Int. J. Nonlinear Anal. Appl. 13 (2022) 1, 3463-3474
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.6109

Fibrewise multi-topological spaces

M. H. Jaber[a,*], Y.Y. Yousif[a]

[a]*Department of Mathematics, College of Education for Pure Sciences, University of Baghdad, Ibn Al-Haitham, Baghdad, Iraq*

(Communicated by Madjid Eshaghi Gordji)

Abstract

We define and study new ideas of fibrewise topological space on D namely fibrewise multi-topological space on D. We also submit the relevance of fibrewise closed and open topological space on D. Also fibrewise multi-locally sliceable and fibrewise multi-locally section able multi-topological space on D. Furthermore, we propose and prove a number of statements about these ideas.

Keywords: fibrewise upper topological space, fibrewise lower topological space, fibrewise multi-topological spaces, fibrewise upper locally sliceable
2021 MSC: 54C08, 54C10, 55R70

1. Introduction

To begin with we work in the category of Fibrewise (briefly F.W.) sets on a given set, named the base set. If the base set is stated by D then a F.W. set on D consists of a set E together with a function$X$ *is* $X : E \to D$, named the projection (briefly, project.). For every point d of D the fibre on d is the subset $E_d = X^{-1}(d)$ of E; fibres will be empty let we do not require X to be surjection, also for every subset D^* of D we regard $E_{D^*} = X^{-1}(D^*)$ as a F.W. set on D^* with the project. determined by X. A multi-function [2] Ω of a set E in to F is a correspondence such that Ω (e) is a nonempty subset of F for every $e \in E$. We will denote such a multi-function by $\Omega : E \to F$. For a multi-function Ω, the upper and lower inverse set of a set V of F, will be denoted by Ω^+(V) and Ω^-(V) respectively that is Ω^+(V) = {e ∈ E : Ω(e) ⊆ V} and Ω^-(V) = {e ∈ E: Ω(e) ∩ V ≠ ∅ }.

Definition 1.1. *[5] Suppose that E and F are F.W sets on D, with project.* $X_E : E \to D$ *and* $Y_F : F \to D$*, respectively, a function* $\Omega : E \to F$ *is named to be F.W. if* $Y_F \circ \Omega = X_E$*, that is to say if* $\Omega(X_d) \subset F^d$ *for every point d of D.*

*Corresponding author
Email addresses: maged.hameed1203a@ihcoedu.uobaghdad.edu.iq (M. H. Jaber), yousif.y.y@ihcoedu.uobaghdad.edu.iq (Y.Y.Yousif)

Received: November 2021 *Accepted:* January 2022

It should be noted that a F.W. function $\Omega : \ E \ \rightarrow \ F$ on D determines, by restriction, F.W. function $\Omega D^* : \ E \ \rightarrow \ FD^*$ onD^* for every D^* of D.

Let E_r be an indexed family of F.W. sets on D the F.W. product $\prod_D E_r$ is stated, as a F.W. set on D, and comes included with the family of F.W. projection $\pi_r : \prod_D E_r \rightarrow E_r$. Specifically the F.W. product is stated as the subset of the ordinary product $\prod E_r$ where in the fibres are the products of the relevant fibers of the factors E_r. The F.W. product is recognized by the following Cartesian property: for every F.W. set E on D the F.W. functions $\Omega : E \rightarrow \prod_r E_r$ correspond precisely to the families of F.W. functions $\{\Omega_r\}$, with $\Omega_r = \pi_r \circ \Omega : E \rightarrow E_r$. For example if $E^r = E$ for every index r the diagonal $\Delta : E \rightarrow \prod_D E$, is stated so that $\pi_r \circ \Delta = idE$ for every r. If $\{E^r\}$ is as before,the F.W. coproduct $\coprod_D E^r$ is with stated, as F.W. set on D, and comes included with the family of F.W. insertions $\sigma : \ E_r \rightarrow \coprod_D E_r$, specifically the F.W. coproduct synchronize, as a set, with the ordinary coproduct (disjoint union), the fibres being the coproducts of the relevant fipers of the summands E_r. The F.W. coproduct is recognized by the following Cartesian property, for every F.W. set E on D the F.W. functions $\varphi : \coprod_D E_r \rightarrow E$ correspond precisely to the families of F.W. functions $\{\varphi_r\}$, *where in* $\varphi_r = \varphi o \sigma_r$: $E_r \rightarrow E$. For example, if $E_r = E$ for every index r the codiagonal $\nabla \ : \ \coprod_D E \rightarrow E$ is stated so that, $\nabla \circ \sigma_r = idE$ for every r. The notation E$\times_D$ F is used for the F.W. product in the case of the family $\{E, F\}$, of two F.W. sets and similarity for finite families generally. As well as, we builte on some of the result in [1, 7, 6]. For other notions or notations that are not defined here we follow closely [5, 4, 3].

Definition 1.2. *[5] Let D be topological space, the F.W. topology space (briefly, F.W.T.S.) on a F.W. set E on D, mean any topology on E for which the project. X is continuous.*

Remark 1.3. *[5]*

(a) *The coarsest like topology is the topology trace by X, where in the open sets of E are precisely the pre image of the open sets of D, this is named the F.W. indiscrete topology.*
(b) *The F.W.T.S. on D is stated to be a F.W. set on D with a F.W.T.S.*

Definition 1.4. *[5] The F.W. functions $\Omega : E \rightarrow F$; E and F are F.W. spaces on D is named:*

(a) *Continuous (briefly, cont.) if each $e \in E^d; d \in D$, the $\Omega^{-1}(e)$ is open set of e.*
(b) *Open if for every $e \in E_d, d \in D$, the direct image of each open set of e is an open set of $\Omega\left(e\right)$.*

Definition 1.5. *[5] The F.W.T.S. E on D is named F.W. closed (resp., open) if the project. X is closed (resp., open) functions.*

Definition 1.6. *[2] Let $\Omega : E \rightarrow F$ be a multi-function. Then Ω is upper cont. (briefly, U. cont.) iff $\Omega^+(V)$ open in E for all V open in F. That is, $\Omega^+(V) = \{x \in E : \Omega(x) \subseteq V\}.V \subseteq F$.*

Definition 1.7. *[2] Let $\Omega : E \rightarrow F$ be a multi-function. Then Ω is lower cont. (briefly, L. cont.) iff $\Omega^-(V)$ open in E for all V open in F. That is, $\Omega^-(V) = \{e \in E : \Omega(e) \cap V \neq \varnothing\}.V \subseteq F$.*

Let $\Omega : E \rightarrow F$ be a multi-function. Then Ω is mult cont. (briefly, M. cont.) iff it is U. cont. and L. cont.

2. Fibrewise Multi-Topological spaces

In this part, we submit the ideas of fibrewise multi-topology several Topological on the obtained fibrewise multi-topology are studies.

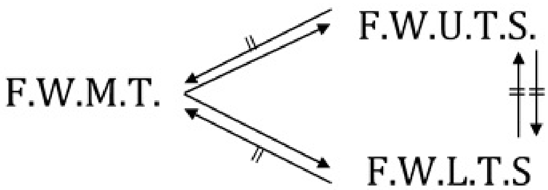
Definition 2.1. *Let D be topologicalspace, the F.W. upper topology space(briefly, F.W.U.T.S.) on a F.W. set E on D mean any topology on E for which the project. X is U. cont.*

Definition 2.2. *Let D be topologicalspace the F.W. lower topology space (briefly, F.W.L.T.S.) on a F.W. set E on D mean any topology on E for which the project. X is L. cont.*

Let D be topological space the fibrewise multi-topology space (briefly, F.W.M.T.S.) iff it is F.W.U.T.S. and F.W.L.T.S.

Remark 2.3.

a) *Every F.W.M.T.S. is F.W.U.T.S. but the convers is not true.*
b) *Every F.W.M.T.S. is F.W.L.T.S. but the convers is not true.*
c) *The F.W.U.T.S. and F.W.L.T.S. are independence*



Example 2.4.

(a) *Let* $E = D = \{a, b, c\}$. *let* $\tau_{(E)} = \{E, \varnothing, \{a\}\}$. *and* ρ = *discrete topology. Define the project.* $X : \left(E, \tau_{(E)}\right) \to (D,\ \rho)$ *by* $X(a) = X(b) = X(c) = \{a\}$.
Then X is U. cont., L. cont. and M. cont. Thus, E is F.W.U.T.S., F.W.L.T.S., and F.W.M.T.S.

(b) *Let* $E = \mathbb{R}$ *with the usual topology* τ *and let* $D = \{a, b, c\}$ *with the topology* $\rho\ = \{\varnothing, Y, \{a\}\}$. *Define multi-function*

$$X\ : (\mathbb{R}, \tau) \to (D, \rho\ ) \quad by \quad X\,(x) = \begin{cases} \{a\}\,; & x \leq 0 \\ \{a, c\}\,; & x > 0 \end{cases}$$

Then X is L. cont. but is not U. cont. and is not M. cont. Thus, E is F.W. L.T.S. But not F.W.U.T.S. And not F.W.M.T.S.

(c) *Let* $E = \mathbf{R}$ *with the usual topology* τ *and* $D = \{a, b, c\}$ *with the topology* $\rho\ = \{\varnothing, Y, \{a\}\}$. *Define multi-function*

$$X\ : (\mathbf{R}, \tau) \to (D, \rho\ ) \quad by \quad X\,(x) = \begin{cases} \{a\}\,; & x \leq 0 \\ \varnothing; & x > 0 \end{cases}$$

Then X is U. cont. but not L. cont. and is not M. cont. Thus, E is F.W.U.T.S., but not F.W.L.T.S. and not F.W.M.T.S.

(d) *Let $E = \mathbf{R}$ with the usual topology τ and $D = \{a,b,c\}$ with the topology$\rho = \{\varnothing, Y, \{a\}\}$. Define multi-function*

$$X \ : (\mathbf{R}, \tau) \to (D, \rho\ ) \ \ by \ \ X(x) = \begin{cases} \{a\}; & x < 0 \\ \{a,b\}; & x = 0 \\ \{c\}; & x > 0 \end{cases}$$

Then X is not U. cont., not L. cont. and is not M. cont. Thus, E is not F.W.U.T.S., not F.W.L.T.S. and not F.W.M.T.S.

Definition 2.5. *A F.W. function $\Omega : E \to F$; E and F are F.W.T.S. on D is named*

(a) *Upper continuous (briefly, F.W.U. cont.) if for every $e \in E_d$; $d \in D$ and each open set M of $\Omega(e)$ we have $\Omega(e) \subset M$*
(b) *Lower continuous (briefly, F.W.L. cont.) if for every $e \in E_d$; $d \in D$ and each open set M of $\Omega(e)$ we have $\Omega(e) \cap M \neq \phi$*
(c) *multi continuous (briefly, F.W.M. cont.) if $\Omega(e)$ is upper cont. and (lower cont.).*

Let $\Omega :\ E \ \longrightarrow\ F$ be a F.W. function where in E is a F.W. set and F is a F.W.T.S. on D. we are able to give E the trace topology, in the normal sense, and this is necessarily a F.W.T. We may refer to it, as the trace F.W.T.

Proposition 2.6. *Let $\Omega : E \to F$ be a F.W. function, where in F is a F.W.U.T.S. (resp., F.W.L.T.S.) on D and E is a F.W. set has the trace F.W.T. If for every F.W.U.T.S. (resp., F.W.L.T.S.) N, a fibrewise function $\varphi : N \to E$ is U. cont. (resp., L cont.) iff the composition $\Omega \circ \varphi : N \to F$ is U. cont. (resp., L cont.).*
Proof .
$\Longrightarrow$ *Suppose that φ is U. cont. (resp., L. cont.). Let $n \in N_d$; $d \in D$ and M open set of $(\Omega \circ \varphi)(n) = f \ \in F_d$ in F. Let Ω is U. cont. (resp., L. cont.), $\Omega^+(M)$ (resp. $\Omega^-(M)$ is an open set containing $\varphi(n) = e \in E_b$ in E. Let φ is U. cont. (resp., L. cont.), then $\varphi^+(\Omega^+(M)$ (resp., $\varphi^-(\Omega^-(M)$ is an open set containing $n \in N_d$ in N, and $\varphi^+(\Omega^+(M)) = (\varphi o \varphi)^+(M)$ (resp., $\varphi^-(\Omega^-(M)) = (\varphi o \Omega)^-(M)$ is an open set containing $n \ \in N_d$ in N, then $\Omega \circ \varphi : N \to F$ is U. cont. (resp., L. cont.).*
$\Longleftarrow$ *Suppose that $\Omega \circ \varphi$ is U. cont. (resp., L. cont.). Let n$\in N_d$; $d \in D$ and V open set of $\varphi(n) = e \in E_d$ in E. Since Ω is open, $\Omega(V)$ is an open set containing $\Omega(e) = \Omega(\varphi(n)) = f \in F_d$ in F. Since $\Omega \circ \varphi$ is U. cont. (resp., L. cont.), then $(\Omega \circ \varphi)^+(\Omega(V)) = \varphi^+(V)$ (resp. $(\Omega \circ \varphi)^-(\Omega(V)) = \varphi^-(V)$ is an open set containing $n \in N_d$ in N, then $\varphi(V)$ is U. cont. (resp., L. cont.).* □

Corollary 2.7. *Let $\Omega : E \to F$ be a F.W. function; F is a F.W.M.T.S. on D and E is a fibrewise set has the trace F.W.T. If for every F.W.M.T.S. N, a F.W. function $\varphi : N \to E$ is M. cont. iff the composition $\Omega \circ \varphi : N \to F$ is M. cont.*

Proposition 2.8. *Let $\Omega : E \to F$ be a F.W. function, where in F is a F.W.T.S. on D and E is a F.W. set has the trace F.W.T. If for every a F.W.T.S. N, a F.W. function $\varphi : N \to E$ is open, surjection iff the composition $\Omega \circ \varphi : N \to F$is open.*
Proof . *Clear* □

Let us pass of general cases of Propositions 2.6 as follows: likewise, in the case of families $\{\Omega_r\}$ of F.W. function; $\Omega_r : E \to F_r$ with F_r F.W.U.T.S. (resp., F.W.L.T.S.) on D for every r. In particular, given a family $\{E_r\}$ of F.W.U.T.S. (resp., F.W.L.T.S.) on D, the F.W.U.T.S. (resp., F.W.L.T.S.) product

$\prod_D E_r$ is stated to be F.W. product with the F.W.T. trace be the family of project. $\pi_r : \prod_D E_r \to E_r$. Then for every F.W.U.T.S. (resp., F.W.L.T.S.) N on D a F.W. function $\alpha : N \to \prod_D E_r$ is U. cont. (resp., L. cont.) iff each of the F.W. function $\pi_r \circ \alpha : N \to E_r$ is U. cont. (resp. ,L. cont.). For example, when $E_r =$ E for every index r we see that the diagonal $\Delta : E \to \prod_r E$ is U. cont. (resp., L. cont.) iff the composition $\pi_r \circ \Delta = idx$ is U. cont.(resp., L. cont.).
Again if $\{E_r\}$ is a family of F.W.U.T.S. (resp. ,F.W.L.T.S.) on D and $\varphi : \coprod_D E_r \to E$ is a F.W. function; E a F.W.T. on D and $\coprod_D E_r$ is F.W.U.T.S. coproduct at the set-theoretic level with the ordinary coproduct topology, also for every F.W.T. E_r with the family of F.W. insertions $\sigma_r : E_r \to \coprod_D E_r$ is U cont. (resp., L. cont.), iff the composition $\varphi_r = \varphi \circ \sigma_r : E_r \to E$ is U. cont. (resp., L. cont.). For example, when $E_r = E$ for every index r we see that the codiagonal $\nabla : \coprod_D E \to E$ is U. cont. (resp., L. cont.).
Now, we study the ideas of F.W. closed and open topology spaces. topology property on the obtained ideas are studies.

Definition 2.9. *A F.W.U.T.S. (resp., F.W.L.T.S.) E on D is named F.W. closed if the project. function X_E is closed.*

A F.W.M.T.S. E on D is named F.W. closed if the project. X_E is closed.

Proposition 2.10. *Let $\Omega : E \to F$ be a closed F.W. function; E and F are F.W.U.T.S. (resp., F.W.L.T.S.) on D, if F is F.W. closed, then E is F.W. closed.*
Proof . *Suppose that $\Omega : E \to F$ is closed F.W. function and F is F.W. closed i.e., the project. function $Y_F : F \to D$ is closed. To show that E is F.W. closed i.e., the project. function $X_E : E \to D$ is closed. Now let G be a closed subset of E_d, where in $d \in D$, let Ω is closed, then $\Omega(G)$ is closed subset of F_d. Since Y_F is closed, then $Y_F\Omega(G)$ is closed in D, but $Y_F\Omega(G)) = (Y_F \circ \Omega)(G) = X_E(G)$ is closed in D. Thus X_E is closed and E is F.W. closed.* □

Corollary 2.11. *Let $\Omega : E \to F$ be a closed F.W. function, where in E and F are F.W.M.T.S. on D, if F is F.W. closed, then E is F.W. closed.*

Proposition 2.12. *Let E be a F.W.U.T.S. (resp., F.W.L.T.S.) on D, suppose that E_i is F.W. closed for every member E_i of a finite covering of E. Then E is F.W. closed.*
Proof . *Let E be a F.W.U.T.S. (resp., F.W.L.T.S.) on D, then the project. function $X_E : E \to D$ exist. To show that X_E is closed.*
Now, since E_i is F.W. closed, then the project. function $X_{Ei} : E_i \to D$ is closed for every member E_i of a finite covering of E. Let G be a closed subset of E, then $X_E(G) = \cup X_{Ei}(E_i \cap G)$ that is a finite union of closed set and hence X_E is closed. Thus, E is F.W. closed. □

Corollary 2.13. *Let E be a F.W.M.T.S. on D, suppose that E_i is F.W. closed for every member E_i of a finite covering of E. Then E is F.W. closed.*

Proposition 2.14. *Let E be a F.W.U.T.S. (resp., F.W.L.T.S.) on D. Then E is F.W. closed iff for every F.W. E_d of E and each open set H of E_d in E, there exists an open set O of d such that $E_o \subset H$.*
Proof .
$\Rightarrow$ Suppose that E is F.W. closed, the project. function $X_E : E \to D$ is closed.
Now, let $d \in D$ and H open set of E_d in E, then $E - H$ is closed in E, this implies $X_E(E - H)$ is closed in D, let $O = D - X_E(E - H)$, then O an open set of d in D and $E_o = X_E^+(O) = E - X_E{}^+(X_E(E - H))$(resp. $E_o = X_E^-(O) = E - X_E{}^-(X_E(E - H))) \subset H$.

$\Leftarrow$ *Suppose that the assumption holds and $X_E : E \to D$. Now, let G be a closed subset of E and d $\in D-\ X_E(G)$ and each open set H of fibre E_d in E. By assumption there exists an open O of d such that $E_O \subset H$. It is easy to show that $O \subset D-\ X_E(G)$, hence $D-\ X_E(G)$ is open in D and this implies $X_E(G)$ is closed in D and X_E is closed. Thus X_E is F.W. closed.* $\square$

Corollary 2.15. *Let E be a F.W.M.T.S. on D. Then E is F.W. closed iff for every fibre E_d of E and each open set H of E_d in E, there exists an open set O of d such that $E_o \subset U$*

Definition 2.16. *A F.W.U.T.S. (resp., F.W.L.T.S.) E on D is named F.W. open if the project. function X_E is open.*

A F.W.M.T.S. E on D is named F.W. open if the project. function X_E is open.

Proposition 2.17. *Let $\Omega : E \to F$ be an open F.W. function; E and F are F.W.U.T.S. (resp., F.W.L.T.S.) on D, if F is F.W. open, then E is F.W. open.*
Proof . *Suppose that $\Omega : E \to F$ is open F.W. function and F is F.W. open, the project. function $Y_F : F \to D$ and open. To show that E is F.W. open, the project. function $X_E : E \to D$ is open. Now let O is open subset of $E_d, ; d \in D$, since Ω is open, then $\Omega(O)$ is open subset of F_d, let Y_F is open, then $Y_F(\Omega(O))$ is open in D, but $Y_F(\Omega(O)) = (Y_F o \Omega)(O) =\ X_E(O)$ is open in D. Thus X_E is open and E is F.W. open.* $\square$

Corollary 2.18. *Let $\Omega : E \to F$ be an open F.W. function; E and F are F.W.M.T.S. on D, if F is F.W. open, then E is F.W. open.*

Proposition 2.19. *Let $\{E_r\}$ be a finite family of F.W. open on D. Thus, the F.W.U.T.S. (resp., F.W.L.T.S.) product $E = \prod_D E_r$ is also open.*
Proof . *Suppose that $E = \prod_D E_r$ is a F.W.U.T.S. (resp., F.W.L.T.S) on D, then $X : E = \prod_D E_r \to D$ is exist. To show that X is open. Now, let $\{E_r\}$ be a finite family of F.W. open spaces on D, then the project. function $X_r :\ E_r \to D$ is open for every r. Let O be an open subset of E, then $X(O) = X(\prod_D(E_r\,(O)) = \prod_D X_r\,(E_r \cap O)$ that is a finite product of open set and hence X is open. Thus, the F.W.U.T.S. (resp., F.W.L.T.S.) product $E = \prod_D E_r$ is a F.W. is open.* $\square$

Corollary 2.20. *Let $\{E_r\}$ be a finite family of F.W. open on D. Thus, the F.W.M.T.S. product $E = \prod_D E_r$ is also open.*

That is to say, the class of F.W. open topological space is finitely multi plicativr. In fact, Proposition 2.19 and Corollary 2.20 remains true for infinite families provided each member of the family is F.W. nonempty in the sense that project. is surjective.

Remark 2.21. *If E is F.W. open then the second projection function $\pi_r :\ E_{\times D}F \to F$ is open for all F.W.U.T.S. (resp., F.W.L.T.S. and F.W.M.T.S.) F. because for every non-empty open set $M_{1\times D}M_2 \subset E_{\times D}F$, we have $\pi_2(M_{1\times D}M_2) = M_2$ is open. We use this in the proof of the following results.*

Proposition 2.22. *Let $\Omega : E \to F$ be a F.W. function, where in E and F are F.W.U.T.S. (resp., F.W.L.T.S.) on D. Let $id_E \times \Omega : E \times_D E\ \to E \times_D F$, if $id_E \times \Omega$ is open and that is F.W. open. Then Ω itself is open*
Proof . *Consider the following commutative diagram:*

$$\begin{tikzcd}
E \times_D E \arrow[r, "\mathrm{Id}_E \times \Omega"'] \arrow[d, "\pi_2"] & E \times_D F \arrow[d, "\pi_2"'] \\
E \arrow[r, "\Omega"] & F
\end{tikzcd}$$

The projection function on the left is surjection and open, since F is F.W. open, while the projection function on the right is open, since E is F.W. open. Therefore $\pi_2 \circ (id_E \times \Omega) = \Omega \circ \pi_2$is open, and so Ω is open, by Proposition 2.17 as asserted. $\square$

Corollary 2.23. *Let $\Omega : E \to F$ be a F.W. function; E and F are F.W.M.T.S. on D. Let $id_E \times \Omega : E \times_D E \ \to E \times_D F$, if $id_E \times \Omega$ is open and that is F.W. open. Then Ω itself is open*

Our next three results apply equally to F.W. closed and the F.W. open.

Proposition 2.24. *Let $\Omega : E \to F$ be a cont. F.W. surjection; E and F are F.W.U.T.S. (resp., F.W.L.T.S.) on D. If E is F.W. closed (resp. open), then F is F.W. closed (resp. open).*
Proof . *Suppose that $\Omega : E \to F$ is cont. F.W. surjection and E is F.W. closed (resp. open), the projection function $X : E \to D$ is closed (resp. open). To show that F is F.W. closed (resp. open), the projection function $Y : F \to D$ is closed (resp. open). Let G be a closed (resp. open) subset of F_d, wherein $d \in D$. Since Ω is cont. F.W., then $\Omega^{-1}(G)$ is closed (resp. open) subset of E_d. Since X is closed (resp. open), then $X\,(\Omega^{-1}(G))$ is closed (resp. open) in D, but $X(\Omega^{-1}(G)) = (X \circ \Omega^{-1}(G) = Y(G)$ is closed (resp. open) in D. Thus, Y is closed (resp. open) and Y is F.W. closed (resp. open).* $\square$

Corollary 2.25. *Let $\Omega : E \to F$ be a cont. F.W. surjection, where in E and F are F.W.M.T.S. on D. If E is F.W. closed (resp. open), then F is F.W. closed (resp. open).*

Proposition 2.26. *Let E be a F.W.U.T.S. (resp., F.W.L.T.S.) on D. Suppose that E is F.W. closed (resp., open) on D. Then $E^{+}_{D^*}$ (resp., $E^{-}_{D^*}$) is F.W. closed (resp., open) on D^* for every subspace D^* of D.*
Proof . *Suppose that E is a F.W. closed (resp. open), the projection $X : E \to D$ is closed (resp., open). To show that $E^{+}_{D^*}$ (resp., $E^{-}_{D^*}$) , the projection function X_{D^*} : $E^{+}_{D^*}$ (resp.,$E^{-}_{D^*}$) $\to D^*$ is closed (resp., open). Now, let G be a closed (resp., open) subset of E, then $G \cap E^{+}_{D^*}$ (resp., $E^{-}_{D^*}$) is closed (resp., open) in subspace $E^{+}_{D^*}$ (resp., $E^{-}_{D^*}$) and $X_{D^*}(G \cap E^{+}_{D^*}$ (resp., $E^{-}_{D^*}$)) $= X(G \cap E^{+}_{D^*}$ (resp., $E^{-}_{D^*}$)) $= X(G) \cap D^*$ that is closed (resp., open) set in D^*. Thus X_{D^*} is closed (resp., open) and $E^{+}_{D^*}$ (resp., $E^{-}_{D^*}$)is F.W. closed (resp., open) on D^*.* $\square$

Corollary 2.27. *Let E be a F.W.M.T.S. on D. Suppose that E is F.W. closed (resp., open) on D. Then $E^{+}_{D^*}$ (resp., $E^{-}_{D^*}$) is F.W. closed (resp., open) on D^* for every subspace D^* of D.*

Proposition 2.28. *Let E be a F.W.U.T.S. (resp., F.W.L.T.S.) on D. Suppose that $E^{+}_{D^*}$ (resp., $E^{-}_{D^*}$) is F.W. closed (resp., open) on D_i for every member D_i of an open covering of D. Then E is F.W. closed (resp., open).*

Proof . *Suppose that E is a F.W.U.T.S. (resp., F.W.L.T.S.) on D, then the projection function* $X : E \to D$ *exists. To show that X is open. Now, since* $E^+_{Di^*}$ *(resp.,* $E^-_{Di^*}$*) is F.W. open on* D_i*, then the projection function* X^+_{Di}*:* $E^+_{Di} \to D_i$ *(resp.,* $X^-_{Di} : E^-_{Di} \to D_i$ *) is open for every member* D_i *of an open covering of D. Let G be an open subset of E, then we have* $X(G) = \cup X^+_{Di}\ (E^+_{Di} \cap G)$ *(resp.,* $\cup X^-_{Di}\ (E^-_{Di} \cap G)$*) that is a union of open sets and hence X is open. Thus, E is F.W. open on D.* $\square$

Corollary 2.29. *Let E be a F.W.M.T.S. on D. Suppose that* E_{D^*} *is F.W. closed (resp., open) on* D_i *for every member* D_i *of an open covering of D. Then E is F.W. closed (resp., open).*

3. Fibrewise Multi-Locally Sliceable and Fibrewise Multi-Locally Sectionable Multi-topological spaces

We present the ideas of fibrewise upper (resp., fibrewise lower and fibrewise multi) locally sliceable and fibrewise upper (resp., fibrewise lower and fibrewise multi) locally sectionable multi-topologicalspaces on (D, ρ), several topologicalproperties on the obtained ideas are studied.

Definition 3.1. *The F.W.U.T.S. (resp., F.W.L.T.S.)* (E, τ) *on* (D, ρ) *is named upper (resp., lower) locally sliceable if for every point* $e \in E_d; d \in D$*, there exist an open set W of d and a section* $s^+ : W \to E^+_W$ *(resp.,* $(s^- : W \to E^-_W)$ *and* $s^+(d) = e$ *(resp.,* $s^-(d) = e$*).*

The F.W.M.T,S. (E, τ) *on* (D, ρ) is named multi-locally sliceable if for every point $e \in E_d; d \in D$, there exist an open set W of d and a section $s^+ : W \to E^+_W$ (resp. $s^- : W \to E^-_W$) and $s^+(d) = e(resp.s^-(d) = e)$.

Definition 3.2. *The F.W.U.T.S. (resp., F.W.L.T.S.)* (E, τ) *on* (D, ρ) *is named upper (resp., lower) locally sliceable if for every point* $e \in E_d; d \in D$*, there exist an open set W of d and a section s:* $W \to E_W$ *and s(d) = e.*

The F.W.M.T.S. (E, τ) *on* (D, ρ) is named multi-locally sliceable if for every point e $\in E_d; d \in D$, there exist an open set W of d and a section $s : W \to E_W$ *and* $s(d) = e$.
The condition lead to X is open for U is an open set of e in E, then $s^{-1}(E^+_W$ (resp., E^-_W) $\cap U \subset X(U)$ is an open set of d in W and hence in D. The class of multi-locally sliceable multi-topologicalspace is finitely multiplicative stated in.

Proposition 3.3. *Let* $\{(W^+_r(resp., W^-_r), \tau_r)\}^k_{r=1}$ *be a finite family of upper (resp., lower) locally sliceable upper (resp., lower) space on* (D, ρ)*. The F.W.U.T. (resp., F.W.L.T.) product* $W = \Pi^D W^r$ *is upper (resp., lower) locally sliceable.*
Proof . *Let* $e = (e^r)$ *be a point of* E^+_d *(resp.,* $E^-_d; d \in D,$*) so that* $e^r = \pi_r(e)$ *for every index r. Sense* E^+_r *(resp.,* E^-_r*) is upper (resp., lower) locally sliceable upper (resp., lower) topologicalspace there is an open set* N^r *of d and a section* $s^+_r : W^r \to E^+_r|W^r(resp., s^-_r : W^r \to E^-_r|W^r)$ *where in* $s^+_r(d)$ *(resp.,* $s^-_r(d) = e^r$*.) Then the intersection* $W = E^1 \cap \ldots \cap E^n$ *is an open set of d and a section* $s : W \to E^+_N$ *(resp.,* E^-_N*) is given by* $\pi_r \circ s(w) = s^r(w)$ *for every index r and every point* $w \in W$. $\square$

Corollary 3.4. *Let* $\{(E^r, \tau_r)\}^k_{r=1}$ *be a finite family of multi-locally sliceable multi space on* (D, ρ)*. The F.W.M.T. product* $E = \Pi^D E^r$ *is multi-locally sliceable.*

Proposition 3.5. *Let $\Omega : E \to F$ be cont. F.W. surjection, where in (E, τ) and (F, ω) are F.W.U.T.S. (resp., F.W.L.T.S.) on (D, ρ). If E is upper (resp., lower) locally sliceable, then F is so.*

Proof . *Let $f \in F^d; d \in D$. Then $f = \Omega(e)$, for some $e \in E_d^+$ (resp., E_d^-). If E^+ (resp., E^-) is upper (resp., lower) locally sliceable then, there is an open set W of d and a section $s^+ : N \to E_N^+$ $\big($resp., $s^- : N \to E_N^-\big)$ is section such that $s^+(d)$ (resp., $s^-(d) = w$).* □

Corollary 3.6. *Let $\Omega : E \to F$ be cont. F.W. surjection, where in (E, τ) and (F, ω) are F.W.M.T.S. on (D, ρ). If E is multi-locally sliceable, then F is so.*

Definition 3.7. *The F.W.U.T.S. (resp., F.W.L.T.S.) (E, τ) on (D, ρ) is named F.W.U. (resp., F.W.L.) discrete if the projection function X is upper (resp., lower) a local homeomorphism.*

The F.W.M.T.S. (E, τ) on (D, ρ) is named F.W. multi-discrete if the projection function X is multi-local homeomorphism.
This means, we recall that for every point d of D and every e of E_d^+ (resp., E_d^-) there is τ-open set V of e in E and a ρ open set N of d in D where in X maps V homomorphically onto N. It is clear that F.W. multi-discrete topologicalspace and multi-locally sliceable there for F.W. open.
The class of F.W. multi-discrete topologicalspaces are finitely multiplicative.

Proposition 3.8. *let $\{(E^r, \tau_r)\}_{r=1}^k$ be a finite family of F.W. upper (resp., lower) discrete topologicalspace on (D, ρ). Then the F.W.U.T. (resp., F.W.L.T.) product $E = \Pi^D E^r, \tau$ is F.W. upper (resp., lower) discrete.*

Proof . *Given a point $e \in E_d^+$ (resp., E_d^-); $d \in D$, there is for every index r a τ open set U_r of $\pi_r(e)$ inE^r, where in the projection function $X^r = X \circ \pi_r^{-1}$ maps U_r homomorphically onto the ρ-open $X^r(U_r) = W^r$ of d. Then the τ_r-open $\Pi_D U_r$ of e is mapped homomorphically intersection $W = \cap E_r^+$ (resp., E_r^-) which, a ρ open of d.* □

Corollary 3.9. *let $\{(E^r, \tau^r)\}_{r=1}^k$ be a finite family of F.W. multi-discrete topologicalspace on (D, ρ). Then the F.W.M.T. product $E = \Pi^D E^r, \tau$ is F.W. multi-discrete.*

An attractive characterization of F.W. multi-discrete topologicalspaces are given by the following:

Proposition 3.10. *If (E, τ^r) F.W.U.T.S. (resp., F.W.L.T.S.) on (D, ρ). Then, E is F.W. upper (resp., lower) discrete iff:*

(a) *E is F.W. open*
(b) *The diagonal embedding $\Delta : E \to E \times_D E$ is open.*

Proof .
$\Leftarrow$ *Suppose that (a) and (d) are satisfied. Let $e \in E^d; d \in D$, then $\Delta(e) = (e, e)$ admits a $\tau^r \times \tau^r$-open set in $E \times_D E$that is entirely contained in $\Delta(E)$. Without really lacking in general we may suppose the $\tau^r \times \tau^r$-open set of the form $U \times_D U$, where in U is a τ^r-open set of e in E. Then $X|U$is homeomorphism. There for, E is F.W. upper (resp., lower) discrete.*
$\Rightarrow$ *Assum that E is F.W. upper (resp., lower) discrete. We have already seen that E is F.W. open. To prove that Δ is open it is sufficient to prove that $\Delta(E)$ is $\tau^r \times \tau^r$-open in $E \times_D E$. So, let e $\in E^d; d \in D$, and let U be a τ^r-open set of e in E, where in $N = X(U)$ is a ρ-open set of d in D and X maps U homomorphically onto N. Then $U \times_D U$ is contained in $\Delta(E)$ since if not then there exist distinct $\mathcal{E}, \mathcal{E}^* \in E^N$, where in $n \in N$ and $\mathcal{E}, \mathcal{E}^* \in U$, that is absurd.* □

Open subset of F.W. upper (resp., lower) discrete topologicalspace is also F.W. upper (resp., lower) discrete, actually, we have.

Corollary 3.11. *If (E, τ^r) F.W.M.T.S. on (D, ρ). Then, E is F.W. multi-discrete iff:*

(a) *E is F.W. open*
(b) *The diagonal embedding $\Delta : E \to E \times_D E$ is open.*

Proposition 3.12. ***3.5.*** *Assume that $\Omega : E \to F$ is a cont. F.W. injection, where in (E, τ), (F, ω) are F.W. open upper (resp., lower) topologicalspaces on (D, ρ). If F is F.W. upper (resp., lower) discrete. Then E is so.*
Proof . *Consider the diagram shown below.*

$$\begin{tikzcd} E \arrow[r, "\Delta"] \arrow[d, "\Omega"'] & E \times_D E \arrow[d, "\Omega \times \Omega"] \\ F \arrow[r, "\Delta"] & F \times_D F \end{tikzcd}$$

Since Ω is cont. so is $\Omega \times \Omega$. Now $\Delta(E)$ $\omega \times$ ω-open in $N \times_D N$, by Proposition 3.10. Since F is F.W., $\Delta(E) = \Delta(\Omega^+(F))$ $(resp., \Omega^-(F)) = (\Omega \times \Omega)^+$ $(resp., (\Omega \times \Omega)^-)(\Delta(F))\tau$ $\times$ τ-open in $E \times_D E$. Thus, (Proposition 3.13) follows from (Proposition 3.10). □

Corollary 3.13. *Assume that $\Omega : E \to F$ is a cont. F.W. injection, where in (E, τ), (F, ω) are F.W. open multi-topological space on (D, ρ). If F is F.W. multi-discrete. Then E is so.*

Proposition 3.14. ***3.6.*** *Assume that $\Omega : E \to F$ be an open F.W. surjection, where in (E, τ), (F, ω) are F.W. open upper (resp., lower) topologicalspaces on (D, ρ). If E is F.W. upper (resp., lower) discrete. Then F is so.*
Proof .. *In the above figure, with these fresh hypotheses on Ω is E is F.W.U (resp., F.W.L.) discrete .then $\Delta(E)$ is τ $\times$ τ-open in $E \times_D E$ by Proposition 3.10, so $\Delta(F) = \Delta\Omega(M) = (\Omega \times \Omega)(\Delta(E))$ is $(\omega \times \omega)$-open in $F \times_D F$. Thus, Proposition* **??** *follows from Proposition 3.10 again.* □

Corollary 3.15. ***3.6.*** *Assume that $\Omega : E \to F$ be an open F.W. surjection, where in $(E, \tau), (F, \omega)$ are F.W. open multi-topologicalspaces on (D, ρ). If E is F.W. multi-discrete. Then F is so.*

Proposition 3.16. *If $\Omega, \psi : E \to F$ is cont. F.W. function, where in (E, τ) is F.W.U.T. (resp., F.W.L.T.) and (F, ω) is F.W. upper (resp., lower) discrete topological space on (D, ρ). Then the coincidence set $K(\Omega, \psi)$ of Ω and ψ is open in E.*
Proof . *The coincidence set is precisely $\Delta^-(\Omega \times \psi)^-$ $(resp., \Delta^+(\Omega \times \psi)^+)\Delta(N)$, where:*
$E \overset{\Delta}{\to}$ E $\times_D E \overset{\Omega \times \Psi}{\to} F \times_D F \overset{\Delta}{\leftarrow} F$. Hence Proposition 3.16 follows at once from Proposition 3.10. In particular take $E = F$, take $\Omega = id^E$ and take $\psi = s \circ X$ wherein s is a section, we conclude that s is an open embedding when E is F.W. upper (resp., lower)discrete. □

Corollary 3.17. *If $\Omega, \psi : E \to F$ is cont. F.W. function, where in (E, τ) is F.W.M.T.S. and (F, ω) is F.W. multi-discrete topological space on (D, ρ). Then the coincidence set $K(\Omega, \psi)$ of Ω and ψ is open in E.*

Proposition 3.18. ***3.8.*** *If $\Omega : E \to F$ is cont. F.W. function, where in (E, τ) is F.W. open and (F, ω) is F.W. upper (resp., lower) discrete topologicalspace on (D, ρ). Then the F.W. graph:*

$$\begin{tikzcd} E \arrow[r, "\Gamma"] \arrow[d, "\Omega"'] & E \times_D F \arrow[d, "\Omega \times id_F"] \\ F \arrow[r, "\Delta"] & F \times_D F \end{tikzcd}$$

$$\Gamma : E \ \to \ E \times_D F$$

Of Ω is an open embedding.

Proof . *The F.W. graph is defend in the same way as the ordinary graph, but with values in the F.W.U.T. (resp., F.W.L.T.) product, therefore the diagram shown below is commutative.*
Since $\Delta(F)$ is ω-open in $F \times^B F$, by (Proposition 3.10), so $\Gamma(E) = (\Omega \times id^F)^-$ (resp., $(\Omega \times id^F)^+(\Delta(F)))$ is $(\tau \times \omega)$-open in $E \times_D F$. □

Corollary 3.19. *If $\Omega : E \to F$ is cont. F.W. function, where in (E, τ) is F.W. open and (F, ω) is F.W. multi-discrete topologicalspace on (D, ρ). Then the F.W. graph above of Ω is an open embedding.*

Remark 3.20. *If (E, τ) is F.W. upper (resp., lower) discrete topological space on (D, ρ) then for every point $e \in E^d; d \in D$, there is a ρ-open set W of d a unique section $s : W \to E^W$ exist satisfying $s(d) = e$, we may refer to s as the section through m.*
Also, if (E, τ) is F.W. multi-discrete topologicalspace on (D, ρ) then for every point $e \in E^d; d \in D$, there is a ρ-open set W of d a unique section $s : W \to E^W$ exist satisfying $s(d) = e$, we may refer to s as the section through m.

Definition 3.21. *The F.W.U.T.S. (resp., F.W.L.T.S.) (E, τ) on (D, ρ) is named upper (resp., lower) locally sectionable if for every point $d \in D$, admits an open set W and a section $s : W \to E^W$.*

The F.W.M.T.S. (E, τ) on (D, ρ) is named multi-locally sectionable if for every point d ∈ D, admits an open set W and a section $s : W \to E^W$.

Remark 3.22. *The F.W. non-empty upper (resp., lower) locally sliceable upper (resp., lower) topological spaces are upper (resp., lower) locally sectionable, but the converse is false. In fact, upper (resp., lower, multi) locally sectionable upper (resp., lower, multi) topological spaces are not necessarily F.W. open, for example take $E = (-1, 1] \subset R$ with $(E, \tau) : \tau_1 = \tau_2$, the natural projection onto $D = R|Z; (D, \rho) : \rho_1\rho_2$.*
The class of upper (resp., lower, multi) locally sectionable upper (resp., lower, multi) topological spaces is finitely multiplicative.

Proposition 3.23. *If $\{(E^{r,}\tau^r)\}$ is a finite family of upper (resp., lower) locally sectionable upper (resp., lower) topologicalspaces on (D, ρ). The F.W.U.T. (resp., F.W.L.T.) product $E = \Pi^D M^r$ is upper (resp., lower) locally sectionable.*

Proof . *Given a point d of D, there exist ρ-open set N^r of d and a section $s^r : N^r \to M^r|N^r$ for every index r. Since there are finite number of index the intersection N of the ρ-open sets N^r is also a ρ-open set of d, and a section $s^r : N \to (\Pi_D M^r)^N$ is given by $\pi^r \circ s(N) = s^r(N)$, for $n \in N$.* □

Our last two result apply equally well to every of the above three propositions.

Corollary 3.24. *If $\{(E^{r,}\tau^r)\}$ is a finite family of multi-locally sectionable multi-topologicalspaces on (D, ρ). The F.W.M.T. product $E = \Pi_D M^r$ is multi-locally sectionable.*

Proposition 3.25. *If $\{(E^{r,}\tau^r)\}$ is a F.W. upper (resp., lower) discrete topologicalspace on (D, ρ). Suppose that $(E^{r,}\tau^r)$ is upper (resp., lower) locally sliceable, F.W. upper (resp., lower) discrete or upper (resp., lower) locally sectionable on (D, ρ). Then so is E_{D^*} on D^* for every ρ-open set D^* of D.*

Corollary 3.26. *If $\{(E^{r,}\tau^r)\}$ is a F.W. multi-discrete topological space on (D, ρ). Suppose that $(E^{r,}\tau^r)$ is multi-locally sliceable, F.W. multi-discrete or multi-locally sectionable on (D, ρ). Then so is E_{D^*} on D^* for every ρ-open set D^* of D.*

Proposition 3.27. *Let $(E^{r,}\tau^r)$ be a F.W.U.T.S. (resp. F.W.L.T.S.) on (D, ρ). Assume that E_{D_j} is upper (resp., lower) locally sliceable F.W. upper (resp., lower) discrete or upper (resp., lower) locally sectionable on D^j for every member D^j of a ρ-open covering of D. So is E over*

Corollary 3.28. *Let $(E^{r,}\tau^r)$ be F.W.M.T.S. on (D, ρ). Assume that E_{D_j} is multi-locally sliceable F.W. multi-discrete or multi-locally sectionable on D^j for every member D^j of a ρ- open covering of D. So is E on D.*

Remark 3.29. *It is not difficult to give example of different F.W. multi-discrete multi-topology on the same F.W. set that are in equivalent, as (F.W.M.T.). For this reason, we must be careful not to say the F.W. multi-discrete topology.*

References

[1] A.A. Khadra, S.S. Mahmoud and Y.Y. Yousif, *Fibrewise near topological spaces*, J. Comput. 4(5) (2012) 223–230.
[2] T. Banzaru, *Multi-functions and M-product spaces*, Bull. Stin. Tech. Inst. Politech. Timisoara, Ser. Mat. Fiz. Mer. Teor. Apl., 17(31) (1972) 17–23.
[3] N. Bourbaki, *General Topology, Part 1*, Addison Wesley, Reading, Mass, 1996.
[4] R. Englking, *Outline of General Topology*, Amsterdam, 1989.
[5] I.M. James, *Fibrewise Topology*, Cambridge University Press, London, 1989.
[6] Y.Y. Yousif, *Some result on fibrewise topological spaces*, Ibn Al-Haitham J. Pure Appl. Sci. 21(2) (2008) 118–132.
[7] Y.Y. Yousif, *Some result on fibrewise Lindelöf and locally Lindelöf topological spaces*, Ibn Al-Haitham J. Sci. 22(3) (2009) 191–198.