrals obtained by Liu *et al.* [7].

The organization of the paper is as follows: In Section 2, we recall some basic results which are necessary for our main results. In Section 3, we prove a new form of H-H inequalities using left and right-sided ψ-R-L fractional integrals for preinvexity. We present two essential results of ψ-R-L fractional integral identities using the first-order derivative of a preinvex function. These results will be used to obtain some fractional H-H inequalities and give some examples which satisfy the theorems. Further, we discuss some applications for special means with the help of these results.

2. Preliminaries

In this section, we recall some basic definitions and results required for this manuscript.

Hermite established the most popular inequalities known as H-H integral inequality.

$$\xi\left(\frac{\varrho_1 + \varrho_2}{2}\right) \leq \frac{1}{\varrho_2 - \varrho_1} \int_{\varrho_1}^{\varrho_2} \xi(w) dw \leq \frac{\xi(\varrho_1) + \xi(\varrho_2)}{2},$$

where $\xi : [\varrho_1, \varrho_2] \subset \mathbb{R} \to \mathbb{R}$ is a convex function [9].

Definition 2.1. *[16] The set $A \subseteq \mathbb{R}^n$ is said to be invex with respect to vector function $\hbar : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, if*

$$z + \delta\hbar(w, z) \in A, \quad \forall w, z \in A, \quad \delta \in [0, 1].$$

It is well known that every convex set is preinvex with respect to $\hbar(w, z) = w - z$, but not conversely.

Definition 2.2. *[16] The function ξ on the invex set A is said to be preinvex with respect to $\hbar$, if*

$$\xi(z + \delta\hbar(w, z)) \leq (1 - \delta)\xi(z) + \delta\xi(w), \quad \forall w, z \in A, \quad \delta \in [0, 1].$$

It is well known that every convex function is a preinvex with respect to $\hbar(w, z) = w - z$, but not conversely.

CONDITION C [10] Let $A \subseteq \mathbb{R}^n$ be an open invex subset with respect to $\hbar : A \times A \to \mathbb{R}$. The function $\hbar$ satisfies the condition C if for any $w, z \in A$ and any $\delta \in [0, 1]$,

$$\hbar(z, z + \delta\hbar(w, z)) = -\delta\hbar(w, z),$$

$$\hbar(w, z + \delta\hbar(w, z)) = (1 - \delta)\hbar(w, z).$$

Note that $\forall w, z \in A$ and $\delta \in [0, 1]$, then from Condition C, we have

$$\hbar(z + \delta_2\hbar(w, z), z + \delta_1\hbar(w, z)) = (\delta_2 - \delta_1)\hbar(w, z).$$

Theorem 2.3. *[12] Let $\xi : A = [\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1)] \to (0, \infty)$ be a preinvex function on the interval of real numbers $int(A)$ and $\varrho_1, \varrho_2 \in int(A)$ with $\varrho_1 < \varrho_1 + \hbar(\varrho_2, \varrho_1)$. Then,*

$$\xi\left(\frac{2\varrho_1 + \hbar(\varrho_2, \varrho_1)}{2}\right) \leq \frac{1}{\hbar(\varrho_2, \varrho_1)} \int_{\varrho_1}^{\varrho_1+\hbar(\varrho_2,\varrho_1)} \xi(w)dw \leq \frac{\xi(\varrho_1) + \xi(\varrho_2)}{2}. \tag{2.1}$$

Definition 2.4. *[6] Let (ϱ_1, ϱ_2) $(-\infty \leq \varrho_1 < \varrho_2 \leq \infty)$ be an interval of the real line $\mathbb{R}$ and $\alpha > 0$. Also let $\psi(w)$ be an increasing and positive monotone function on $(\varrho_1, \varrho_2]$, having a continuous derivative $\psi'(w)$ on (ϱ_1, ϱ_2). The left and right-sided ψ-R-L fractional integrals of a function ξ with respect to an other function ψ on $[\varrho_1, \varrho_2]$ are defined by*

$$I_{\varrho_1^+}^{\alpha:\psi}\xi(w) = \frac{1}{\Gamma(\alpha)} \int_{\varrho_1}^{w} \psi'(\mu)(\psi(w) - \psi(\mu))^{\alpha-1}\xi(\mu)d\mu,$$

$$I_{\varrho_2^-}^{\alpha:\psi}\xi(w) = \frac{1}{\Gamma(\alpha)} \int_{w}^{\varrho_2} \psi'(\mu)(\psi(\mu) - \psi(w))^{\alpha-1}\xi(\mu)d\mu$$

respectively.

3. Main results

In this section, first, we prove H-H inequalities for ψ-R-L fractional integrals via preinvexity.

Theorem 3.1. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\hbar : A \times A \to \mathbb{R}$ and $\varrho_1, \varrho_2 \in A$ with $\varrho_1 < \varrho_1 + \hbar(\varrho_2, \varrho_1)$. If $\xi : [\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1)] \to (0, \infty)$ is a preinvex function, $\xi \in L[\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1)]$ and $\hbar$ satisfies Condition C. Also suppose $\psi(w)$ is an increasing and positive monotone function on $(\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1))$, having a continuous derivative $\psi'(w)$ on $(\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1))$ and $\alpha \in (0, 1)$. Then,*

$$\begin{aligned}\xi\left(\varrho_1 + \frac{1}{2}\hbar(\varrho_2, \varrho_1)\right) &\leq \frac{\Gamma(\alpha+1)}{2\hbar^\alpha(\varrho_2, \varrho_1)} \Big[I_{\psi^{-1}(\varrho_1)^+}^{\alpha:\psi}(\xi \circ \psi)\psi^{-1}(\varrho_1 + \hbar(\varrho_2, \varrho_1)) \\ &\quad + I_{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))^-}^{\alpha:\psi}(\xi \circ \psi)\psi^{-1}(\varrho_1)\Big] \\ &\leq \frac{\xi(\varrho_1) + \xi(\varrho_1 + \hbar(\varrho_2, \varrho_1))}{2} \leq \frac{\xi(\varrho_1) + \xi(\varrho_2)}{2}.\end{aligned}$$

Proof . By definition of invex set $w, z \in A$, then $w + \delta\hbar(z, w) \in A, \quad \forall \delta \in [0, 1]$.
From Theorem 2.3, we get

$$\xi\left(w + \frac{1}{2}\hbar(z, w)\right) \leq \frac{\xi(w) + \xi(z)}{2}. \tag{3.1}$$

Using $w = \varrho_1 + (1 - \delta)\hbar(\varrho_2, \varrho_1)$ and $z = \varrho_1 + \delta\hbar(\varrho_2, \varrho_1)$ in (3.1), we have

$$\begin{aligned}\xi(\varrho_1 + (1 - \delta)\hbar(\varrho_2, \varrho_1) &+ \frac{1}{2}\hbar(\varrho_1 + \delta\hbar(\varrho_2, \varrho_1), \varrho_1 + (1 - \delta)\hbar(\varrho_2, \varrho_1))) \\ &\leq \frac{\xi(\varrho_1 + (1 - \delta)\hbar(\varrho_2, \varrho_1)) + \xi(\varrho_1 + \delta\hbar(\varrho_2, \varrho_1))}{2}.\end{aligned} \tag{3.2}$$

Applying Condition C in (3.2), we have

$$\xi\left(\varrho_1 + \frac{1}{2}\hbar(\varrho_2, \varrho_1)\right) \leq \frac{\xi(\varrho_1 + (1 - \delta)\hbar(\varrho_2, \varrho_1)) + \xi(\varrho_1 + \delta\hbar(\varrho_2, \varrho_1))}{2}. \tag{3.3}$$

Multiplying (3.3) by $\delta^{\alpha-1}$ on both sides and integrating the resultant with respect to δ over $[0,1]$, we have

$$\frac{2}{\alpha}\xi\left(\varrho_1+\frac{1}{2}\hbar(\varrho_2,\varrho_1)\right)\leq\int_0^1\delta^{\alpha-1}\xi(\varrho_1+(1-\delta)\hbar(\varrho_2,\varrho_1))d\delta+\int_0^1\delta^{\alpha-1}\xi(\varrho_1+\delta\hbar(\varrho_2,\varrho_1))d\delta. \tag{3.4}$$

Next,

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2\hbar^\alpha(\varrho_2,\varrho_1)}\left[I^{\alpha:\psi}_{\psi^{-1}(\varrho_1)^+}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))+I^{\alpha:\psi}_{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))^-}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1)\right]\\
&=\frac{\alpha}{2\hbar^\alpha(\varrho_2,\varrho_1)}\left[\int_{\psi^{-1}(\varrho_1)}^{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))}(\varrho_1+\hbar(\varrho_2,\varrho_1)-\psi(\mu))^{\alpha-1}(\xi\,\mathrm{o}\,\psi)(\mu)\psi'(\mu)d\mu\right.\\
&\quad\left.+\int_{\psi^{-1}(\varrho_1)}^{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))}(\psi(\mu)-\varrho_1)^{\alpha-1}(\xi\,\mathrm{o}\,\psi)(\mu)\psi'(\mu)d\mu\right]\\
&=\frac{\alpha}{2}\left[\int_0^1\delta^{\alpha-1}\xi(\varrho_1+(1-\delta)\hbar(\varrho_2,\varrho_1))d\delta+\int_0^1\delta^{\alpha-1}\xi(\varrho_1+\delta\hbar(\varrho_2,\varrho_1))d\delta\right].
\end{aligned}\tag{3.5}$$

From (3.4) and (3.5), we have

$$\begin{aligned}
\xi\left(\varrho_1+\frac{1}{2}\hbar(\varrho_2,\varrho_1)\right)\leq\frac{\Gamma(\alpha+1)}{2\hbar^\alpha(\varrho_2,\varrho_1)}&\left[I^{\alpha:\psi}_{\psi^{-1}(\varrho_1)^+}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))\right.\\
&\left.+I^{\alpha:\psi}_{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))^-}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1)\right].
\end{aligned}$$

Now, we prove the second pair inequality of the theorem.

$$\begin{aligned}
\xi(\varrho_1+\delta\hbar(\varrho_2,\varrho_1))&=\xi(\varrho_1+\hbar(\varrho_2,\varrho_1)+(1-\delta)\hbar(\varrho_1,\varrho_1+\hbar(\varrho_2,\varrho_1)))\\
&\leq\delta\xi(\varrho_1+\hbar(\varrho_2,\varrho_1))+(1-\delta)\xi(\varrho_1).
\end{aligned}\tag{3.6}$$

Similarly,

$$\begin{aligned}
\xi(\varrho_1+(1-\delta)\hbar(\varrho_2,\varrho_1))&=\xi(\varrho_1+\hbar(\varrho_2,\varrho_1)+\delta\hbar(\varrho_1,\varrho_1+\hbar(\varrho_2,\varrho_1)))\\
&\leq(1-\delta)\xi(\varrho_1+\hbar(\varrho_2,\varrho_1))+\delta\xi(\varrho_1).
\end{aligned}\tag{3.7}$$

From (3.6) and (3.7), we have

$$\xi(\varrho_1+\delta\hbar(\varrho_2,\varrho_1))+\xi(\varrho_1+(1-\delta)\hbar(\varrho_2,\varrho_1))\leq\xi(\varrho_1)+\xi(\varrho_1+\hbar(\varrho_2,\varrho_1)). \tag{3.8}$$

Multiplying both sides by $\delta^{\alpha-1}$ in (3.8), then integrating with respect to δ over 0 to 1, we have

$$\int_0^1\delta^{\alpha-1}\xi(\varrho_1+\delta\hbar(\varrho_2,\varrho_1))d\delta+\int_0^1\delta^{\alpha-1}\xi(\varrho_1+(1-\delta)\hbar(\varrho_2,\varrho_1))d\delta\leq\frac{\xi(\varrho_1)+\xi(\varrho_1+\hbar(\varrho_2,\varrho_1))}{\alpha}. \tag{3.9}$$

From (3.5) and (3.9), we have

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2\hbar^\alpha(\varrho_2,\varrho_1)}\left[I^{\alpha:\psi}_{\psi^{-1}(\varrho_1)^+}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))+I^{\alpha:\psi}_{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))^-}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1)\right]\\
&\leq\frac{\xi(\varrho_1)+\xi(\varrho_1+\hbar(\varrho_2,\varrho_1))}{2}\leq\frac{\xi(\varrho_1)+\xi(\varrho_2)}{2}.
\end{aligned}$$

This completes the proof. □

Now, we present results of ψ-R-L fractional integral identities including the first-order derivative of a preinvex function.

Lemma 3.2. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\hbar : A \times A \to \mathbb{R}$ and $\varrho_1, \varrho_2 \in A$ with $\varrho_1 < \varrho_1 + \hbar(\varrho_2, \varrho_1)$. Suppose that $\xi : A \to \mathbb{R}$ is a differentiable function. If ξ' is preinvex function on A and $\xi' \in L[\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1)]$, $\psi(w)$ is an increasing and positive monotone function on $(\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1))$, having a continuous derivative $\psi'(w)$ on $(\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1))$ and $\alpha \in (0, 1)$. Then,*

$$\begin{aligned}
&\frac{\xi(\varrho_1) + \xi(\varrho_1 + \hbar(\varrho_2, \varrho_1))}{2} - \frac{\Gamma(\alpha+1)}{2\hbar^\alpha(\varrho_2, \varrho_1)} \Big[I^{\alpha:\psi}_{\psi^{-1}(\varrho_1)^+} (\xi \circ \psi)\psi^{-1}(\varrho_1 + \hbar(\varrho_2, \varrho_1)) \\
&\quad + I^{\alpha:\psi}_{\psi^{-1}(\varrho_1 + \hbar(\varrho_2, \varrho_1))^-} (\xi \circ \psi)\psi^{-1}(\varrho_1) \Big] \\
&= \frac{1}{2\hbar^\alpha(\varrho_2, \varrho_1)} \int_{\psi^{-1}(\varrho_1)}^{\psi^{-1}(\varrho_1 + \hbar(\varrho_2, \varrho_1))} [(\psi(\mu) - \varrho_1)^\alpha - (\varrho_1 + \hbar(\varrho_2, \varrho_1) - \psi(\mu))^\alpha](\xi' \circ \psi)(\mu)\psi'(\mu)d\mu \\
&= \frac{\hbar(\varrho_2, \varrho_1)}{2} \int_0^1 ((1-\delta)^\alpha - \delta^\alpha)\xi'(\varrho_1 + (1-\delta)\hbar(\varrho_2, \varrho_1))d\delta.
\end{aligned}$$

Proof . Let $M_1 = \frac{\Gamma(\alpha+1)}{2\hbar^\alpha(\varrho_2,\varrho_1)} I^{\alpha:\psi}_{\psi^{-1}(\varrho_1)^+} (\xi \circ \psi)\psi^{-1}(\varrho_1 + \hbar(\varrho_2, \varrho_1))$
and
$M_2 = \frac{\Gamma(\alpha+1)}{2\hbar^\alpha(\varrho_2,\varrho_1)} I^{\alpha:\psi}_{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))^-} (\xi \circ \psi)\psi^{-1}(\varrho_1)$.

$$\begin{aligned}
M_1 &= \frac{\alpha}{2\hbar^\alpha(\varrho_2, \varrho_1)} \int_{\psi^{-1}(\varrho_1)}^{\psi^{-1}(\varrho_1 + \hbar(\varrho_2, \varrho_1))} (\varrho_1 + \hbar(\varrho_2, \varrho_1) - \psi(\mu))^{\alpha-1}(\xi \circ \psi)(\mu)\psi'(\mu)d\mu \\
&= -\frac{1}{2\hbar^\alpha(\varrho_2, \varrho_1)} \int_{\psi^{-1}(\varrho_1)}^{\psi^{-1}(\varrho_1 + \hbar(\varrho_2, \varrho_1))} (\xi \circ \psi)(\mu)d(\varrho_1 + \hbar(\varrho_2, \varrho_1) - \psi(\mu))^\alpha \\
&= -\frac{1}{2\hbar^\alpha(\varrho_2, \varrho_1)} \left[-\xi(\varrho_1)\hbar^\alpha(\varrho_2, \varrho_1) - \int_{\psi^{-1}(\varrho_1)}^{\psi^{-1}(\varrho_1 + \hbar(\varrho_2, \varrho_1))} (\varrho_1 + \hbar(\varrho_2, \varrho_1) - \psi(\mu))^\alpha(\xi' \circ \psi)(\mu)\psi'(\mu)d\mu \right] \\
&= \frac{\xi(\varrho_1)}{2} + \frac{1}{2\hbar^\alpha(\varrho_2, \varrho_1)} \int_{\psi^{-1}(\varrho_1)}^{\psi^{-1}(\varrho_1 + \hbar(\varrho_2, \varrho_1))} (\varrho_1 + \hbar(\varrho_2, \varrho_1) - \psi(\mu))^\alpha(\xi' \circ \psi)(\mu)\psi'(\mu)d\mu.
\end{aligned}$$

$$\begin{aligned}
M_2 &= \frac{\alpha}{2\hbar^\alpha(\varrho_2, \varrho_1)} \int_{\psi^{-1}(\varrho_1)}^{\psi^{-1}(\varrho_1 + \hbar(\varrho_2, \varrho_1))} (\psi(\mu) - \varrho_1)^{\alpha-1}(\xi \circ \psi)(\mu)\psi'(\mu)d\mu \\
&= \frac{1}{2\hbar^\alpha(\varrho_2, \varrho_1)} \int_{\psi^{-1}(\varrho_1)}^{\psi^{-1}(\varrho_1 + \hbar(\varrho_2, \varrho_1))} (\xi \circ \psi)(\mu)d(\psi(\mu) - \varrho_1)^\alpha \\
&= \frac{1}{2\hbar^\alpha(\varrho_2, \varrho_1)} \left[\xi(\varrho_1 + \hbar(\varrho_2, \varrho_1))\hbar^\alpha(\varrho_2, \varrho_1) - \int_{\psi^{-1}(\varrho_1)}^{\psi^{-1}(\varrho_1 + \hbar(\varrho_2, \varrho_1))} (\psi(\mu) - \varrho_1)^\alpha(\xi' \circ \psi)(\mu)\psi'(\mu)d\mu \right] \\
&= \frac{\xi(\varrho_1 + \hbar(\varrho_2, \varrho_1))}{2} - \frac{1}{2\hbar^\alpha(\varrho_2, \varrho_1)} \int_{\psi^{-1}(\varrho_1)}^{\psi^{-1}(\varrho_1 + \hbar(\varrho_2, \varrho_1))} (\psi(\mu) - \varrho_1)^\alpha(\xi' \circ \psi)(\mu)\psi'(\mu)d\mu.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{\xi(\varrho_1) + \xi(\varrho_1 + \hbar(\varrho_2, \varrho_1))}{2} - M_1 - M_2 \\
&= \frac{1}{2\hbar^\alpha(\varrho_2, \varrho_1)} \int_{\psi^{-1}(\varrho_1)}^{\psi^{-1}(\varrho_1 + \hbar(\varrho_2, \varrho_1))} [(\psi(\mu) - \varrho_1)^\alpha - (\varrho_1 + \hbar(\varrho_2, \varrho_1) - \psi(\mu))^\alpha](\xi' \circ \psi)(\mu)\psi'(\mu)d\mu. \quad (3.10)
\end{aligned}$$

Next, we prove the second pair equality of the lemma.

$$
\begin{aligned}
\text{Let}\quad M_3 &= \frac{\hbar(\varrho_2,\varrho_1)}{2}\int_0^1 ((1-\delta)^\alpha-\delta^\alpha)\xi'(\varrho_1+(1-\delta)\hbar(\varrho_2,\varrho_1))d\delta \\
&= \frac{\hbar(\varrho_2,\varrho_1)}{2}\left[\int_0^1 (1-\delta)^\alpha\xi'(\varrho_1+(1-\delta)\hbar(\varrho_2,\varrho_1))d\delta-\int_0^1 \delta^\alpha\xi'(\varrho_1+(1-\delta)\hbar(\varrho_2,\varrho_1))d\delta\right] \\
&= \frac{\xi(\varrho_1)+\xi(\varrho_1+\hbar(\varrho_2,\varrho_1))}{2}-\frac{\alpha}{2}\left[\int_0^1 \delta^{\alpha-1}\xi(\varrho_1+(1-\delta)\hbar(\varrho_2,\varrho_1))d\delta\right. \\
&\quad \left.+\int_0^1 \delta^{\alpha-1}\xi(\varrho_1+\delta\hbar(\varrho_2,\varrho_1))d\delta\right] \\
&= \frac{\xi(\varrho_1)+\xi(\varrho_1+\hbar(\varrho_2,\varrho_1))}{2} \\
&\quad -\frac{\alpha}{2\hbar^\alpha(\varrho_2,\varrho_1)}\left[\int_{\psi^{-1}(\varrho_1)}^{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))}(\varrho_1+\hbar(\varrho_2,\varrho_1)-\psi(\mu))^{\alpha-1}(\xi\,\mathrm{o}\,\psi)(\mu)\psi'(\mu)d\mu\right. \\
&\quad \left.+\int_{\psi^{-1}(\varrho_1)}^{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))}(\psi(\mu)-\varrho_1)^{\alpha-1}(\xi\,\mathrm{o}\,\psi)(\mu)\psi'(\mu)d\mu\right] \\
&= \frac{\xi(\varrho_1)+\xi(\varrho_1+\hbar(\varrho_2,\varrho_1))}{2}-\frac{\Gamma(\alpha+1)}{2\hbar^\alpha(\varrho_2,\varrho_1)}\left[I^{\alpha:\psi}_{\psi^{-1}(\varrho_1)^+}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))\right. \\
&\quad \left.+I^{\alpha:\psi}_{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))^-}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1)\right]. \qquad (3.11)
\end{aligned}
$$

This completes the proof. □

Remark 3.3. *When $\hbar(\varrho_2,\varrho_1)=\varrho_2-\varrho_1$, then above lemma reduces to Lemma 3.1 of [7].*

Lemma 3.4. *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\hbar: A\times A\to\mathbb{R}$ and $\varrho_1,\varrho_2\in A$ with $\varrho_1<\varrho_1+\hbar(\varrho_2,\varrho_1)$. Suppose that $\xi: A\to\mathbb{R}$ is a differentiable function. If ξ' is preinvex function on A and $\xi'\in L[\varrho_1,\varrho_1+\hbar(\varrho_2,\varrho_1)]$, $\psi(w)$ is an increasing and positive monotone function on $(\varrho_1,\varrho_1+\hbar(\varrho_2,\varrho_1))$, having a continuous derivative $\psi'(w)$ on $(\varrho_1,\varrho_1+\hbar(\varrho_2,\varrho_1))$ and $\alpha\in(0,1)$. Then,*

$$
\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2\hbar^\alpha(\varrho_2,\varrho_1)}\left[I^{\alpha:\psi}_{\psi^{-1}(\varrho_1)^+}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))+I^{\alpha:\psi}_{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))^-}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1)\right] \\
&\quad -\xi\left(\varrho_1+\frac{1}{2}\hbar(\varrho_2,\varrho_1)\right) \\
&= \int_{\psi^{-1}(\varrho_1)}^{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))} k(\xi'\,\mathrm{o}\psi)(\mu)\psi'(\mu)d\mu \\
&\quad +\frac{1}{2\hbar^\alpha(\varrho_2,\varrho_1)}\int_{\psi^{-1}(\varrho_1)}^{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))}[(\varrho_1+\hbar(\varrho_2,\varrho_1)-\psi(\mu))^\alpha-(\psi(\mu)-\varrho_1)^\alpha](\xi'\,\mathrm{o}\,\psi)(\mu)\psi'(\mu)d\mu,
\end{aligned}
$$

where

$$
k=\begin{cases}\frac{1}{2}, & \psi^{-1}\left(\varrho_1+\frac{1}{2}\hbar(\varrho_2,\varrho_1)\right)\le\mu\le\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1)),\\ -\frac{1}{2}, & \psi^{-1}(\varrho_1)<\mu<\psi^{-1}\left(\varrho_1+\frac{1}{2}\hbar(\varrho_2,\varrho_1)\right).\end{cases}
$$

Proof . Let

$$N_1 = -\frac{1}{2}\int_{\psi^{-1}(\varrho_1)}^{\psi^{-1}\left(\varrho_1+\frac{1}{2}\hbar(\varrho_2,\varrho_1)\right)} (\xi' \circ \psi)(\mu)\psi'(\mu)d\mu = \frac{1}{2}\xi(\varrho_1) - \frac{1}{2}\xi\left(\varrho_1 + \frac{1}{2}\hbar(\varrho_2,\varrho_1)\right), \tag{3.12}$$

$$N_2 = \frac{1}{2}\int_{\psi^{-1}\left(\varrho_1+\frac{1}{2}\hbar(\varrho_2,\varrho_1)\right)}^{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))} (\xi' \circ \psi)(\mu)\psi'(\mu)d\mu = \frac{1}{2}\xi\left(\varrho_1 + \hbar(\varrho_2,\varrho_1)\right) - \frac{1}{2}\xi\left(\varrho_1 + \frac{1}{2}\hbar(\varrho_2,\varrho_1)\right), \tag{3.13}$$

$$\begin{aligned} N_3 &= \frac{1}{2\hbar^{\alpha}(\varrho_2,\varrho_1)}\int_{\psi^{-1}(\varrho_1)}^{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))} (\varrho_1+\hbar(\varrho_2,\varrho_1)-\psi(\mu))^{\alpha}(\xi' \circ \psi)(\mu)\psi'(\mu)d\mu \\ &= -\frac{1}{2}\xi(\varrho_1) + \frac{\alpha}{2\hbar^{\alpha}(\varrho_2,\varrho_1)}\int_{\psi^{-1}(\varrho_1)}^{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))} (\varrho_1+\hbar(\varrho_2,\varrho_1)-\psi(\mu))^{\alpha-1}(\xi \circ \psi)(\mu)\psi'(\mu)d\mu \\ &= -\frac{1}{2}\xi(\varrho_1) + \frac{\Gamma(\alpha+1)}{2\hbar^{\alpha}(\varrho_2,\varrho_1)} I^{\alpha:\psi}_{\psi^{-1}(\varrho_1)^+}(\xi \circ \psi)\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1)), \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} N_4 &= -\frac{1}{2\hbar^{\alpha}(\varrho_2,\varrho_1)}\int_{\psi^{-1}(\varrho_1)}^{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))} (\psi(\mu)-\varrho_1)^{\alpha}(\xi' \circ \psi)(\mu)\psi'(\mu)d\mu \\ &= -\frac{1}{2}\xi(\varrho_1+\hbar(\varrho_2,\varrho_1)) + \frac{\alpha}{2\hbar^{\alpha}(\varrho_2,\varrho_1)}\int_{\psi^{-1}(\varrho_1)}^{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))} (\psi(\mu)-\varrho_1)^{\alpha-1}(\xi \circ \psi)(\mu)\psi'(\mu)d\mu \\ &= -\frac{1}{2}\xi(\varrho_1+\hbar(\varrho_2,\varrho_1)) + \frac{\Gamma(\alpha+1)}{2\hbar^{\alpha}(\varrho_2,\varrho_1)} I^{\alpha:\psi}_{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))^-}(\xi \circ \psi)\psi^{-1}(\varrho_1). \end{aligned} \tag{3.15}$$

Adding (3.12), (3.13), (3.14) and (3.15), we have

$$\begin{aligned} N_1+N_2+N_3+N_4 = \frac{\Gamma(\alpha+1)}{2\hbar^{\alpha}(\varrho_2,\varrho_1)} &\Big[I^{\alpha:\psi}_{\psi^{-1}(\varrho_1)^+}(\xi \circ \psi)\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1)) \\ &+ I^{\alpha:\psi}_{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))^-}(\xi \circ \psi)\psi^{-1}(\varrho_1)\Big] - \xi\left(\varrho_1+\frac{1}{2}\hbar(\varrho_2,\varrho_1)\right). \end{aligned}$$

This completes the proof. □

Remark 3.5. *When $\hbar(\varrho_2,\varrho_1) = \varrho_2 - \varrho_1$, then above lemma reduces to Lemma 3.2 of [7].*

Now, we derive some fractional H-H inequalities using the above results.

Theorem 3.6. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\hbar : A \times A \to \mathbb{R}$ and $\varrho_1, \varrho_2 \in A$ with $\varrho_1 < \varrho_1 + \hbar(\varrho_2,\varrho_1)$ such that $\xi' \in L[\varrho_1, \varrho_1+\hbar(\varrho_2,\varrho_1)]$. Suppose that $\xi : A \to \mathbb{R}$ is a differentiable function. If $|\xi'|$ is preinvex function on A, $\psi(w)$ is an increasing and positive monotone function on $(\varrho_1, \varrho_1+\hbar(\varrho_2,\varrho_1))$, having a continuous derivative $\psi'(w)$ on $(\varrho_1, \varrho_1+\hbar(\varrho_2,\varrho_1))$ and $\alpha \in (0,1)$. Then,*

$$\begin{aligned} \Bigg|\frac{\xi(\varrho_1)+\xi(\varrho_1+\hbar(\varrho_2,\varrho_1))}{2} - \frac{\Gamma(\alpha+1)}{2\hbar^{\alpha}(\varrho_2,\varrho_1)} &\Big[I^{\alpha:\psi}_{\psi^{-1}(\varrho_1)^+}(\xi \circ \psi)\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1)) \\ + I^{\alpha:\psi}_{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))^-}(\xi \circ \psi)\psi^{-1}(\varrho_1)\Big]\Bigg| &\le \frac{\hbar(\varrho_2,\varrho_1)}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)[|\xi'(\varrho_1)|+|\xi'(\varrho_2)|]. \end{aligned}$$

Proof . From Lemma 3.2 and definition of preinvexity, we have

$$\begin{aligned}
&\Bigg|\frac{\xi(\varrho_1)+\xi(\varrho_1+\hbar(\varrho_2,\varrho_1))}{2}-\frac{\Gamma(\alpha+1)}{2\hbar^\alpha(\varrho_2,\varrho_1)}\Big[I^{\alpha:\psi}_{\psi^{-1}(\varrho_1)^+}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))\\
&\quad+I^{\alpha:\psi}_{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))^-}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1)\Big]\Bigg|\\
&=\left|\frac{1}{2\hbar^\alpha(\varrho_2,\varrho_1)}\int_{\psi^{-1}(\varrho_1)}^{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))}[(\psi(\mu)-\varrho_1)^\alpha-(\varrho_1+\hbar(\varrho_2,\varrho_1)-\psi(\mu))^\alpha](\xi'\,\mathrm{o}\,\psi)(\mu)\psi'(\mu)d\mu\right|\\
&\leq\frac{\hbar(\varrho_2,\varrho_1)}{2}\int_0^1|(1-\delta)^\alpha-\delta^\alpha||\xi'(\varrho_1+(1-\delta)\hbar(\varrho_2,\varrho_1))|d\delta\\
&\leq\frac{\hbar(\varrho_2,\varrho_1)}{2}\int_0^1|(1-\delta)^\alpha-\delta^\alpha|[\delta|\xi'(\varrho_1)|+(1-\delta)|\xi'(\varrho_2)|]d\delta\\
&=\frac{\hbar(\varrho_2,\varrho_1)}{2}\Bigg[|\xi'(\varrho_1)|\left(\int_0^{1/2}\delta(1-\delta)^\alpha d\delta-\int_0^{1/2}\delta^{\alpha+1}d\delta+\int_{1/2}^1\delta^{\alpha+1}d\delta-\int_{1/2}^1\delta(1-\delta)^\alpha d\delta\right)\\
&\quad+|\xi'(\varrho_2)|\left(\int_0^{1/2}(1-\delta)^{\alpha+1}d\delta-\int_0^{1/2}(1-\delta)\delta^\alpha d\delta+\int_{1/2}^1(1-\delta)\delta^\alpha d\delta-\int_{1/2}^1(1-\delta)^{\alpha+1}d\delta\right)\Bigg]\\
&=\frac{\hbar(\varrho_2,\varrho_1)}{2(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)[|\xi'(\varrho_1)|+|\xi'(\varrho_2)|].
\end{aligned}$$

This completes the proof. □

Remark 3.7. *When $\hbar(\varrho_2,\varrho_1)=\varrho_2-\varrho_1$, then above theorem reduces to Theorem 3.4 of [7].*

Example 3.8. *Let $\varrho_1=0,\varrho_2=2,\xi(w)=w,\psi(w)=w$ and $\hbar(\varrho_2,\varrho_1)=\frac{\varrho_2-2\varrho_1}{2}$. Then all the assumptions of above theorem are satisfied. Clearly,*

$$\frac{\xi(\varrho_1)+\xi(\varrho_1+\hbar(\varrho_2,\varrho_1))}{2}=\frac{\xi(\varrho_1)+\xi(\varrho_1+\frac{\varrho_2-2\varrho_1}{2})}{2}=\frac{1}{2},\tag{3.16}$$

and

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2\hbar^\alpha(\varrho_2,\varrho_1)}\Big[I^{\alpha:\psi}_{\psi^{-1}(\varrho_1)^+}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))+I^{\alpha:\psi}_{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))^-}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1)\Big]\\
&=\frac{\alpha}{2}\left[\int_0^1(1-\mu)^{\alpha-1}\mu d\mu+\int_0^1\mu^{\alpha-1}\mu d\mu\right]=\frac{\alpha}{2}\left[\frac{\Gamma\alpha\Gamma 2}{\Gamma(\alpha+2)}+\frac{1}{\alpha+1}\right]=\frac{1}{2}.
\end{aligned}\tag{3.17}$$

From (3.16) and (3.17), we get

$$\begin{aligned}
&\Bigg|\frac{\xi(\varrho_1)+\xi(\varrho_1+\hbar(\varrho_2,\varrho_1))}{2}-\frac{\Gamma(\alpha+1)}{2\hbar^\alpha(\varrho_2,\varrho_1)}\Big[I^{\alpha:\psi}_{\psi^{-1}(\varrho_1)^+}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))\\
&\quad+I^{\alpha:\psi}_{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))^-}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1)\Big]\Bigg|=0
\end{aligned}$$

Next,

$$\begin{aligned}
\frac{\hbar(\varrho_2,\varrho_1)}{2(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)[|\xi'(\varrho_1)|+|\xi'(\varrho_2)|]&=\frac{\varrho_2-2\varrho_1}{2(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\\
&=\frac{1}{(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)>0.
\end{aligned}$$

Theorem 3.9. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\hbar : A \times A \to \mathbb{R}$ and $\varrho_1, \varrho_2 \in A$ with $\varrho_1 < \varrho_1 + \hbar(\varrho_2, \varrho_1)$ such that $\xi' \in L[\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1)]$. Suppose that $\xi : A \to \mathbb{R}$ is a differentiable function. If $|\xi'|$ is preinvex function on A, $\psi(w)$ is an increasing and positive monotone function on $(\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1))$, having a continuous derivative $\psi'(w)$ on $(\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1))$ and $\alpha \in (0, 1)$. Then,*

$$\Bigg|\frac{\Gamma(\alpha+1)}{2\hbar^\alpha(\varrho_2,\varrho_1)}\left[I^{\alpha:\psi}_{\psi^{-1}(\varrho_1)^+}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1)) + I^{\alpha:\psi}_{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))^-}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1)\right]$$
$$- \xi\left(\varrho_1 + \frac{1}{2}\hbar(\varrho_2,\varrho_1)\right)\Bigg| \le \frac{|\xi(\varrho_2)-\xi(\varrho_1)|}{2} + \frac{\hbar(\varrho_2,\varrho_1)}{2(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)[|\xi'(\varrho_1)| + |\xi'(\varrho_2)|].$$

Proof . Using Lemma 3.4, we have

$$\Bigg|\frac{\Gamma(\alpha+1)}{2\hbar^\alpha(\varrho_2,\varrho_1)}\left[I^{\alpha:\psi}_{\psi^{-1}(\varrho_1)^+}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1)) + I^{\alpha:\psi}_{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))^-}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1)\right]$$
$$- \xi\left(\varrho_1 + \frac{1}{2}\hbar(\varrho_2,\varrho_1)\right)\Bigg| \le \left|\int_{\psi^{-1}(\varrho_1)}^{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))} k(\xi'\,\mathrm{o}\,\psi)(\mu)\psi'(\mu)d\mu\right|$$
$$+ \left|\frac{1}{2\hbar^\alpha(\varrho_2,\varrho_1)}\int_{\psi^{-1}(\varrho_1)}^{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))}[(\varrho_1+\hbar(\varrho_2,\varrho_1)-\psi(\mu))^\alpha - (\psi(\mu)-\varrho_1)^\alpha](\xi'\,\mathrm{o}\,\psi)(\mu)\psi'(\mu)d\mu\right|.$$

Using previous theorem and definition of preinvexity, we have

$$\Bigg|\frac{\Gamma(\alpha+1)}{2\hbar^\alpha(\varrho_2,\varrho_1)}\left[I^{\alpha:\psi}_{\psi^{-1}(\varrho_1)^+}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1)) + I^{\alpha:\psi}_{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))^-}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1)\right]$$
$$- \xi\left(\varrho_1 + \frac{1}{2}\hbar(\varrho_2,\varrho_1)\right)\Bigg| \le \frac{|\xi(\varrho_2)-\xi(\varrho_1)|}{2} + \frac{\hbar(\varrho_2,\varrho_1)}{2(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)[|\xi'(\varrho_1)| + |\xi'(\varrho_2)|].$$

This completes the proof. □

Remark 3.10. *When $\hbar(\varrho_2, \varrho_1) = \varrho_2 - \varrho_1$, then above theorem reduces to Theorem 3.5 of [7].*

Example 3.11. *Let $\varrho_1 = 0, \varrho_2 = 1, \xi(w) = w/2, \psi(w) = w$ and $\hbar(\varrho_2, \varrho_1) = \varrho_2 - 3\varrho_1$. Then all the assumptions of above theorem are satisfied. Clearly,*

$$\frac{\Gamma(\alpha+1)}{2\hbar^\alpha(\varrho_2,\varrho_1)}\left[I^{\alpha:\psi}_{\psi^{-1}(\varrho_1)^+}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1)) + I^{\alpha:\psi}_{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))^-}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1)\right]$$
$$= \frac{\alpha}{4}\left[\int_0^1 (1-\mu)^{\alpha-1}\mu d\mu + \int_0^1 \mu^{\alpha-1}\mu d\mu\right] = \frac{\alpha}{4}\left[\frac{\Gamma\alpha\Gamma 2}{\Gamma(\alpha+2)} + \frac{1}{\alpha+1}\right] = \frac{1}{4}, \tag{3.18}$$

and

$$\xi\left(\varrho_1 + \frac{1}{2}\hbar(\varrho_2,\varrho_1)\right) = \xi\left(\varrho_1 + \frac{1}{2}(\varrho_2 - 3\varrho_1)\right) = \frac{1}{4}. \tag{3.19}$$

From (3.18) and (3.19), we get

$$\Bigg|\frac{\Gamma(\alpha+1)}{2\hbar^\alpha(\varrho_2,\varrho_1)}\left[I^{\alpha:\psi}_{\psi^{-1}(\varrho_1)^+}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1)) + I^{\alpha:\psi}_{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))^-}(\xi\,\mathrm{o}\,\psi)\psi^{-1}(\varrho_1)\right]$$
$$- \xi\left(\varrho_1 + \frac{1}{2}\hbar(\varrho_2,\varrho_1)\right)\Bigg| = 0. \tag{3.20}$$

Next,

$$\frac{|\xi(\varrho_2)-\xi(\varrho_1)|}{2} + \frac{\hbar(\varrho_2,\varrho_1)}{2(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)[|\xi'(\varrho_1)| + |\xi'(\varrho_2)|] = \frac{1}{4} + \frac{1}{2(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right) > 0.$$

Theorem 3.12. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\hbar : A \times A \to \mathbb{R}$ and $\varrho_1, \varrho_2 \in A$ with $\varrho_1 < \varrho_1 + \hbar(\varrho_2, \varrho_1)$ such that $\xi' \in L[\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1)]$. Suppose that $\xi : A \to \mathbb{R}$ is a differentiable function. If $|\xi'|^q$ is preinvex function on A for some fixed $q > 1$, $\psi(w)$ is an increasing and positive monotone function on $(\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1))$, having a continuous derivative $\psi'(w)$ on $(\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1))$ and $\alpha \in (0, 1)$. Then,*

$$\left| \frac{\xi(\varrho_1) + \xi(\varrho_1 + \hbar(\varrho_2, \varrho_1))}{2} - \frac{\Gamma(\alpha+1)}{2\hbar^\alpha(\varrho_2, \varrho_1)} \left[I^{\alpha:\psi}_{\psi^{-1}(\varrho_1)^+} (\xi \,\mathrm{o}\, \psi)\psi^{-1}(\varrho_1 + \hbar(\varrho_2, \varrho_1)) + I^{\alpha:\psi}_{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))^-} (\xi \,\mathrm{o}\, \psi)\psi^{-1}(\varrho_1) \right] \right| \le \frac{\hbar(\varrho_2, \varrho_1)}{2(\alpha p + 1)^{\frac{1}{p}}} \left(\frac{|\xi'(\varrho_1)|^q + |\xi'(\varrho_2)|^q}{2} \right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof . From Lemma 3.2, we have

$$\begin{aligned} &\left| \frac{\xi(\varrho_1) + \xi(\varrho_1 + \hbar(\varrho_2, \varrho_1))}{2} - \frac{\Gamma(\alpha+1)}{2\hbar^\alpha(\varrho_2, \varrho_1)} \left[I^{\alpha:\psi}_{\psi^{-1}(\varrho_1)^+} (\xi \,\mathrm{o}\, \psi)\psi^{-1}(\varrho_1 + \hbar(\varrho_2, \varrho_1)) \right.\right. \\ &\left.\left. + I^{\alpha:\psi}_{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))^-} (\xi \,\mathrm{o}\, \psi)\psi^{-1}(\varrho_1) \right] \right| \\ &\le \frac{\hbar(\varrho_2, \varrho_1)}{2} \int_0^1 |\delta^\alpha - (1-\delta)^\alpha||\xi'(\varrho_1 + \delta\hbar(\varrho_2, \varrho_1))| d\delta. \end{aligned}$$

Using Hölder's inequality and definition of preinvexity, we have

$$\begin{aligned} &\left| \frac{\xi(\varrho_1) + \xi(\varrho_1 + \hbar(\varrho_2, \varrho_1))}{2} - \frac{\Gamma(\alpha+1)}{2\hbar^\alpha(\varrho_2, \varrho_1)} \left[I^{\alpha:\psi}_{\psi^{-1}(\varrho_1)^+} (\xi \,\mathrm{o}\, \psi)\psi^{-1}(\varrho_1 + \hbar(\varrho_2, \varrho_1)) \right.\right. \\ &\left.\left. + I^{\alpha:\psi}_{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))^-} (\xi \,\mathrm{o}\, \psi)\psi^{-1}(\varrho_1) \right] \right| \\ &\le \frac{\hbar(\varrho_2, \varrho_1)}{2} \left(\int_0^1 |\delta^\alpha - (1-\delta)^\alpha|^p d\delta \right)^{\frac{1}{p}} \left(\int_0^1 |\xi'(\varrho_1 + \delta\hbar(\varrho_2, \varrho_1))|^q d\delta \right)^{\frac{1}{q}} \\ &\le \frac{\hbar(\varrho_2, \varrho_1)}{2} \left(\int_0^1 |1 - 2\delta|^{\alpha p} d\delta \right)^{\frac{1}{p}} \left(\int_0^1 ((1-\delta)|\xi'(\varrho_1)|^q + \delta|\xi'(\varrho_2)|^q) \right)^{\frac{1}{q}} \\ &= \frac{\hbar(\varrho_2, \varrho_1)}{2(\alpha p + 1)^{\frac{1}{p}}} \left(\frac{|\xi'(\varrho_1)|^q + |\xi'(\varrho_2)|^q}{2} \right)^{\frac{1}{q}}. \end{aligned}$$

This completes the proof. □

Theorem 3.13. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\hbar : A \times A \to \mathbb{R}$ and $\varrho_1, \varrho_2 \in A$ with $\varrho_1 < \varrho_1 + \hbar(\varrho_2, \varrho_1)$ such that $\xi' \in L[\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1)]$. Suppose that $\xi : A \to \mathbb{R}$ is a differentiable function. If $|\xi'|^q$ is preinvex function on A for some fixed $q > 1$, $\psi(w)$ is an increasing and positive monotone function on $(\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1))$, having a continuous derivative $\psi'(w)$ on $(\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1))$ and $\alpha \in (0, 1)$. Then,*

$$\left| \frac{\xi(\varrho_1) + \xi(\varrho_1 + \hbar(\varrho_2, \varrho_1))}{2} - \frac{\Gamma(\alpha+1)}{2\hbar^\alpha(\varrho_2, \varrho_1)} \left[I^{\alpha:\psi}_{\psi^{-1}(\varrho_1)^+} (\xi \,\mathrm{o}\, \psi)\psi^{-1}(\varrho_1 + \hbar(\varrho_2, \varrho_1)) + I^{\alpha:\psi}_{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))^-} (\xi \,\mathrm{o}\, \psi)\psi^{-1}(\varrho_1) \right] \right| \le \frac{\hbar(\varrho_2, \varrho_1)}{(\alpha+1)} \left(1 - \frac{1}{2^\alpha} \right) \left[\frac{|\xi'(\varrho_1)|^q + |\xi'(\varrho_2)|^q}{2} \right]^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.
Proof . Using Lemma 3.2, Hölder's inequality and definition of preinvexity, we have

$$\begin{aligned}&\Bigg|\frac{\xi(\varrho_1)+\xi(\varrho_1+\hbar(\varrho_2,\varrho_1))}{2}-\frac{\Gamma(\alpha+1)}{2\hbar^{\alpha}(\varrho_2,\varrho_1)}\Big[I^{\alpha:\psi}_{\psi^{-1}(\varrho_1)^+}(\xi\circ\psi)\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))\\&\quad+I^{\alpha:\psi}_{\psi^{-1}(\varrho_1+\hbar(\varrho_2,\varrho_1))^-}(\xi\circ\psi)\psi^{-1}(\varrho_1)\Big]\Bigg|\le\frac{\hbar(\varrho_2,\varrho_1)}{2}\left(\int_0^1|\delta^{\alpha}-(1-\delta)^{\alpha}|d\delta\right)^{\frac{1}{p}}\\&\quad\times\left(\int_0^1|\delta^{\alpha}-(1-\delta)^{\alpha}||\xi'(\varrho_1+\delta\hbar(\varrho_2,\varrho_1))|^q d\delta\right)^{\frac{1}{q}}\\&=\frac{\hbar(\varrho_2,\varrho_1)}{2}\left(\int_0^{1/2}((1-\delta)^{\alpha}-\delta^{\alpha})d\delta+\int_{1/2}^1(\delta^{\alpha}-(1-\delta)^{\alpha})d\delta\right)^{\frac{1}{p}}\\&\quad\times\left(\int_0^{1/2}((1-\delta)^{\alpha}-\delta^{\alpha})((1-\delta)|\xi'(\varrho_1)|^q+\delta|\xi'(\varrho_2)|^q)d\delta\right.\\&\quad\left.+\int_{1/2}^1(\delta^{\alpha}-(1-\delta)^{\alpha})((1-\delta)|\xi'(\varrho_1)|^q+\delta|\xi'(\varrho_2)|^q)d\delta\right)^{\frac{1}{q}}\\&=\frac{\hbar(\varrho_2,\varrho_1)}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left[\frac{|\xi'(\varrho_1)|^q+|\xi'(\varrho_2)|^q}{2}\right]^{\frac{1}{q}}.\end{aligned}$$

□

3.1. ***Application to special means:***

We recall the following means for two real numbers $\beta,\gamma,\ \beta\neq\gamma$:

$$A(\beta,\gamma)=\frac{\beta+\gamma}{2},\quad \beta,\gamma\in\mathbb{R},$$

$$H(\beta,\gamma)=\frac{2}{\frac{1}{\beta}+\frac{1}{\gamma}},\quad \beta,\gamma\in\mathbb{R}\backslash\{0\},$$

$$L(\beta,\gamma)=\frac{\gamma-\beta}{\ln|\gamma|-\ln|\beta|},\quad |\beta|\neq|\gamma|,\ \beta\gamma\neq0,$$

$$L_n(\beta,\gamma)=\left[\frac{\gamma^{n+1}-\beta^{n+1}}{(n+1)(\gamma-\beta)}\right]^{\frac{1}{n}},\quad n\in\mathbb{Z}\backslash\{-1,0\},\quad \beta,\gamma\in\mathbb{R},\ \beta\neq\gamma.$$

Now, using the above results in previous theorems, we have some exciting results.

Proposition 3.14. *Let* $\varrho_1,\varrho_1+\hbar(\varrho_2,\varrho_1)\in\mathbb{R}^+,\varrho_1<\varrho_1+\hbar(\varrho_2,\varrho_1)$. *Then*

$$|A(\varrho_1^n,(\varrho_1+\hbar(\varrho_2,\varrho_1))^n)-L_n^n(\varrho_1,\varrho_1+\hbar(\varrho_2,\varrho_1))|\le\frac{n\hbar(\varrho_2,\varrho_1)}{2(p+1)^{\frac{1}{p}}}\left(\frac{\varrho_1^{(n-1)q}+\varrho_2^{(n-1)q}}{2}\right)^{\frac{1}{q}}.$$

Proof . Applying Theorem 3.12 with $\xi(w)=w^n,\psi(w)=w,\alpha=1$. Then we compute the result easily. □

Proposition 3.15. *Let* $\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1) \in \mathbb{R}^+, \varrho_1 < \varrho_1 + \hbar(\varrho_2, \varrho_1)$. *Then*

$$|A(e^{\varrho_1}, e^{\varrho_1+\hbar(\varrho_2,\varrho_1)}) - L(e^{\varrho_1}, e^{\varrho_1+\hbar(\varrho_2,\varrho_1)})| \leq \frac{\hbar(\varrho_2, \varrho_1)}{2(p+1)^{\frac{1}{p}}} \left(\frac{e^{\varrho_1 q} + e^{\varrho_2 q}}{2}\right)^{\frac{1}{q}}.$$

Proof . Applying Theorem 3.12 with $\xi(w) = e^w, \psi(w) = w, \alpha = 1$. Then we compute the result easily. □

Proposition 3.16. *Let* $\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1) \in \mathbb{R}^+, \varrho_1 < \varrho_1 + \hbar(\varrho_2, \varrho_1)$. *Then*

$$\left|H^{-1}(\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1)) - L^{-1}(\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1))\right| \leq \frac{\hbar(\varrho_2, \varrho_1)}{2(p+1)^{\frac{1}{p}}} \left[\frac{1}{2}\left(\frac{1}{\varrho_1^{2q}} + \frac{1}{\varrho_2^{2q}}\right)\right]^{\frac{1}{q}}.$$

Proof . Applying Theorem 3.12 with $\xi(w) = \frac{1}{w}, \psi(w) = w, \alpha = 1$. Then we compute the result easily. □

Proposition 3.17. *Let* $\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1) \in \mathbb{R}^+, \varrho_1 < \varrho_1 + \hbar(\varrho_2, \varrho_1)$. *Then*

$$|A(\varrho_1^n, (\varrho_1 + \hbar(\varrho_2, \varrho_1))^n) - L_n^n(\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1))| \leq \frac{n\hbar(\varrho_2, \varrho_1)}{4} \left(\frac{\varrho_1^{(n-1)q} + \varrho_2^{(n-1)q}}{2}\right)^{\frac{1}{q}}.$$

Proof . Applying Theorem 3.13 with $\xi(w) = w^n, \psi(w) = w, \alpha = 1$. Then we compute the result easily. □

Proposition 3.18. *Let* $\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1) \in \mathbb{R}^+, \varrho_1 < \varrho_1 + \hbar(\varrho_2, \varrho_1)$. *Then*

$$|A(e^{\varrho_1}, e^{\varrho_1+\hbar(\varrho_2,\varrho_1)}) - L(e^{\varrho_1}, e^{\varrho_1+\hbar(\varrho_2,\varrho_1)})| \leq \frac{\hbar(\varrho_2, \varrho_1)}{4} \left(\frac{e^{\varrho_1 q} + e^{\varrho_2 q}}{2}\right)^{\frac{1}{q}}.$$

Proof . Applying Theorem 3.13 with $\xi(w) = e^w, \psi(w) = w, \alpha = 1$. Then we compute the result easily. □

Proposition 3.19. *Let* $\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1) \in \mathbb{R}^+, \varrho_1 < \varrho_1 + \hbar(\varrho_2, \varrho_1)$. *Then*

$$\left|H^{-1}(\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1)) - L^{-1}(\varrho_1, \varrho_1 + \hbar(\varrho_2, \varrho_1))\right| \leq \frac{\hbar(\varrho_2, \varrho_1)}{4} \left[\frac{1}{2}\left(\frac{1}{\varrho_1^{2q}} + \frac{1}{\varrho_2^{2q}}\right)\right]^{\frac{1}{q}}.$$

Proof . Applying Theorem 3.13 with $\xi(w) = \frac{1}{w}, \psi(w) = w, \alpha = 1$. Then we compute the result easily. □

4. Conclusion

We established a new form of Hermite-Hadamard inequality having left and right-sided ψ-Riemann-Liouville fractional integrals for preinvex functions and results of ψ-Riemann-Liouville fractional integral identities including the first-order derivative of a preinvex function and derived some fractional Hermite-Hadamard inequalities. Some applications for special means are also discussed.

5. Acknowledgments

The authors are grateful to the referee's for valuable comments and suggestions that helped us improve this article.

6. Funding

The publication of this article was funded by the Qatar National Library.

References

[1] A. Barani, A.G. Ghazanfari and S.S. Dragomir, *Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex*, J. Inequal. Appl. 2012(1) (2012) 247.

[2] A. Ben-Israel and B. Mond, *What is invexity?*, J. Aust. Math. Soc. Series B. Appl. Math. 28(1) (1986) 1–9.

[3] S.S. Dragomir and C.E.M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, Victoria University, Australia, 2000.

[4] M. A. Hanson, *On sufficiency of the Kuhn-Tucker conditions*, J. Math. Anal. Appl. 80(2) (1981) 545–550.

[5] I. Işcan, *Hermite-Hadamard's inequalities for preinvex functions via fractional integrals and related fractional inequalities*, arXiv:1204.0272 (2012).

[6] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science Limited, 2006.

[7] K. Liu, J. Wang and D. O'Regan, *On the Hermite-Hadamard type inequality for ψ-Riemann-Liouville fractional integrals via convex functions*, J. Inequal. Appl. 2019(1) (2019) 27.

[8] S.K. Mishra and N. Sharma, *On strongly generalized convex functions of higher order*, Math. Inequal. Appl. 22(1) (2019) 111–121.

[9] D.S. Mitrinović and I.B. Lacković, *Hermite and convexity*, Aequ. Math. 28(3) (1985) 229–232.

[10] S.R. Mohan and S.K. Neogy, *On invex sets and preinvex functions*, J. Math. Anal. Appl. 189 (1995) 901–908.

[11] M.A. Noor, *Hadamard integral inequalities for product of two preinvex functions*, Nonl. A. Forum 14 (2009) 167–173.

[12] M.A. Noor, *Hermite-Hadamard integral inequalities for log-preinvex functions*, J. Math. Anal. Approx. Theory 2(2) (2007) 126–131.

[13] M.A. Noor, *On Hadamard integral inequalities involving two log-preinvex functions*, J. Inequal. Pure. Appl. Math. 8(3) (2007) 1–14.

[14] R. Pini, *Invexity and generalized convexity*, Optim. 22(4) (1991) 513–525.

[15] J. Wang and M. Fečkan, *Fractional Hermite-Hadamard Inequalities*, De Gruyter, Berlin, 2018.

[16] T. Weir and B. Mond, *Preinvex functions in multiple objective optimization*, J. Math. Anal. Appl. 136 (1998) 29–38.