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Shape preserving approximation using convex smooth piecewise polynomials for functions in L_p quasi normed spaces

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Abstract

Many papers used the algebraic polynomials to approximate functions in L_p space for $0 < p < 1$. Few are introduced for the convex algebraic polynomials best approximation. But no one proves direct Theorems for constrained convex approximation using smooth interpolatory piecewise polynomials for functions in L_p, $0 < p < 1$. That is what we shall introduce here.

Keywords: L_p-space, piecewise, convex approximation, derivative

1. Introduction and Notation

Define $L_p(I)=\{\mathcal{F}: I\to R \ : f \in L_p\}$, where I is closed interval between -1,1 and $L_p^{\mathfrak{r}}(I) = \{\mathcal{F} : I\to R \ :\mathcal{F}^{\ \mathfrak{r}}\in L_p\}$ with $\|\mathcal{F}\|_{Lp}=\left(\int_{-1}^{1}|\mathcal{F}(x)|^{p}\right)^{\frac{1}{p}}$. For $\kappa \in N$ and interval I,

$$\Delta_{\mathfrak{u}}^{\kappa}(\mathcal{F},x,I) := \begin{cases} \sum_{\mathfrak{i}=0}^{\kappa}(-1)^{\mathfrak{i}}\binom{\kappa}{\mathfrak{i}}\mathcal{F}\left(x+\left(\frac{\kappa}{2}-\mathfrak{i}\right)\mathfrak{u}\right), & x\mp\frac{\kappa\mathfrak{u}}{2}\in I \\ 0, & \text{otherwise.} \end{cases}$$

Then $w_\kappa(\mathcal{F},\mathfrak{t},I) := sup_{0<\mathfrak{u}<\mathfrak{t}}\|\Delta_{\mathfrak{u}}^{\kappa}(\mathcal{F},.;I)\|_p$ is a measure of the smoothness modulus of f on I. $w_\kappa(\mathcal{F},\mathfrak{t}) := w_\kappa(\mathcal{F},\mathfrak{t},I)$, $L_p^{\mathfrak{r}}=L_p^{\mathfrak{r}}(I)$, for any interval I, we write $w_\kappa(\mathcal{F},\delta,I)$. We use $\vartheta(x)=\sqrt{1+x^2}$ and $\Omega_n(x)=\vartheta(x)n^{-1}+n^{-2}$, $\mathfrak{n}\in N$, $\Omega_0\equiv1$. $\Pi_{\mathfrak{n}}$ symbolizes the space of algebraic polynomial of degree$\leq n$.

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A function $\mathcal{F}:[a,b]\to R$ is said to be κ-monotone,$\kappa\geq 1$ *on* $[a,b]$ if and only if for all choices of $\kappa+1$ distinct points $x_0,x_1,\ldots,x_k\in[a,b]$ the inequality $\mathcal{F}[x_0,x_1,\ldots,x_k]>0$ holds, where

$$\mathcal{F}[x_0,x_1,\ldots,x_k]=\sum_{j=0}^{\kappa}\frac{\mathcal{F}(x_j)}{w'(x_j)}$$

$$I_j:=I_{j,n}:=[x_j,x_{j-1}],\ h_j:=h_{j,n}:=|I_{j,n}|=x_{j-1}-x_j$$

$$I_{i,j}:=\bigcup_{\kappa=\min\{i,j\}}^{\max\{i,j\}}I_\kappa=\left[x_{\max\{i,j\}},x_{\min\{i,j\}-1}\right\},1\leq i,j\leq n$$

(the shortest interval containing both I_i and I_j), $x_j:=x_{j,i}:=\cos\left(\frac{j\pi}{n}\right),0\leq j\leq n,$ 1, for $j<0$ and -1 for$j>n$ (Chebyshev knots)

$$h_{i,j}:=|I_{i,j}|=\sum_{\kappa=min\{i,j\}}^{\max\{i,j\}}h_\kappa=x_{\min\{i,j\}-1}-x_{\max\{i,j\}}$$

$$\mathcal{T}_j:=\mathcal{T}(x):=\frac{|I_j|}{(|x-x_j|+|I_j|)},\ \delta_n(x):=\min\{1,n\vartheta(x)\}$$

$$\Phi^\kappa:=\{\mathcal{T}\in c[0,\infty]|\mathcal{T}\uparrow,\mathcal{T}(0)=0\text{ and }\mathfrak{t}_2{}^{-\kappa}\mathcal{T}(\mathfrak{t}_2)\leq t_1{}^{-\kappa}\mathcal{T}(\mathfrak{t}_1)\quad\text{for}\quad 0\leq\mathfrak{t}_1\leq\mathfrak{t}_2\}.$$

Note: If $\mathcal{F}\in L_p^{\mathfrak{r}}$, then $\Gamma(\mathfrak{t}):=\mathfrak{t}^r w_\kappa(\mathcal{F}^{(\mathfrak{r})},\mathfrak{t})_p$ is equivalent to a function from $\Phi^{\kappa+\mathfrak{r}}$. $\sum_\kappa:=\sum_{\kappa,n}$ denoted the x_j, $1\leq j\leq n-1$ piecewise polynomials of degree not exceeding $\kappa-1$ that are continuous. $\sum_\kappa^{(1)}=\sum_{\kappa,\mathfrak{n}}^{(1)}$ denote the set of all x_j, $1\leq j\leq n-1$ piecewise polynomials that have continuous derivatives. $\mathcal{P}_j:=\mathcal{P}_j(s):=\mathbb{S}|I_j$, $1\leq j\leq n$ ($\mathbb{S}$ is a piecewise polynomial of pieces $\mathcal{P}_j(x)$, $x\in I_j$, $1\leq j\leq n-1$, and write $\mathbb{S}|I_j$. $\mathfrak{b}_{i,j}(s,\Gamma):=\frac{\|\mathcal{P}_i-\mathcal{P}_j\|_p}{\Gamma(h_j)}\left(\frac{h_j}{h_{i,j}}\right)^\kappa$, where $\Gamma\in\Phi^\kappa$, $\Gamma\not\equiv 0$ and $\mathbb{S}\in\sum_\kappa$. $\mathfrak{b}_\kappa(s,\Gamma,B):=\max_{1\leq i,j\leq n}\{\mathfrak{b}_{i,j}(s,\Gamma)\,|I_i\subset B\text{ and }I_j\subset B\}$, where an interval $B\subseteq[-1,1]$ contains at least one interval I_v

$$\mathfrak{b}_\kappa(s,\Gamma):=\mathfrak{b}(s,\Gamma,I)=\max_{1\leq i,j\leq n}\mathfrak{b}_{i,j}(s,\Gamma),$$

$c(p):=$ is an absolute constant depending on p,and is different from one step to others and $c(\kappa,p):=$ positive constant that are either absolute or may only depend on the parameters k and p.

$$\mathfrak{L}_\kappa^L(\mathcal{F},x,[a,\mathfrak{b}])=\min_{1\leq m\leq\kappa}\Delta^m_{(x-a)^{\frac{1}{m}}(\mathfrak{b}-a)^{m-1/m}}\quad,x\in[a,\mathfrak{b}]$$

$$\mathfrak{L}_\kappa^R(\mathcal{F},x,[a,\mathfrak{b}])=\min_{1\leq m\leq\kappa}\Delta^m_{(\mathfrak{b}-x)^{\frac{1}{m}}(\mathfrak{b}-a)^{m-1/m}},\ x\in[a,\mathfrak{b}]$$

If $\kappa\in N$, $\mathfrak{r}\in N_0$ and $\mathcal{F}\in C^{\mathfrak{r}}$, then for all $n\geq\kappa+\mathfrak{r}-1$. There is a polynomial $\mathcal{P}_n\in\prod_n$ satisfies

$$|\mathcal{F}(x)-\mathcal{P}_n(x)|\leq c(\kappa,\mathfrak{r})\Omega_n^{\mathfrak{r}}(x)\,w_k(\mathcal{F}^{\mathfrak{r}},\Omega_n(x)),\quad x\epsilon[-1,1]\tag{1.1}$$

and, moreover

$$|\mathcal{F}(x)-\mathcal{P}_n(x)|\leq c(\mathfrak{r},\kappa)\,\vartheta^{2\mathfrak{r}}(x)\,w_k\left(\mathcal{F},\vartheta^{\frac{2}{k}}(x)\,n^{-\frac{2(\kappa-1)}{\kappa}}\right),\text{ if }1-n^{-2}\leq|x|\leq 1\tag{1.2}$$

Recently, we were able to show [13] that (1.1) and (1.2) hold for monotone approximation ($q=1$) if $\mathfrak{r}\in N$, $\kappa=2$ and $n\geq\mathcal{N}(\mathcal{F},\mathfrak{r})$. In fact, we follow similar ideas and apply some of the construction in [13]. But there are some additional rather significant technical difficulties that we have to overcome in this case (for example. Proofs in the cases for $\mathfrak{r}=1$ and $\mathfrak{r}\geq 2$ turn out to be completely different). Also, one of the important tools that we are using is our recent result [14] on convex approximation of $\mathcal{F}\in\mathcal{C}^{\mathfrak{r}}\cap\Delta^{(2)}$, by convex piecewise polynomials (Theorem 3.1).

2. The Auxiliary Lemma

Lemma 2.1. *Let* $\Gamma\in\Phi^{\kappa}$, $\kappa\in N,\mathcal{F}\in L_p(I)$ *and* $\mathbb{S}\in\sum_{\kappa,n}$. *If* $w_\kappa(\mathcal{F},\mathfrak{t})_p \leq c(p)\Gamma(\mathfrak{t})$ *and* $\|\mathcal{F}-\mathbb{S}\|_p\leq c(p)\Gamma(\Omega_n(x))$, *then*

$$\mathfrak{b}_\kappa(s,\Gamma)\leq c(\kappa,p).$$

Theorem 2.2. *[6] For every* $\mathfrak{r}\in N$ *there is a constant* $c=c(p,\mathfrak{r})$ *with the following property, for each convex function* $\mathcal{F}\in L_{p^{\mathfrak{r}}}[a,b]$, *there is a number* $\mathcal{H}>0$, *such that for every partition* $\mathcal{X}=\{x_j\}_{j=0}^{n}$ *of* $[a,\mathfrak{b}]$ *satisfying* $x_1-a\leq\mathcal{H}$ *and* $\mathfrak{b}-x_{n-1}\leq\mathcal{H}$.

There is a convex piecewise polynomial $s\in\mathbb{S}(\mathcal{X},\mathfrak{r}+2)$ such that

$$|\mathcal{F}(x)-s(x)|\leq c(x-a)^{\mathfrak{r}}\mathfrak{L}_2^L\left(\mathcal{F}^{(\mathfrak{r})},x;[a,x_1]\right),x\in[a,x_1],$$

$|\mathcal{F}(x)-s(x)|\leq c(\mathfrak{b}-x)^{\mathfrak{r}}\mathfrak{L}_2^R(\mathcal{F}^{(\mathfrak{r})},x;[x_{n-1},\mathfrak{b}],x\in[x_{n-1},\mathfrak{b}]$, and, for each $j=2,\ldots,n-1$ and $x\in[x_{j-1},x_j]$

$$|\mathcal{F}(x)-s(x)|\leq c(x_j-x_{j-1})^{\mathfrak{r}}\Delta^2_{x_j-x_{j-1}}\left(\mathcal{F}^{(\mathfrak{r})}\right),\ x\in[x_{j-1},x_j]+$$

$$c(x_1-a)^{\mathfrak{r}}\Delta^2_{x_1-a}\left(\mathcal{F}^{(\mathfrak{r})}\right),\ \ x\in[a,x_1]\ +c(\mathfrak{b}-x_{n-1})^{\mathfrak{r}}\Delta^2_{\mathfrak{b}-x_{n-1}}(\mathcal{F}^{(\mathfrak{r})},\mathfrak{b}-x_{n-1};[x_{n-1},\mathfrak{b}].$$

Lemma 2.3. *[6]* $|\mathcal{F}(x)-s(x)|\leq c(x-a)^{\mathfrak{r}}\mathfrak{L}_2^L\left(\mathcal{F}^{(\mathfrak{r})},x;[a,x_1]\right),x\in[a,x_1]$.

Lemma 2.4. *[6]* $|\mathcal{F}(x)-s(x)|\leq c(\mathfrak{b}-x)^{\mathfrak{r}}\mathfrak{L}_2^R\left(\mathcal{F}^{(\mathfrak{r})},x;[x_{n-1},\mathfrak{b}]\right),\ x\in[x_{n-1},\mathfrak{b}]$.

Lemma 2.5. *[16] Let* $\mathfrak{r}\in N$, $Z_m:=(\mathcal{Z}_i)_{i=0}^m$, $a=:\mathcal{Z}_0<\mathcal{Z}_1<\cdots<\mathcal{Z}_{m-1}<\mathcal{Z}_m:=\mathfrak{b}$ *be a partition of* $[a,\mathfrak{b}]$, *let* $s\in\Delta^{(2)}\cap\mathcal{Y}_{\mathfrak{r}+2}(Z_m)$. *Then there exists* $\widetilde{s}\in\Delta^{(2)}\cap\mathcal{Y}_{\mathfrak{r}+2}(Z_m)\cap L_{p^1}[a,\mathfrak{b}]$ *such that, for any* $1\leq j\leq m-1$,

$$\|s-\widetilde{s}\|_{[\mathcal{Z}_{j-1},\mathcal{Z}_{j+1}]}\leq c(\mathfrak{r},\mathcal{O}(Z_m))\,w_{\mathfrak{r}+2}(s,\mathcal{Z}_{j+2}-\mathcal{Z}_{j-2};[\mathcal{Z}_{j-2},\mathcal{Z}_{j+2}]),$$

where $\mathcal{Z}_j:=\mathcal{Z}_0$, $j<0$ *and* $\mathcal{Z}_j:=\mathcal{Z}_m$, $j>m$. *Moreover,* $\widetilde{s}^{(v)}(a)=s^{(v)}(a)$ *and* $\widetilde{s}^{(v)}(\mathfrak{b})=s^{(v)}(\mathfrak{b})$. $v=0,1$.

Lemma 2.6. *[16]* $(h_{j\pm1}<3h_j)$.

Theorem 2.7. *[14]*$\mathfrak{b}_\kappa(s,\Gamma)\leq1$ *and, additionally*

1. *If* $d_+>0$, *then* $d_+|I_2|^{\mathfrak{r}-2}\leq\min_{x\in I_2}\mathbb{S}'(x)$.
2. *If* $d_+=0$, $\mathbb{S}^{(i)}(1)=0$, *for all* $2\leq i\leq\kappa-2$.
3. *If* $d_->0$, *then* $d_-|I_{n-1}|^{\mathfrak{r}-2}\leq\min_{x\in I_{n-1}}\mathbb{S}'(x)$.
4. *If* $d_-=0$, *then* $\mathbb{S}^{(i)}(-1)=0$, *for all* $2\leq i\leq\kappa-2$.

then there exists a polynomial $P\in\Delta^{(1)}\cap\prod_{Cn}$ *satisfying, for all* $x\in[-1,1]$,

$$\|\mathbb{S}-\mathcal{P}\|_p\leq c(p,\kappa,\mathcal{S})\,\delta_n^{\mathcal{S}}(x)\,\Gamma(\Omega_n(x)),\quad \text{if } d_+>0 \text{ and } d_->0, \tag{2.1}$$

$$\|\mathbb{S}-\mathcal{P}\|_p\leq c(p,\kappa,\mathcal{S})\,\delta_n^{\min\{\mathcal{S},2\kappa-2\}}(x)\,\Gamma(\Omega_n(x)),\quad \text{if } \min\{d+,d-\}=0. \tag{2.2}$$

3. Main theorems

Theorem 3.1. *Let $\mathcal{F}$ be a convex function in L_p[-1,1], then $\mathfrak{r}\in N$, there is a constant $c(p,\mathfrak{r})$, there exist $\mathcal{N}(\mathcal{F},\mathfrak{r})$ and a convex piecewise polynomials $\mathbb{S}\in\sum_{\mathfrak{r}+2,n}\cap\Delta^{(2)}$ of degree $\mathfrak{r}$+1, and has Chebyshev partition knots T_n:*

$$\|\mathcal{F}(x)-\mathbb{S}(x)\|_p\leq c(p,\mathfrak{r})\left(\frac{\vartheta(x)}{n}\right)^{\mathfrak{r}}w_2\left(\mathcal{F}^{(\mathfrak{r})},\frac{\vartheta(x)}{n}\right)_p, x\in[-1,1] \tag{3.1}$$

and

$$\|\mathcal{F}(x)-\mathbb{S}(x)\|_{L_{p[-1.-1+n^{-2}]\cup[1-n^{-2},1]}}\leq c(p,\mathfrak{r})\,\vartheta^{2\mathfrak{r}}(x)\,w_2\left(\mathcal{F}^{(\mathfrak{r})},\frac{\vartheta}{n}\right)_{L_{p[-1,-1+n^{-2}]\cup[1-n^{-2},1]}},$$
$$x\in\left[-1,-1+n^{-2}\right]\cup[1-n^{-2},1] \tag{3.2}$$

and

$$\|\mathcal{F}(x)-\mathbb{S}(x)\|_{L_{p[-1,-1+n^{-2}]\cup[1-n^{-2},1]}}\leq c(p,\mathfrak{r})\vartheta^{2\mathfrak{r}}(x)w_1(\mathcal{F}^{(\mathfrak{r})},\vartheta^2(x))_{L_{p[-1,-1+n^{-2}]\cup[1-n^{-2}.1]}},$$
$$x\in[-1,-1+n^{-2}]\cup[1-n^{-2},1] \tag{3.3}$$

Proof . By Theorem 2.2, if we let x be the Chebyshev partition $T_n=\{\mathfrak{t}_j\}$, where $n\geq\mathcal{N}:=3/\sqrt{\mathcal{H}}$, then by $x_1-a\leq\mathcal{H}$ and $\mathfrak{b}-x_{n-1}\leq\mathcal{H}$), the proof holds, because from $\sin\pi/2n\leq\pi/2n$, we have

$$\mathfrak{t}_1+1=1-\mathfrak{t}_{n-1}=2\sin^2\left(\frac{\pi}{2n}\right)\leq\frac{\pi^2}{2n^2}\leq\frac{5}{\mathcal{N}}\leq\mathcal{H}.$$

Since $\frac{\vartheta(x)}{n}\sim\Omega_n(x)\sim\mathfrak{t}_j-\mathfrak{t}_{j-1}$,for $x\in[\mathfrak{t}_{j-1},\mathfrak{t}_j]$. From lemmas 2.3, 2.4 and the above discussion, we get

$$\|\mathcal{F}(x)-s(x)\|_p\leq:=(p,a)^{\mathfrak{r}}\,\mathfrak{L}_2^L(\mathcal{F}^{(\mathfrak{r})},x;[a,x_1]_{L_p[a,x_1]},$$
$$\|\mathcal{F}(x)-s(x)\|_p\leq c(p,\mathfrak{b})^{\mathfrak{r}}\left\|\mathfrak{L}_2^R(\mathcal{F}^{(\mathfrak{r})},x;[x_{n-1},\mathfrak{b}]\right\|_{L_p[x_{n-1},\mathfrak{b}]}.$$

Thus, (3.1),(3.2) and(3.3) are satisfied. □

Theorem 3.2. *If $\mathcal{F}$ be a convex function in $L_p[-1,1]$, for $\mathfrak{r}\in N$, there is a number $\mathcal{N}(\mathcal{F},\mathfrak{r})$ satisfies for any $n\geq\mathcal{N}$, we can find a continuous and differentiable convex piecewise polynomials $\mathbb{S}$ of degree $\mathfrak{r}+1$ with Chebyshev partition knots T_n, satisfying (3.1),(3.2) and (3.3). Let $\mathcal{Y}_{\mathfrak{r}}(Z_m)$ denoted the space of all piecewise polynomial function (ppf) of degree $\mathfrak{r}-1$ (order $\mathfrak{r}$) with the knots $Z_m:=(\mathcal{Z}_i)_{i=0}^m$, $a=:\mathcal{Z}_0<\mathcal{Z}_1<\cdots<\mathcal{Z}_{n-1}<\mathcal{Z}_m:=\mathfrak{b}$. Also, the scale of the partition Z_m is denoted by*

$$\mathcal{O}(Z_m):=max_{0\leq j\leq m-1}\frac{|J_{j\pm1}|}{|J_j|}\ ,where J_j=[\mathcal{Z}_j,\mathcal{Z}_{j+1}], \tag{3.4}$$

where $|J_j|$ is the length of the interval J_j.

Proof . For a large number n and let $\mathbb{S}_0$ be a convex in $\sum_{\mathfrak{r}+2,n}\cap\Delta^{(2)}$ and also a piecewise polynomial using Theorem 3.1 for which estimates(3.1)-(3.3) hold. Let $a:=x_{2n-1,2n}$, $\mathfrak{b}:=x_{1,2n}$ and let $Z_n=(\mathcal{Z}_i)_{i=0}^n$ be such that $\mathcal{Z}_0:=a$, $\mathcal{Z}_n:=\mathfrak{b}$ and $\mathcal{Z}_i:=x_{n-i}$, $1\leq i\leq n-1$ (note that $Z_n\subset T_{2n}$).

Obviously, $\mathbb{S}_0 \in \mathcal{Y}_{\mathfrak{r}+2}(Z_n)$, $\mathcal{O}(Z_n) \sim 1$, and by Lemma 2.5 implies that

$$\left\|\mathbb{S}_0 - \widetilde{\mathbb{S}}_0\right\|_{L_p(\widetilde{I}_j)} \le c(p,\mathfrak{r})\, w_{\mathfrak{r}+2}(\mathbb{S}_0, h_j, J_j)_{L_p(\widetilde{I}_j)}, \text{ where } \widetilde{I}_j := I_j \cap [a,\mathfrak{b}] \text{ and } J_j := [x_{j+2}, x_{j-2}] \cap [a,\mathfrak{b}], \tag{3.5}$$

and

$$\widetilde{\mathbb{S}}_0^{(v)}(a) = \widetilde{\mathbb{S}}_0^{(v)}(a) \text{ and } \widetilde{\mathbb{S}}_0^{(v)}(\mathfrak{b}) = \widetilde{\mathbb{S}}_0^{(v)},\ v = 0, 1. \tag{3.6}$$

Let

$$\mathbb{S}(x) := \begin{cases} \mathbb{S}_0(x), & \text{if } x \in [-1,1] \setminus [a,\mathfrak{b}], \\ \widetilde{\mathbb{S}}_0(x), & \text{if } x \in [a,\mathfrak{b}]. \end{cases}$$

Then $\mathbb{S} \in \sum_{\mathfrak{r}+2,2n}^{(1)} \cap \Delta^{(2)}$, so inequalities (3.2) and(3.3) are satisfied, if we put instead of $2n, n$ and (3.1) also satisfied. Since $\frac{\vartheta(x)}{n} \sim h_j$, ,for any $x \in J_j$, $1 \le j \le n$, for $x \in \widetilde{I}_j$, $1 \le j \le n$, we get

$$\begin{aligned} \|\mathcal{F} - \mathbb{S}\|_{L_p(\widetilde{I}_j)} &\le \|\mathcal{F}(x) - \mathbb{S}_0(x)\|_{L_p(\widetilde{I}_j)} + \left\|\mathbb{S}_0(x) - \widetilde{\mathbb{S}}_0(x)\right\|_{p(\widetilde{I}_j)} \\ &\le c(p)\, \|\mathcal{F} - \mathbb{S}_0\|_{p(J_j)} + c(p)\, w_{\mathfrak{r}+2}(\mathcal{F}, h_j; J_j)_{p(J_j)} \\ &\le c(p) h_j^{\mathfrak{r}} w_2\left(\mathcal{F}^{(\mathfrak{r})}, h_j\right)_p \le c(p) \left(\frac{\vartheta(x)}{n}\right)^{\mathfrak{r}} w_2\left(\mathcal{F}^{(\mathfrak{r})}, \frac{\vartheta(x)}{n}\right)_p. \end{aligned}$$

□

Theorem 3.3. *Let $\mathcal{F}$ be a convex function in $L_p[-1,1]$,then for $\mathfrak{r} \in N$, there exist a constant $c(p,\mathfrak{r})$ and $\mathcal{N}(\mathcal{F},\mathfrak{r})$, such that for every $n \ge \mathcal{N}$, there is a $\mathcal{P}_n \in \Pi_n \cap \Delta^{(2)}$ satisfying*

$$\|\mathcal{F}(x) - \mathcal{P}_n(x)\|_p \le c(p,\mathfrak{r}) \left(\frac{\vartheta(x)}{n}\right)^{\mathfrak{r}} w_2\left(\mathcal{F}^{(\mathfrak{r})}, \frac{\vartheta(x)}{n}\right)_p, x \in [-1,1]. \tag{3.7}$$

The following strong estimates are valid:

$$\|\mathcal{F}(x) - \mathcal{P}_n(x)\|_{L_p[-1,-1+n^{-2}] \cup [1-n^{-2},1]} \le c(p,\mathfrak{r}) \vartheta^{2\mathfrak{r}}(x) w_2\left(\mathcal{F}^{(\mathfrak{r})}, \frac{\vartheta(x)}{n}\right)_{L_p[-1,-1+n^{-2}] \cup [1-n^{-2},1]}, \tag{3.8}$$

and

$$\|\mathcal{F}(x) - \mathcal{P}_n(x)\|_{L_p[-1,-1+n^{-2}] \cup [1-n^{-2},1]} \le c(p,\mathfrak{r}) \vartheta^{2\mathfrak{r}}(x) w_1\left(\mathcal{F}^{(\mathfrak{r})}, \vartheta^2(x)\right)_{L_p[-1,-1+n^{-2}] \cup [1-n^{-2},1]}), \tag{3.9}$$

for $x \in [-1, -1+n^{-2}] \cup [1-n^{-2}, 1]$.

Proof . In the case $\mathfrak{r} \ge 2$, let $\mathbb{S}$ be the piecewise polynomial satisfies Theorem 3.2 Let us assume that $\mathbb{S}$ has no knots at x_1 and x_{n-1} (we shall treat $\mathbb{S}$ as a piecewise polynomial with knots at the chebyshev partition T_{2n}). Then

$$\mathcal{L}_1(x) := \mathbb{S}(x)|I_1 \cup I_2 = \mathcal{F}(1) + \frac{\mathcal{F}'(1)}{1!}(x-1) + \cdots + \frac{\mathcal{F}^{(\mathfrak{r})}(1)}{\mathfrak{r}!}(x-1)^{\mathfrak{r}} + a_+(n;\mathcal{F})(x-1)^{\mathfrak{r}+1}$$

and

$$\mathcal{L}_n(x) := \mathbb{S}(x)|I_n \cup I_{n-1} = \mathcal{F}(-1) + \frac{\mathcal{F}'(-1)}{1!}(x+1) + \cdots + \frac{\mathcal{F}^{(\mathfrak{r})}(-1)}{\mathfrak{r}!}(x+1)^{\mathfrak{r}} + a_-(n;\mathcal{F})(x+1)^{\mathfrak{r}+1},$$

where $a_+(n,\mathcal{F})$ and $a_-(n,\mathcal{F})$ are constants depending on n and $\mathcal{F}\cdot$ show that

$$n^{-2}\max\left\{\left|a_+\left(n,\mathcal{F}\right)\right|,\left|a_-\left(n,\mathcal{F}\right)\right|\right\}\to 0 \quad \text{as} \quad n\to\infty \tag{3.10}$$

.

From Theorem 3.1 (or Lemma 3.4), for all $x\in I_1\cup I_2$,

$$\begin{aligned}\left\|a_+(n,\mathcal{F})(1-x)\right\|_p &\le c(p)\frac{\left\|\mathcal{L}_1\left(x\right)-\mathcal{F}\left(x\right)\right\|_p}{(1-x)^{\mathfrak{r}}}\\ &\quad+\frac{c(p)}{(1-x)^{\mathfrak{r}}}\left\|\mathcal{F}\left(x\right)-\mathcal{F}\left(1\right)-\frac{\mathcal{F}'\left(1\right)}{1!}\left(x-1\right)-\cdots-\frac{\mathcal{F}^{(\mathfrak{r})}\left(1\right)}{\mathfrak{r}!}(x-1)^{\mathfrak{r}}\right\|_p\\ &\le c\left(p\right)w_1\left(\mathcal{F}^{(\mathfrak{r})},1-x\right)_p+\frac{1}{(\mathfrak{r}-1)\,!(1-x)^{\mathfrak{r}}}\left\|\int_x^1\left(\mathcal{F}^{(\mathfrak{r})}\left(\mathfrak{t}\right)-\mathcal{F}^{(\mathfrak{r})}\left(1\right)\right)(\mathfrak{t}-x)^{\mathfrak{r}-1}d\mathfrak{t}\right\|_p\\ &\le c\left(p\right)w_1\left(\mathcal{F}^{(\mathfrak{r})},1-x\right)_p,\end{aligned}$$

and, in particular, $n^{-2}\|a_+(n,\mathcal{F})\|_p\le c(p)w_1(\mathcal{F}^{(\mathfrak{r})},n^{-2})_p\to 0$ as $n\to\infty$. Similarly for $\|a_-(n,\mathcal{F})\|_p$.

For $\mathcal{F}\in L_{p^{\mathfrak{r}}}$, $\mathfrak{r}\ge 2$, let $i_+\ge 2$ is the small integer $2\le i\le\mathfrak{r}$, If it exists, such that $\mathcal{F}^{(i)}\left(1\right)\ne 0$, and let

$$\mathcal{D}_+\left(\mathfrak{r},\mathcal{F}\right)=\begin{cases}(2\mathfrak{r}!)^{-1}\left|\mathcal{F}^{(i_+)}(1)\right| & \text{if } i_+ \text{ exists,}\\ 0 & \text{otherwise.}\end{cases}$$

Similarly, let $i_-\ge 2$, be the smallest integer $2\le i\le\mathfrak{r}$, if it exists, such that $\mathcal{F}^{(i)}\left(-1\right)\ne 0$, and denote

$$\mathcal{D}_-\left(\mathfrak{r},\mathcal{F}\right)=\begin{cases}(2\mathfrak{r}!)^{-1}\left|\mathcal{F}^{(i_-)}(1)\right| & \text{if } i_-\text{exists,}\\ 0 & \text{otherwise}\end{cases}$$

Hence, if n is sufficiently large, then

$$\mathbb{S}''(x)\ge\mathcal{D}_+(\mathfrak{r},\mathcal{F})(1-x)^{\mathfrak{r}-2},\ x\in\left(x_2,1\right], \tag{3.11}$$

and

$$\mathbb{S}''(x)\ge\mathcal{D}_-(\mathfrak{r},\mathcal{F})(x+1)^{\mathfrak{r}-2}\ ,\ x\in[-1,x_{n-2}]. \tag{3.12}$$

In the case $\mathfrak{r}\ge 2$, let $\mathfrak{r}\in N$, $\mathfrak{r}\ge 2$, and a convex $\mathcal{F}\in L_p^{\mathfrak{r}}$, let $\mathcal{T}\in\Phi^2$ be such that $w_2\left(\mathcal{F}^{(\mathfrak{r})},\mathfrak{t}\right)\sim\mathcal{T}\left(\mathfrak{t}\right)$, let $\Gamma\left(\mathfrak{t}\right):=\mathfrak{t}^{\mathfrak{r}}\mathcal{T}\left(\mathfrak{t}\right)$, and note that $\Gamma\in\Phi^{\mathfrak{r}+2}$. For a large number $\mathcal{N}\in N$ and any $n\ge\mathcal{N}$, we suppose that the piecewise polynomial $\mathbb{S}\in\sum_{\mathfrak{r}+2},n$ of Theorem 3.2 satisfying (3.11),(3.12) and satisfies

$$w_{\mathfrak{r}+2}(\mathcal{F},\mathfrak{t})\le\mathfrak{t}^2w_2(\mathcal{F}^{(\mathfrak{r})},\mathfrak{t})\sim\Gamma(\mathfrak{t}).$$

So then by Lemma 2.1 with $\kappa=\mathfrak{r}+2$, we conclude that

$$\mathfrak{b}_{\mathfrak{r}+2}(\mathbb{S},\Gamma)\le c.$$

There using (3.5) and Lemma 2.6

$$\min_{x\in I_2}\mathbb{S}''(x)\ge\mathcal{D}_+(\mathfrak{r},\mathcal{F})|I_1|^{\mathfrak{r}-2}\ge 3^{-\mathfrak{r}+2}\mathcal{D}_+(\mathfrak{r},\mathcal{F})|I_2|^{\mathfrak{r}-2}.$$

Similarly, (3.6) yields

$$\min_{x\in I_2}\mathbb{S}''(x)\ge 3^{-\mathfrak{r}+2}\mathcal{D}_-(\mathfrak{r},\mathcal{F})|I_{n-1}|^{\mathfrak{r}-2}.$$

Then by Theorem 2.7 if $\kappa=\mathfrak{r}+2$, $d_{+}:=3^{-\mathfrak{r}+2}\mathcal{D}_{+}(\mathfrak{r},\mathcal{F})$, $d_{-}:=3^{-\mathfrak{r}+2}\mathcal{D}_{-}(\mathfrak{r},\mathcal{F})$ and $\mathcal{S}=2\kappa-2=2\mathfrak{r}+2$, so that there exists a polynomial $\mathcal{P}\in\Pi_{cn}\cap\Delta^{(2)}$such that:

$$\|\mathbb{S}(x)-\mathcal{P}(x)\|_{p}\leq c(p)\,\delta_{n}^{2\mathfrak{r}+2}(x)\,\Omega_{n}^{\mathfrak{r}}(x)\,\mathcal{T}(\Omega_{n}(x)),\quad x\in[-1,1]. \tag{3.13}$$

So for $x\in I_{1}\cup I_{n}$, $x\neq-1,1$, by $\Omega_{n}(x)\sim n^{-2}$for these x, and $\mathfrak{t}^{-2}\mathcal{T}(\mathfrak{t})$ is non-increasing we have

$$\begin{aligned}\|\mathbb{S}(x)-\mathcal{P}(x)\|_{p}&\leq c(p)(n\vartheta(x))^{2\mathfrak{r}+2}\Omega_{n}^{\mathfrak{r}}(x)\mathcal{T}(\Omega_{n}(x))\\&\leq c(p)n^{2}\vartheta^{2\mathfrak{r}+2}(x)(\frac{n\Omega_{n}(x)}{\vartheta(x)})^{2}\mathcal{T}(\frac{\vartheta(x)}{n})\\&\leq c(p)\vartheta^{2\mathfrak{r}}(x)w_{2}(\mathcal{F}^{(\mathfrak{r})},\frac{\vartheta(x)}{n})_{p}.\end{aligned} \tag{3.14}$$

In turn, this implies for $x\in I_{1}\cup I_{n}$, that

$$\|\mathbb{S}(x)-\mathcal{P}(x)\|_{p}\leq c(p)\left(\frac{\vartheta}{n}\right)^{\mathfrak{r}}w_{2}\left(\mathcal{F}^{(\mathfrak{r})},\frac{\vartheta(x)}{n}\right)_{p},x\in[-1,1]. \tag{3.15}$$

Now , (3.15) together with (3.1) yield

$$\|\mathcal{F}(x)-\mathcal{P}_{n}(x)\|_{p}\leq c(p,\mathfrak{r})\left(\frac{\vartheta(x)}{n}\right)^{\mathfrak{r}}w_{2}\left(\mathcal{F}^{(\mathfrak{r})},\frac{\vartheta(x)}{n}\right)_{p},\ x\in[-1,1].$$

and (3.14) together with (3.8) yield

$$\|\mathcal{F}(x)-\mathcal{P}_{n}(x)\|_{p}\leq c(p,\mathfrak{r})\,\vartheta^{2\mathfrak{r}}(x)\,w_{2}\left(\mathcal{F}^{(\mathfrak{r})},\frac{\vartheta(x)}{n}\right)_{p},\ x\in[-1,1].$$

Now to prove $\|\mathcal{F}(x)-\mathcal{P}_{n}(x)\|_{p}\leq c(p,\mathfrak{r})\vartheta^{2\mathfrak{r}}(x)w_{1}(\mathcal{F}^{(\mathfrak{r})},\vartheta^{2}(x))_{p}$, using that $\mathfrak{t}^{-1}w_{1}(\mathcal{F}^{(\mathfrak{r})},\mathfrak{t})$ is non-increasing we have, for $x\in I_{1}\cup I_{n}$, $x\neq-1,1$,

$$\begin{aligned}\|\mathbb{S}(x)-\mathcal{P}(x)\|_{p}&\leq c(p)(n\vartheta(x))^{2\mathfrak{r}+2}\Omega_{n}^{\mathfrak{r}}(x)w_{1}(\mathcal{F}^{(\mathfrak{r})},\Omega_{n}(x))_{p}\\&\leq c(p)n^{2}\vartheta^{2\mathfrak{r}+2}(x)\frac{\Omega_{n}(x)}{\vartheta^{2}(x)}w_{1}(\mathcal{F}^{(\mathfrak{r})},\vartheta^{2}(x))_{p}\end{aligned}$$

$$\|\mathbb{S}(x)-\mathcal{P}(x)\|_{p}\leq c(p)\,\vartheta^{2\mathfrak{r}}(x)\,w_{1}\left(\mathcal{F}^{(\mathfrak{r})},\vartheta^{2}(x)\right)_{p}.$$

In case $\mathfrak{r}$=1, let us define a convex polynomial $\mathcal{P}_{n}$ the approximates the quadratic spline $\mathbb{S}$ from Theorem 3.1 (with $\mathfrak{r}=1$) so that

$$\|\mathbb{S}(x)-\mathcal{P}_{n}(x)\|_{p}\leq c(p)w_{3}(\mathcal{F},\Omega_{n}(x)),$$

and

$$\mathcal{P}_{n}(\pm1)=\mathbb{S}(\pm1)\quad\text{and}\quad p'_{n}(\pm1)=\mathbb{S}'(\pm1). \tag{3.16}$$

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To construct the above polynomial, we shall use away similar to that in [2] by replacing $\mathcal{F}$ by $\mathbb{S}$ and n by $2n$. □

References

[1] R.A. DeVore and G.G. Lorentz, *Constructive Approximation*, Grundlehren Math. Wiss. [Fundam. Principles Math. Sci.], vol. 303 Springer-Verlag, Berlin 1993.

[2] K.A. Kopotun, *Pointwise and uniform estimates for convex approximation of functions by algebraic polynomial*, Constr. Approx. 10 (1994) 153–178.

[3] D. Leviatan and I.A. Shevchuk, *Monotone approximation estimates involving the third modulus of smoothness*, Approx. Theory IX, Vol. I. (Nashville, TN, 1998) Innov. Appl. Math. Vanderbilt Univ. Press Nashville, (1998) 223–230.

[4] L.P. Yushchenko, *On one counterexample in convex approximation*, Ukrain. Math. J. 52(12) (2000) 1956–1963.

[5] A.V. Bondarenko and A.V. Prymak, *Negative results in shape-preserving higher-order approximations*, Mat. Zametki 76 (2004) 6 812–823.

[6] K.A .Kopotun, *Interpolatory estimates for convex piecewise polynomial approximation*, J. Math. Anal. Appl. 474(1) (2019) 467–479.

[7] H.H. Gonska, D. Leviatan, I.A. Shevchuk and H.-J. Wenz, *Interpolatory pointwise estimates for polynomial approximations*, Constr. Approx. 16 (2000) 603–629.

[8] K.A. Kopotun, D. Leviatan, A. Prymak and I.A. Shevchuk, *Uniform and pointwise shape preserving approximation by algebraic polynomials*, Surv. Approx. Theory 6 (2011) 24–74.

[9] T.O. Petrova, *A counterexample to convex interpolation approximation*, Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos 35 (2002): 107–112.

[10] R.A. DeVore and X.M. Yu, *Pointwise estimates for monotone polynomial approximation*, Constr. Approx. 1 (1985) 323–331.

[11] D. Leviatan, *Pointwise estimates for convex polynomial approximation*, Proc. Amer. Math. Soc. 98(3)(1986) 471–474.

[12] X.M. Yu, *Pointwise estimates for convex polynomial approximation*, Approx. Theory Appl. 1 4 (1985) 65–74.

[13] J.D. Cao and H.H. Gonska, *Pointwise estimates for higher order convexity preserving polynomial approximation*, J. Aust. Math. Soc. Ser. B 36(2) (1994) 213–233.

[14] K.A. Kopotun, D. Leviatan and I.A. Shevchuk, *Interpolatory pointwise estimates for monotone polynomial approximation*, J. Math. Anal. Appl. 459(2) (2018) 1260–1295.

[15] G.A. Dzyubenko, D. Leviatan and I.A. Shevchuk, *Pointwise estimates of coconvex approximation*, Jaen J. Approx. 6 (2014) 261–295.

[16] K. A. Kopotun, D. Leviatan, I.L. Petrova and I.A. Shevchuk, *Interpolatory pointwise estimates for convex polynomial approximation*, Acta Math. Hungar. 163(1) (2020) 85–117.