Int. J. Nonlinear Anal. Appl. 4 (2013) No. 2, 89-101
ISSN: 2008-6822 (electronic)
http://www.ijnaa.semnan.ac.ir

A fixed point approach to the Hyers-Ulam stability of an AQ functional equation in probabilistic modular spaces

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(Communicated by Themistocles M. Rassias)

Abstract

In this paper, we prove the Hyers-Ulam stability in β-homogeneous probabilistic modular spaces via fixed point method for the functional equation

$$f(x+ky)+f(x-ky)=f(x+y)+f(x-y)+\frac{2(k+1)}{k}f(ky)-2(k+1)f(y)$$

for fixed integers k with $k \neq 0, \pm 1$.

Keywords: Hyers-Ulam Stability, AQ functional Equation, Fixed Point, Probabilistic Modular Space.
2010 MSC: 39B52, 47H10, 46S50, 54E70, 39B72, 47H09.

1. Introduction and preliminaries

More than a half century ago, Ulam [32] posed the famous Ulam stability problem which was partially solved by Hyers [10] in the framework of Banach spaces. The Hyers-theorem was generalized by

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Received: October 2012 *Revised:* May 2013

Aoki [2] for additive mappings. In the year 1978, a generalized solution for approximately linear mappings was provided by Rassias [29]. He considered a mapping $f : X \to Y$ satisfying the condition $\|f(x_1 + x_2) - f(x_1) - f(x_2)\| \leq \epsilon(\|x_1\|^p + \|x_2\|^p)$ for all $x_1, x_2 \in X$, where $\epsilon \geq 0$ and $0 \leq p < 1$. This theorem was later extended for $p \neq 1$ and generalized by several mathematicians (cf. Gajda [6], Rassias and Semrl [30] and [3, 4, 7, 8, 11, 12, 13, 14, 15, 18, 28]).
The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.1}$$

is related to a symmetric bi-additive function (cf. [1, 16]). It is natural that (1.1) is called a quadratic functional equation. In particular, every solution of the quadratic (1.1) is said to be a quadratic function. For other types of quadratic functional equations (cf. [3, 5, 13]).
Now, we consider a mapping $f : X \to Y$ that satisfies the following general mixed additive and quadratic ("AQ" for short) functional equation:

$$f(x + ky) + f(x - ky) = f(x + y) + f(x - y) + \frac{2(k + 1)}{k} f(ky) - 2(k + 1)f(y), \tag{1.2}$$

for fixed integers k with $k \neq 0, \pm 1$. It is easy to see that the function $f(x) = ax^2 + bx$ is a solution of the functional equation (1.2).
In the present paper, we will study the Hyers-Ulam stability of the given equation (1.2) from linear spaces into *probabilistic modular spaces* by applying the fixed point theorem in *modular spaces* (see Theorem 1.8).

Definition 1.1. *Let $\mathcal{X}$ be an arbitrary vector space.*
(a) A functional $\rho : \mathcal{X} \to [0, \infty]$ is called modular if for arbitrary $x, y \in \mathcal{X}$,
(i) $\rho(x) = 0$ if and only if $x = 0$,
(ii) $\rho(\alpha x) = \rho(x)$ for every scaler α with $|\alpha| = 1$,
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,
(b) if (iii) is replaced by
(iii)$'$ $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,
then we say that ρ is convex modular.

The vector subspace $\mathcal{X}_\rho = \{x \in \mathcal{X} : \quad \rho(\nu x) \to 0 \text{ as } \nu \to 0\}$ of $\mathcal{X}$ is called a modular space. Let ρ be a convex modular, the modular space $\mathcal{X}_\rho$ can be equipped with a norm called the Luxemburg norm, defined by

$$\|x\|_\rho = \inf \left\{ \nu > 0 \quad ; \quad \rho\left(\frac{x}{\nu}\right) \leq 1 \right\}.$$

A function modular is said to satisfy the Δ_2-condition if there exists $\ell > 0$ such that $\rho(2x) \leq \ell\rho(x)$ for all $x \in \mathcal{X}_\rho$.

Definition 1.2. *Let $\{x_n\}$ and x be in $\mathcal{X}_\rho$. Then*
(i) the sequence $\{x_n\}$, with $x_n \in \mathcal{X}_\rho$, is ρ-convergent to x and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.
(ii) The sequence $\{x_n\}$, with $x_n \in \mathcal{X}_\rho$, is called ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.
(iii) A subset $\mathcal{B}$ of $\mathcal{X}_\rho$ is called ρ-complete if and only if any ρ-Cauchy sequence is ρ-convergent to an element of $\mathcal{B}$.

The modular ρ has the Fatou property if and only if $\rho(x) \leq \liminf_{n\to\infty} \rho(x_n)$ whenever the sequence $\{x_n\}$ is ρ-convergent to x.

Remark 1.3. *Note that ρ is an increasing function. Suppose $0 < a < b$. Then the property (iii) of Definition 1.1 with $y = 0$ shows that $\rho(ax) = \rho\left(\frac{a}{b}bx\right) \leq \rho(bx)$ for all $x \in \mathcal{X}$. Moreover, if ρ is a convex modular on $\mathcal{X}$ and $|\alpha| \leq 1$, then $\rho(\alpha x) \leq \alpha\rho(x)$ and also $\rho(x) \leq \frac{1}{2}\rho(2x)$ for all $x \in \mathcal{X}$.*

The theory of modular spaces was, in fact, initiated by Nakano [25] and generalized by Musielak [24] and Orlicz [27]. for more, the reader is referred to [19, 20, 21, 22, 31, 33]. On the other hand, in 1942 a generalization of the notion of metric space was introduced by Menger [23] under the name of statistical metric space which is now called probabilistic metric space. Such a probabilistic generalization of metric space appears when there is an uncertainty about the distance between the points and we know only the probabilities of possible values this distance. After the appearance of Menger's paper, the study of probabilistic metric spaces was performed rapidly by many authors in theory and application, and many concepts and results in classical functional analysis obtained some generalizations and counterparts in probabilistic functional analysis. In [9], after introducing the probabilistic modular, authors then investigated some basic facts in such spaces and study linear operators defined between them.
In the following, Δ stands for the set of all non-decreasing functions $f : \mathbb{R} \to \mathbb{R}_0^+$ satisfying $\inf_{t\in\mathbb{R}} f(t) = 0$, and $\sup_{t\in\mathbb{R}} f(t) = 1$. We also denote the function min by $\wedge$.

Definition 1.4. *A pair (X, μ) is called a probabilistic modular space ($\mathcal{PM}$-space) if X is a real vector space, and μ is a mapping from X into Δ satisfying the following conditions:*
(1) $\mu(x)(0) = 0$;
(2) $\mu(x)(t) = 1$ *for all* $t > 0$, *if and only if* $x = \theta$ (θ *is the null vector in* X);
(3) $\mu(-x)(t) = \mu(x)(t)$;
(4) $\mu(\alpha x + \beta y)(s + t) \geq \mu(x)(s) \wedge \mu(y)(t)$, *for all* $x, y \in X$, *and* $\alpha, \beta, s, t \in \mathbb{R}_0^+$, $\alpha + \beta = 1$.

We say (X, μ) is β-homogeneous, where $\beta \in (0, 1]$ if,

$$\mu(\alpha x)(t) = \mu(x)\left(\frac{t}{|\alpha|^\beta}\right)$$

for every $x \in X$, $t > 0$, and $\alpha \in \mathbb{R} \setminus \{0\}$.

Example 1.5. *Suppose that X is a real vector space and ρ is a modular on X. Define*

$$\mu(x)(t) = \begin{cases} 0, & t \leq 0 \\ \frac{t}{t+\rho(x)}, & t > 0 \end{cases}.$$

Then (X, μ) is a probabilistic modular space.

Definition 1.6. *Let (X, μ) be a $\mathcal{PM}$-space, $\{x_n\}$ a sequence in X and $x \in X$. Then*
(i) the sequence $\{x_n\}$, with $x_n \in (X, \mu)$, is μ-convergent to x and write $x_n \xrightarrow{\mu} x$, if for every $t > 0$ and $\lambda \in (0, 1)$, there exists a positive integer n_0 such that $\mu(x_n - x)(t) > 1 - \lambda$ for all $n \geq n_0$.
(ii) the sequence $\{x_n\}$, with $x_n \in (X, \mu)$, is μ-Cauchy, if for every $t > 0$ and $\lambda \in (0, 1)$, there exists a positive integer n_0 such that $\mu(x_n - x_m)(t) > 1 - \lambda$ for all $m, n \geq n_0$.

By [9], every μ-convergent sequence in a $\mathcal{PM}$-space is a μ-Cauchy sequence. If each μ-Cauchy sequence is μ-convergent in a $\mathcal{PM}$-space (X, μ), then (X, μ) is called a *μ-complete $\mathcal{PM}$-space*.

A $\mathcal{PM}$-space (X,μ) possesses Fatou property if for any sequence $\{x_n\}$ of X μ-converging to x, we have

$$\mu(x)(t) \geq \limsup_{n\geq 1} \mu(x_n)(t)$$

for each $t > 0$.

Remark 1.7. *Note that for any $x \in X$, $\mu(x)(.)$ is an increasing function, since $\mu(x) \in \Delta$. Moreover, if μ is a β-homogeneous probabilistic modular on X and $x, y \in X$, then the property (4) of Definition 1.4 shows that*

$$\mu(x+y)\left(2^\beta(s+t)\right) = \mu\left(\frac{1}{2}x + \frac{1}{2}y\right)(s+t) \geq \mu(x)(s) \wedge \mu(y)(t).$$

For more details about the $\mathcal{PM}$-space, the readers refer to [26].
Our aim is based on the fixed point approach:

Theorem 1.8 ([17]). *Let X_ρ be a modular space such that ρ satisfies the Fatou property. Let $\mathcal{C}$ be a ρ-complete nonempty subset of X_ρ and let $T : \mathcal{C} \to \mathcal{C}$ be a quasicontraction, that is, there exists a $K < 1$ such that*

$$\rho(T(x) - T(y)) \leq K \max\{\rho(x-y), \rho(x - T(x)), \rho(y - T(y)), \rho(x - T(y)), \rho(y - T(x))\}.$$

Let $x \in \mathcal{C}$ such that

$$\delta_\rho(x) := \sup\{\rho(T^n(x) - T^m(x)) : m, n \in \mathbb{N}\} < \infty.$$

Then $T^n(x)$ ρ-converges to $\omega \in \mathcal{C}$. Moreover, if $\rho(\omega - T(\omega)) < \infty$ and $\rho(x - T(\omega)) < \infty$, then the ρ-limit of $T^n(x)$ is a fixed point of T. Furthermore, if ω^* is any fixed point of T in $\mathcal{C}$ such that $\rho(\omega - \omega*) < \infty$, then one has $\omega = \omega*$.

In the rest of this paper, we will assume that μ is a probabilistic modular on X with the Fatou property (in the probabilistic modular sense) and (X,μ) is a μ-complete β-homogeneous $\mathcal{PM}$-space with $\beta \in (0,1]$.

2. Approximate Mixed Additive and Quadratic Mappings

Theorem 2.1. *Let $\jmath \in \{-1, 1\}$ be fixed. Let $\mathcal{E}$ be a linear space and (X,μ) a μ-complete β-homogeneous $\mathcal{PM}$-space. Suppose that an odd mapping $f : \mathcal{E} \to (X,\mu)$ satisfies the condition $f(0) = 0$ and an inequality of the form*

$$\begin{aligned}\mu\Big(f(x+ky) + f(x-ky) - f(x+y) - f(x-y) - \frac{2(k+1)}{k} f(ky) \\ + 2(k+1)f(y)\Big)(t) \geq \psi(x,y)(t),\end{aligned} \tag{2.1}$$

for all $x, y \in \mathcal{E}$, where $\psi : \mathcal{E} \times \mathcal{E} \to \Delta$ is a given function such that

$$\psi(0, k^\jmath x)(k^{\beta\jmath} Lt) \geq \psi(0,x)(t)$$

for all $x \in \mathcal{E}$ and has the property

$$\lim_{n\to\infty} \psi(k^{\jmath n}x, k^{\jmath n}y)(k^{\beta\jmath n}t) = 1 \tag{2.2}$$

for all $x, y \in \mathcal{E}$ and a constant $0 < L < \frac{1}{2^\beta}$. Then there exists a unique additive mapping $A : \mathcal{E} \to (X, \mu)$ satisfies (1.2) *and*

$$\mu\Big(A(x) - f(x)\Big)\left(\frac{t}{(k+1)^\beta L^{\frac{\jmath-1}{2}}(1-2^\beta L)}\right) \geq \psi(0,x)(t) \tag{2.3}$$

for all $x \in \mathcal{E}$.

Proof . Putting $x = 0$ in (2.1), we get by the oddness of f,

$$\mu\left(\frac{2(k+1)}{k} f(ky) - 2(k+1)f(y)\right)(t) \geq \psi(0,y)(t)$$

for all $y \in \mathcal{E}$. Therefore,

$$\begin{aligned}\mu\left(\frac{1}{k}f(ky) - f(y)\right)(t) = \mu\Big(&\tfrac{2(k+1)}{k}f(ky) - 2(k+1)f(y)\Big)(2^\beta(k+1)^\beta t)\\ &\geq \psi(0,y)(2^\beta(k+1)^\beta t)\end{aligned}$$

for all $y \in \mathcal{E}$. Replacing y by x in the above inequality, we have

$$\mu\left(\frac{1}{k}f(kx) - f(x)\right)(t) \geq \psi(0,x)\left(2^\beta(k+1)^\beta t\right) \tag{2.4}$$

for all $x \in \mathcal{E}$. Replacing x by $k^{-1}x$ in (2.4), we obtain

$$\begin{aligned}\mu\left(\frac{f(k^{-1}x)}{k^{-1}} - f(x)\right)(t) &= \mu\left(\frac{f(x)}{k} - f(k^{-1}x)\right)\left(\frac{t}{k^\beta}\right) \\ &\geq \psi(0,k^{-1}x)\left(2^\beta(k+1)^\beta L^{-1}\frac{Lt}{k^\beta}\right) \\ &\geq \psi(0,x)\left(2^\beta(k+1)^\beta L^{-1}t\right).\end{aligned} \tag{2.5}$$

By (2.4) and (2.5),

$$\mu\left(\frac{f(k^\jmath x)}{k^\jmath} - f(x)\right)(t) \geq \varphi(x)(t) := \psi(0,x)\left(2^\beta(k+1)^\beta L^{\frac{\jmath-1}{2}}t\right) \tag{2.6}$$

for all $x \in \mathcal{E}$. Consider the set $\mathcal{N} := \{g : \mathcal{E} \to (X,\mu) | \quad g(0) = 0\}$ and introduce the modular ρ on $\mathcal{N}$ as follows,

$$\rho(g) = \inf\{r > 0 : \mu(g(x))(rt) \geq \varphi(x)(t)\}.$$

It is clear that ρ is even and $\rho(0) = 0$. If $\rho(g) = 0$, then for each $r > 0$, $\mu(g(x))(rt) \geq \varphi(x)(t)$ for all $t > 1$ and all $x \in \mathcal{E}$. If we let $t \to +\infty$ and $\epsilon = rt$ is fixed, then $\mu(g(x))(\epsilon) = 1$, which implies that $g = 0$. Now let $\varepsilon > 0$ be given. Then there exist $r_1 > 0$ and $r_2 > 0$ such that

$$r_1 \leq \rho(g) + \varepsilon; \quad \mu(g(x))(r_1 t) \geq \varphi(x)(t)$$

and

$$r_2 \leq \rho(h) + \varepsilon; \quad \mu(h(x))(r_2 t) \geq \varphi(x)(t).$$

If $\alpha+\beta=1$ and $\alpha,\beta\geq 0$, then we have

$$\mu(\alpha g(x)+\beta h(x))(r_1t+r_2t)\geq \mu(g(x))(r_1t)\wedge \mu(h(x))(r_2t)\geq \varphi(x)(t),$$

and so

$$\rho(\alpha g+\beta h)\leq r_1+r_2\leq \rho(g)+\rho(h)+2\varepsilon.$$

This shows that $\rho(\alpha g+\beta h)\leq \rho(g)+\rho(h)$ if $\alpha+\beta=1$ and $\alpha,\beta\geq 0$.
Now, we are going to prove that ρ satisfies the Δ_2-condition with $\ell=2^{\beta}$. For $\varepsilon>0$ given, there exists $r>0$ such that

$$r\leq \rho(g)+\varepsilon;\quad \mu(g(x))(r\,t)\geq \varphi(x)(t).$$

Since (X,μ) is a β-homogeneous $\mathcal{PM}$-space, we get

$$\mu(2g(x))(2^{\beta}r\,t)=\mu(g(x))(r\,t)\geq \varphi(x)(t),$$

and so $\rho(2g)\leq 2^{\beta}r\leq 2^{\beta}\,\rho(g)+2^{\beta}\,\varepsilon$. Hence $\rho(2g)\leq 2^{\beta}\,\rho(g)$. Thus ρ satisfies the Δ_2-condition with $\ell=2^{\beta}$.
Next, we show that ρ satisfies the Fatou property (in the modular sense). To do this, let $\{g_n\}$ be a ρ-convergent in $\mathcal{N}$ to g. We can easily show that $\{g_n(x)\}$ μ-converging to $g(x)$ for any $x\in\mathcal{E}$. Let $\wp:=\liminf_{n\to\infty}\rho(g_n)<\infty$ and $\rho(g)>\wp$. We have $\mu(g(x))(\wp t)<\varphi(x)(t)$ for all $t>0$. Since μ satisfies the Fatou property (in the probabilistic modular sense), we see that

$$\limsup_{n\geq 1}\mu(g_n(x))(\wp t)\leq \mu(g(x))(\wp t)<\varphi(x)(t),\qquad \forall t>0.$$

This shows that there exists a positive integer $n_0\in\mathbb{N}$ such that $\mu(g_n(x))(\wp t)<\varphi(x)(t)$ and so $\rho(g_n)>\wp$ for all $n\geq n_0$. Thus $\liminf_{n\to\infty}\rho(g_n)>\wp$, which is a contradiction. Therefore, ρ satisfies the Fatou property.
Let $\delta>0$ and $0<\lambda<1$. Since $\varphi(x)\in\Delta$, there exists $t_0>0$ such that $\varphi(x)(t_0)>1-\lambda$. Let $\{A_n\}$ be a ρ-Cauchy sequence in $\mathcal{N}_\rho$ and let $\varepsilon<\frac{\delta}{t_0}$ be given. There exists a positive integer $n_0\in\mathbb{N}$ such that $\rho(A_n-A_m)\leq\varepsilon$ for all $n,m\geq n_0$. By definition of the modular ρ, we obtain

$$\mu\Big(A_n(x)-A_m(x)\Big)(\delta)\geq \mu\Big(A_n(x)-A_m(x)\Big)(\varepsilon t_0)\geq \varphi(x)(t_0)>1-\lambda, \tag{2.7}$$

for all $x\in\mathcal{E}$ and $n,m\geq n_0$. Let $x\in\mathcal{E}$ be an arbitrary point. Then (2.7) implies that $\{A_n(x)\}$ is a μ-Cauchy sequence in (X,μ). Since (X,μ) is μ-complete, $\{A_n(x)\}$ is μ-convergent in (X,μ) for all $x\in\mathcal{E}$. Thus, we can define a mapping $A:\mathcal{E}\to(X,\mu)$ by

$$A(x)=\lim_{n\to\infty}A_n(x),$$

for any $x\in\mathcal{E}$. Letting $m\to\infty$, inequality (2.7) implies that

$$\rho(A_n-A)\leq\varepsilon$$

for all $n\geq n_0$, since μ has the Fatou property. Thus $\{A_n\}$ is ρ-convergent sequence in $\mathcal{N}_\rho$. Therefore $\mathcal{N}_\rho$ is ρ-complete.
Now, we consider the mapping $\mathcal{T}:\mathcal{N}_\rho\to\mathcal{N}_\rho$ as follows:

$$\mathcal{T}A(x):=\frac{A(k^{\jmath}x)}{k^{\jmath}}$$

for all $A \in \mathcal{N}_\rho$. Let $g, h \in \mathcal{N}_\rho$ and let $r > 0$ be an arbitrary constant with $\rho(g-h) \le r$. From the definition of ρ, we get

$$\mu\Big(g(x) - h(x)\Big)(r\,t) \ge \varphi(x)(t)$$

for all $x \in \mathcal{E}$. By the assumption and the last inequality, we have

$$\begin{aligned} \mu\left(\mathcal{T}g(x) - \mathcal{T}h(x)\right)(Lr\,t) &= \mu\left(k^{-\jmath}g(k^\jmath x) - k^{-\jmath}h(k^\jmath x)\right)(Lr\,t) \\ &= \mu(g(k^\jmath x) - h(k^\jmath x))(k^{\beta\jmath}Lr\,t) \\ &\ge \varphi(k^\jmath x)(k^{\beta\jmath}L\,t) \\ &\ge \varphi(x)(t) \end{aligned}$$

for all $x \in \mathcal{E}$. Hence $\rho(\mathcal{T}g - \mathcal{T}h) \le L\rho(g-h)$, for all $g, h \in \mathcal{N}_\rho$, that is, $\mathcal{T}$ is a ρ-strict contraction. By replacing x by $k^\jmath x$ in (2.6), we deduce that

$$\mu\Big(k^{-\jmath}f(k^{2\jmath}x) - f(k^\jmath x)\Big)(t) \ge \varphi(k^\jmath x)(t)$$

that is

$$\begin{aligned} &\mu\Big(k^{-2\jmath}f(k^{2\jmath}x) - k^{-\jmath}f(k^\jmath x)\Big)(Lt) \\ &= \mu\Big(k^{-\jmath}f(k^{2\jmath}x) - f(k^\jmath x)\Big)(k^{\beta\jmath}Lt) \\ &\ge \varphi(k^\jmath x)(k^{\beta\jmath}Lt) \ \ge\ \varphi(x)(t). \end{aligned}$$

Hence

$$\begin{aligned} &\mu\left(\tfrac{f(k^{2\jmath}x)}{k^{2\jmath}} - f(x)\right)\left(2^\beta(Lt+t)\right) \\ \ge\ &\mu\left(\tfrac{f(k^{2\jmath}x)}{k^{2\jmath}} - \tfrac{f(k^\jmath x)}{k^\jmath}\right)(Lt) \wedge \mu\left(\tfrac{f(k^\jmath x)}{k^\jmath} - f(x)\right)(t) \\ &\ge \varphi(x)(t) \end{aligned} \tag{2.8}$$

for all $x \in \mathcal{E}$. Replacing x and $2^\beta(Lt+t)$ by $k^\jmath x$ and $k^{\beta\jmath}2^\beta(L^2t + Lt)$ in (2.8), respectively, we find that

$$\begin{aligned} &\mu\Big(k^{-2\jmath}f(k^{3\jmath}x) - f(k^\jmath x)\Big)\left(k^{\beta\jmath}2^\beta(L^2t+Lt)\right) \\ &\ge \varphi(k^\jmath x)\left(k^{\beta\jmath}Lt\right) \ \ge\ \varphi(x)(t). \end{aligned}$$

and so

$$\mu\Big(k^{-3\jmath}f(k^{3\jmath}x) - k^{-\jmath}f(k^\jmath x)\Big)\left(2^\beta(L^2t+Lt)\right) \ge \varphi(x)(t).$$

Hence,

$$\begin{aligned} &\mu\left(\frac{f(k^{3\jmath}x)}{k^{3\jmath}} - f(x)\right)\left(2^\beta\left\{2^\beta(L^2t+Lt)+t\right\}\right) \\ &\ge \mu\left(\frac{f(k^{3\jmath}x)}{k^{3\jmath}} - \frac{f(k^\jmath x)}{k^\jmath}\right)\left(2^\beta(L^2t+Lt)\right) \wedge \mu\left(\frac{f(k^\jmath x)}{k^\jmath} - f(x)\right)(t) \\ &\ge \varphi(x)(t) \end{aligned}$$

for all $x \in \mathcal{E}$. By induction, one can check that

$$\mu\left(\frac{f(k^{\jmath n}x)}{k^{\jmath n}} - f(x)\right)\left(\left\{(2^\beta L)^{n-1} + 2^\beta \sum_{i=1}^{n-1}(2^\beta L)^{i-1}\right\} t\right) \geq \varphi(x)(t),$$

for all $x \in \mathcal{E}$, and so

$$\rho(\mathcal{T}^n f - f) \leq (2^\beta L)^{n-1} + 2^\beta \sum_{i=1}^{n-1}(2^\beta L)^{i-1} \leq 2^\beta \sum_{i=1}^{n}(2^\beta L)^{i-1} \leq \frac{2^\beta}{1-2^\beta L}. \tag{2.9}$$

Next, we show that $\delta_\rho(f) = \sup\{\rho(\mathcal{T}^n(f) - \mathcal{T}^m(f))\,; n, m \in \mathbb{N})\} < \infty$. To do this, Since ρ satisfies the Δ_2-condition with $\ell = 2^\beta$, it follows from (2.9) that

$$\begin{aligned} \rho\left(\mathcal{T}^n f - \mathcal{T}^m f\right) &\leq \tfrac{1}{2}\rho\left(2\mathcal{T}^n f - 2f\right) + \tfrac{1}{2}\rho\left(2\mathcal{T}^m f - 2f\right) \\ &\leq \tfrac{\ell}{2}\rho\left(\mathcal{T}^n f - f\right) + \tfrac{\ell}{2}\rho\left(\mathcal{T}^m f - f\right) \\ &\leq \tfrac{2^{2\beta}}{1-2^\beta L} \end{aligned} \tag{2.10}$$

for all $n, m \in \mathbb{N}$. According to the above inequality, we have $\delta_\rho(f) < \infty$. Due to Theorem 1.8, we get $\{\mathcal{T}^n(f)\}$ is ρ-converges to $A \in \mathcal{N}_\rho$. Since ρ has the Fatou property, the inequality (2.9) gives $\rho(\mathcal{T}A - f) < \infty$.
Setting $m = n + 1$ in the inequality (2.10), we have

$$\rho\left(\mathcal{T}^{n+1} f - \mathcal{T}^n f\right) \leq \frac{2^{2\beta}}{1-2^\beta L}.$$

Hence, we determine that $\rho(\mathcal{T}A - A) \leq (2^{2\beta}/1 - 2^\beta L) < \infty$. By using Theorem 1.8, we have ρ-limit of $\{\mathcal{T}^n(f)\}$ i.e., $A \in \mathcal{N}_\rho$ is a fixed point of the map $\mathcal{T}$. Let us replace x and y by $k^{\jmath n}x$ and $k^{\jmath n}y$ in inequality (2.1), respectively. Then we obtain

$$\mu\Big(\frac{1}{k^{\jmath n}}\Big\{f\left(k^{\jmath n}(x+ky)\right)+f\left(k^{\jmath n}(x-ky)\right)-\left(f\left(k^{\jmath n}(x+y)\right)+f\left(k^{\jmath n}(x-y)\right)\right) \qquad -\frac{2(k+1)}{k}f(k^{\jmath n}(ky))+2(k+1)$$

for all $x, y \in \mathcal{E}$. As n tends to infinity, the right-hand side of the above inequality tends to one; so we conclude that A satisfying the equation (1.2), that is, A is additive. From (2.9), we have

$$\rho(A - f) \leq \frac{2^\beta}{1-2^\beta L}.$$

Hence

$$\mu\Big(A(x) - f(x)\Big)\left(\frac{2^\beta}{1-2^\beta L}\,t\right) \geq \varphi(x)(t) = \psi(0,x)\left(2^\beta(k+1)^\beta L^{\frac{\jmath-1}{2}} t\right),$$

and so

$$\mu\Big(A(x) - f(x)\Big)\left(\frac{t}{(k+1)^\beta L^{\frac{\jmath-1}{2}}(1-2^\beta L)}\right) \geq \psi(0,x)(t).$$

Hence (2.3) holds. Lastly, we prove that A is unique. To do this, let A^* be another fixed point of $\mathcal{T}$. Then

$$\rho(A - A^*) \leq \frac{1}{2}\rho(2\mathcal{T}A - 2f) + \frac{1}{2}\rho(2\mathcal{T}A^* - 2f)$$

$$\leq \frac{\ell}{2}\rho(\mathcal{T}A - f) + \frac{\ell}{2}\rho(\mathcal{T}A^* - f) \leq \frac{2^{2\beta}}{1-2^\beta L} < \infty.$$

Since $\mathcal{T}$ is ρ–strict contraction, we have

$$\rho(A - A^*) = \rho(\mathcal{T}A - \mathcal{T}A^*) \leq L\rho(A - A^*),$$

which implies that $\rho(A - A^*) = 0$ or $A = A^*$, since $\rho(A - A^*) < \infty$. Therefore A is unique. $\square$

Theorem 2.2. *Let $\jmath \in \{-1, 1\}$ be fixed. Let $\mathcal{E}$ be a linear space and (X, μ) a μ-complete β-homogeneous $\mathcal{PM}$-space. Suppose that an even mapping $f : \mathcal{E} \to (X, \mu)$ satisfies the condition $f(0) = 0$ and inequality (2.1). Let $\psi : \mathcal{E} \times \mathcal{E} \to \Delta$ is a given function such that*

$$\psi(0, k^\jmath x)(k^{2\beta\jmath} Lt) \geq \psi(0, x)(t)$$

for all $x \in \mathcal{E}$ and has the property

$$\lim_{n\to\infty} \psi(k^{\jmath n}x, k^{\jmath n}y)(k^{2\beta\jmath n}t) = 1 \tag{2.11}$$

for all $x, y \in \mathcal{E}$ and a constant $0 < L < \frac{1}{2^\beta}$. Then there exists a unique quadratic mapping $Q : \mathcal{E} \to (X, \mu)$ satisfies (1.2) *and*

$$\mu\Big(Q(x) - f(x)\Big)\left(\frac{t}{k^\beta L^{\frac{\jmath-1}{2}}(1-2^\beta L)}\right) \geq \psi(0, x)(t) \tag{2.12}$$

for all $x \in \mathcal{E}$.

Proof . Replacing x by kx in (2.1), we get

$$\begin{aligned}\mu\Big(f(k(x+y)) + f(k(x-y)) \quad & -f(kx+y) - f(kx-y) - \tfrac{2(k+1)}{k}f(ky) \\ & +2(k+1)f(y)\Big)(t) \geq \psi(kx, y)(t)\end{aligned} \tag{2.13}$$

for all $x, y \in \mathcal{E}$. Putting $x = 0$ and replacing y by x in (2.13), we have by the evenness of f,

$$\mu\left(\frac{2}{k}f(kx) - 2kf(x)\right)(t) \geq \psi(0, x)(t)$$

for all $x \in \mathcal{E}$. Therefore,

$$\mu\left(\frac{1}{k^2}f(kx) - f(x)\right)(t) = \mu\left(\frac{2}{k}f(kx) - 2kf(x)\right)(2^\beta k^\beta t) \geq \psi(0, x)\left(2^\beta k^\beta t\right) \tag{2.14}$$

for all $x \in \mathcal{E}$. Now replacing x by $k^{-1}x$ in (2.14), we get

$$\begin{aligned}\mu\left(\frac{f(k^{-1}x)}{k^{-2}} - f(x)\right)(t) &= \mu\left(\frac{f(x)}{k^2} - f(k^{-1}x)\right)\left(\frac{t}{k^{2\beta}}\right) \\ &\geq \psi(0, k^{-1}x)\left(2^\beta k^\beta L^{-1}\frac{Lt}{k^{2\beta}}\right) \\ &\geq \psi(0, x)\left(2^\beta k^\beta L^{-1}t\right)\end{aligned} \tag{2.15}$$

for all $x \in \mathcal{E}$. From inequality (2.14) and (2.15), we have

$$\mu\left(\frac{f(k^{\jmath}x)}{k^{2\jmath}} - f(x)\right)(t) \geq \varphi(x)(t) := \psi(0,x)\left(2^{\beta}k^{\beta}L^{\frac{\jmath-1}{2}}t\right) \tag{2.16}$$

for all $x \in \mathcal{E}$. Consider the set $\mathcal{N}^* := \{g : \mathcal{E} \to (X,\mu)| \quad g(0) = 0\}$ and introduce the functional ρ on $\mathcal{N}^*$ as follows,

$$\rho(g) = \inf\{r > 0 : \mu(g(x))(rt) \geq \varphi(x)(t)\}.$$

Similar to the proof of Theorem 2.1, we can show that ρ modular on $\mathcal{N}^*$ and it satisfies the Fatou property and Δ_2-condition with $\ell = 2^{\beta}$. Furthermore, $\mathcal{N}^*$ is ρ-complete.
Let $x \in \mathcal{E}$ be an arbitrary point. We consider the linear mapping $\mathcal{T} : \mathcal{N}^*_{\rho} \to \mathcal{N}^*_{\rho}$ such that

$$\mathcal{T}Q(x) := \frac{Q(k^{\jmath}x)}{k^{2\jmath}}$$

for all $Q \in {\mathcal{N}^*}_{\rho}$. Proceeding as in the proof of Theorem 2.1, we obtain that $\rho(g-h) \leq r$ implies that $\rho(\mathcal{T}g - \mathcal{T}h) \leq L\rho(g-h)$ for all $g,h \in \mathcal{N}^*_{\rho}$. This means that $\mathcal{T}$ is a ρ-strict contraction.
By substituting x with $k^{\jmath}x$ in (2.16), we arrive

$$\mu\Big(k^{-2\jmath}f(k^{2\jmath}x) - f(k^{\jmath}x)\Big)(t) \geq \varphi(k^{\jmath}x)(t)$$

So

$$\begin{aligned}
&\mu\Big(k^{-2(2\jmath)}f(k^{2\jmath}x) - k^{-2\jmath}f(k^{\jmath}x)\Big)(Lt) \\
&= \mu\Big(k^{-2\jmath}f(k^{2\jmath}x) - f(k^{\jmath}x)\Big)(k^{2\beta\jmath}Lt) \\
&\geq \varphi(k^{\jmath}x)(k^{2\beta\jmath}Lt) \;\geq\; \varphi(x)(t)
\end{aligned}$$

for all $x \in \mathcal{E}$.
So

$$\begin{aligned}
&\mu\left(\tfrac{f(k^{2\jmath}x)}{k^{2(2\jmath)}} - f(x)\right)\left(2^{\beta}(Lt+t)\right) \\
&\geq \mu\left(\tfrac{f(k^{2\jmath}x)}{k^{2(2\jmath)}} - \tfrac{f(k^{\jmath}x)}{k^{2\jmath}}\right)(Lt) \wedge \mu\left(\tfrac{f(k^{\jmath}x)}{k^{2\jmath}} - f(x)\right)(t) \\
&\geq \varphi(x)(t)
\end{aligned} \tag{2.17}$$

for all $x \in \mathcal{E}$. Replacing x and $2^{\beta}(Lt+t)$ by $k^{\jmath}x$ and $k^{2\beta\jmath}2^{\beta}(L^2t+Lt)$ in (2.17), respectively, we get for all $x \in \mathcal{E}$,

$$\begin{aligned}
&\mu\Big(k^{-2(2\jmath)}f(k^{3\jmath}x) - f(k^{\jmath}x)\Big)\left(k^{2\beta\jmath}2^{\beta}(L^2t+Lt)\right) \\
&\geq \varphi(k^{\jmath}x)\left(k^{2\beta\jmath}Lt\right) \;\geq\; \varphi(x)(t).
\end{aligned}$$

It follows from the last inequality that

$$\mu\Big(k^{-3(2\jmath)}f(k^{3\jmath}x) - k^{-2\jmath}f(k^{\jmath}x)\Big)\left(2^{\beta}(L^2t+Lt)\right) \geq \varphi(x)(t),$$

which yields

$$\begin{aligned}
&\mu\left(\frac{f(k^{3\jmath}x)}{k^{3(2\jmath)}} - f(x)\right)\left(2^{\beta}\left\{2^{\beta}(L^2t+Lt)+t\right\}\right) \\
&\geq \mu\left(\frac{f(k^{3\jmath}x)}{k^{3(2\jmath)}} - \frac{f(k^{\jmath}x)}{k^{2\jmath}}\right)\left(2^{\beta}(L^2t+Lt)\right) \wedge \mu\left(\frac{f(k^{\jmath}x)}{k^{2\jmath}} - f(x)\right)(t) \\
&\geq \varphi(x)(t)
\end{aligned}$$

for all $x \in \mathcal{E}$. In general, using induction on a positive integer n, we obtain that

$$\mu\left(\frac{f(k^{\jmath n}x)}{k^{2\jmath n}} - f(x)\right)\left(\left\{(2^\beta L)^{n-1} + 2^\beta \sum_{i=1}^{n-1}(2^\beta L)^{i-1}\right\} t\right) \geq \varphi(x)(t),$$

for all $x \in \mathcal{E}$. Therefore, we have

$$\rho(\mathcal{T}^n f - f) \leq (2^\beta L)^{n-1} + 2^\beta \sum_{i=1}^{n-1}(2^\beta L)^{i-1} \leq 2^\beta \sum_{i=1}^{n}(2^\beta L)^{i-1} \leq \frac{2^\beta}{1-2^\beta L}. \tag{2.18}$$

Proceeding as in the proof of Theorem 2.1 and using Theorem 1.8, we obtain that $\{\mathcal{T}^n(f)\}$ is ρ-converges to $Q \in \mathcal{N}_\rho^*$ and ρ-limit of $\{\mathcal{T}^n(f)\}$ i.e., $Q \in \mathcal{N}_\rho^*$ is a fixed point of the map $\mathcal{T}$.
It follows from (2.18) that

$$\rho(Q - f) \leq \frac{2^\beta}{1-2^\beta L}.$$

This implies that

$$\mu\Big(Q(x) - f(x)\Big)\left(\frac{2^\beta}{1-2^\beta L}t\right) \geq \varphi(x)(t) = \psi(0,x)\left(2^\beta k^\beta L^{\frac{\jmath-1}{2}} t\right).$$

So by the above inequality, we have

$$\mu\Big(Q(x) - f(x)\Big)\left(\frac{t}{k^\beta L^{\frac{\jmath-1}{2}}(1-2^\beta L)}\right) \geq \psi(0,x)(t).$$

This implies that the inequality (2.12) holds.
The rest of the proof is similar of Theorem 2.1. □

Theorem 2.3. *Let $\jmath \in \{-1,1\}$ be fixed. Let $\mathcal{E}$ be a linear space and (X,μ) a μ-complete β-homogeneous $\mathcal{PM}$-space. Suppose that a mapping $f : \mathcal{E} \to (X,\mu)$ satisfies the condition $f(0) = 0$ and inequality (2.1). Let $\psi : \mathcal{E} \times \mathcal{E} \to \Delta$ is a given function such that*

$$\psi(0, sk^\jmath x)(\frac{k^{\imath\beta\jmath}L}{2}t) \geq \psi(0,sx)(\frac{t}{2})$$

and

$$\lim_{n\to\infty} \min\Big\{\psi(k^{\jmath n}x, k^{\jmath n}y)(\frac{k^{\imath\beta\jmath n}}{2}t), \psi(-k^{\jmath n}x, -k^{\jmath n}y)(\frac{k^{\imath\beta\jmath n}}{2}t)\Big\} = 1$$

for all $x, y \in \mathcal{E}$, a constant $0 < L < \frac{1}{2^\beta}$, $s \in \{-1,1\}$ and $\imath \in \{1,2\}$. Then there exist a unique additive mapping $A : \mathcal{E} \to (X,\mu)$ and a unique quadratic mapping $Q : \mathcal{E} \to (X,\mu)$ such that

$$\begin{aligned}\mu\Big(f(x) - A(x) - Q(x)\Big)(t) \geq \min\Big\{&\psi(0,x)\Big(\frac{k^\beta L^{\frac{\jmath-1}{2}}(1-2^\beta L)}{2^{\beta+2}}t\Big),\\ &\psi(0,-x)\Big(\frac{k^\beta L^{\frac{\jmath-1}{2}}(1-2^\beta L)}{2^{\beta+2}}t\Big),\\ &\psi(0,x)\Big(\frac{(k+1)^\beta L^{\frac{\jmath-1}{2}}(1-2^\beta L)}{2^{\beta+2}}t\Big),\\ &\psi(0,-x)\Big(\frac{(k+1)^\beta L^{\frac{\jmath-1}{2}}(1-2^\beta L)}{2^{\beta+2}}t\Big)\Big\}\end{aligned}$$

for all $x \in \mathcal{E}$.

Proof . Let

$$f_o(x) = \frac{1}{2}[f(x) - f(-x)]$$

for all $x \in \mathcal{E}$. Then $f_o(0) = 0$, $f_o(-x) = -f_o(x)$, and

$$\begin{aligned} \mu\Big(f_o(x+ky) &+ f_o(x-ky) - f_o(x+y) - f_o(x-y) - \tfrac{2(k+1)}{k} f_o(ky) \\ &+ 2(k+1) f_o(y)\Big)(t) \geq \min\Big\{\psi(x,y)(\tfrac{t}{2}), \psi(-x,-y)(\tfrac{t}{2})\Big\} \end{aligned} \tag{2.19}$$

for all $x, y \in \mathcal{E}$. Hence, in view of Theorem 2.1, there exists a unique additive function $A : \mathcal{E} \to (X, \mu)$ such that

$$\begin{aligned} \mu\Big(A(x) - f_o(x)\Big)(t) \geq \min\Big\{ &\psi(0,x)\Big(\tfrac{(k+1)^\beta L^{\frac{\jmath-1}{2}}(1-2^\beta L)}{2} t\Big), \\ &\psi(0,-x)\Big(\tfrac{(k+1)^\beta L^{\frac{\jmath-1}{2}}(1-2^\beta L)}{2} t\Big)\Big\}. \end{aligned} \tag{2.20}$$

Let

$$f_e(x) = \frac{1}{2}[f(x) + f(-x)]$$

for all $x \in \mathcal{E}$. Then $f_e(0) = 0$, $f_e(-x) = f_e(x)$, and

$$\begin{aligned} \mu\Big(f_e(x+ky) &+ f_e(x-ky) - f_e(x+y) - f_e(x-y) - \tfrac{2(k+1)}{k} f_e(ky) \\ &+ 2(k+1) f_e(y)\Big)(t) \geq \min\Big\{\psi(x,y)(\tfrac{t}{2}), \psi(-x,-y)(\tfrac{t}{2})\Big\} \end{aligned} \tag{2.21}$$

for all $x, y \in \mathcal{E}$. From Theorem 2.2, it follows that there exists a unique quadratic mapping $Q : \mathcal{E} \to (X, \mu)$ such that

$$\begin{aligned} \mu\Big(Q(x) - f_e(x)\Big)(t) \geq \min\Big\{ &\psi(0,x)\Big(\tfrac{k^\beta L^{\frac{\jmath-1}{2}}(1-2^\beta L)}{2} t\Big), \\ &\psi(0,-x)\Big(\tfrac{k^\beta L^{\frac{\jmath-1}{2}}(1-2^\beta L)}{2} t\Big)\Big\}. \end{aligned} \tag{2.22}$$

Obviously, (2.19) follows from (2.20) and (2.22) . □

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