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Change of measure in fractional stochastic differential equation

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Abstract

Change of measure is a very well known common criterion in both the probability rules and applications. The change of measure is a transformation from actual measure to equivalent measure. We will employ the change of measure in Fractional Stochastic Differential Equations (FSDE), which is a general form of Stochastic Differential Equation (SDE). We will implement our method to some important examples, like, Fractional Brownian Motion (FBM) and Fractional Levy process (FL).

Keywords: Change of measure, Stochastic differential equations, Fractional stochastic differential equations, Fractional Brownian motion, Fractional Levy process.
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1. Introduction

The main feature in probability theory is the change of measure. We will study the change of measure in Fractional Differential Equations (FSDE) which is a general form of Stochastic Differential Equation (SDE). The motivation of our paper is to generated change of measure from Standard Stochastic Differential Equation (SDE) to Fractional Stochastic Differential Equation. The context of our work, we will suggest Fractional Stochastic Differential Equation (FSDE) is compound between Differential Equations and Fractional Brownian motion (FBM).

The Fractional Brownian motion is a general form of Standard Brownian motion (H=0.5). The Fractional coiffecients is between 0 and 1.

The aim of this paper is to study the difference between the original measure to equivalent measure to some popular examples, like from Fractional Brownian motion and Fractional Levy processes mathematically.

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In literature review, Klebaner, Fima C. (2012) [7] presents some features about change of measure such as the changing of measure value for random variables, changing of measure for in general space, changing of measure in processes, changing of measure in wiener measure and changing of measure about point process.

Interestingly, Barndorff. Nielsen, O. E and Shiryaev, Albert (2015) [3]. Show the main facts on change of measure and explain the change of measure in Stochastic Differential Equations particularly based on Levy process. Moreover, Rheinlander, T. (2011) [13] and Miyahara, Y. (2012) [12] show the change of measure for exponential Levy process with application in asset pricing model. Marquardt, T. (2004, 2006) [8] and [9] suggest the Fractional Levy process that is Le'vy process without Brownian motion with application to long memory moving averages processes. Al-Saadony, M.F. and Al-Obaidi, W.J. (2021) [1] present a fractional Levy process that is a fractional Brownian motion for Vasicek Interest Rate process. It is a good example for fractional Stochastic Differential Equation. They estimated the parameters (drift, Volatility and Hurst) for Iraq Sock Market Index (ISX60).

The paper is arranged as follows: in section 2 we illustrated some essential features about changing of measure. In section 3 we apply change of measure for Fractional Stochastic Differential Equation. Finally, in section 4 we conclude some important feature for the change of measure in our processes.

2. Change of measure criteria

Foremost the probability space contains $(\Omega, \mathrm{F_t}, \mathrm{P})$, here Ω is the collection of all available outcomes of a random trial, F_t is the 6-Algerba generated by Ω and interestingly P is a probability measure. Therefore, we will attempt to change probability measure P to another probability measure for some Fractional Stochastic differential Equations as Fractional Le'vy (FL) process and Fractional Geometric Brownian motion (FBM).

Now, we will present some definitions and important properties to help us to understand change of measure in our process as shown in [2]; [11] and [6].

Definition 2.1. *If we have two probability measure P of X_t and Q for Y_t. Then if we attempt to replace the probability measure P to Q which is called change of measure.*

These probability measure have some probability that are follows:-

1- Absolutely Continuous: If P is definitely continuous in regard to Q, hence there will be a positive function $\mathrm{F_t}$, so that:

$$Q_A = \int_A f_1(w) d\mathrm{P}(w) \qquad A \in F \tag{2.1}$$

Simultaneously, If Q is definitely continuous as to P then there will be a positive function f_2, so that

$$\mathrm{P}_A = \int_A f_2(w) d\mathrm{Q}(w) \qquad A \in F \tag{2.2}$$

Where
$\mathrm{f_1}$ is the density of Q regards to P
$\mathrm{f_2}$ is the density of P regards to Q

2- Equivalent probability Measure : P and Q are equaled probability measure if P is definitely continuous with regarding to Q and Q is definitely continuous with respect to P.

3. Change of Measure in Fractional Stochastic Differential Equation

We will employ the change of measure in some popular examples of Fractional Stochastic Differential Equation (FSDE). The differential equation driven by a fractional Le'vy process is called the Fractional Stochastic Differential Equation. The popular stochastic process are.

3.1. Fractional Le'vy process

We suggest that Fractional Le'vy process is division of Le'vy process and Fractional Brownian motion, whereas Marquardt [10] has suggested that a Fractional Le'vy process considers a general Le'vy process without Brownian motion when H $\neq$ 0.5 [4]. The Hurst coefficient H is between 0 and 1 . If H $> 1/2$ is called a long memory process and if H $< 1/2$ is called a short memory process. Now, the representing of Fractional Stochastic Differential Equation is:-

$$\mathrm{dx}_t = \mu x_t d_t + \sigma x_t dL_t^H \qquad H \in (0,1) \tag{3.1}$$

L_t^H is a Fractional Le'vy process at time t, under probability measure P.
We assume a new probability measure Q that equal to P, that is:

$$\mathrm{L}_t^H = w_t^H + \theta t^{2H} + aL_t \tag{3.2}$$

w_t^H is a Fractional Brownian motion.
L_t is a Le'vy process.
a and θ are constant.
So, we substitute the equation (3.2) in (3.1) as follows:-

$$dx_t = \mu x_t d_t + \sigma x_t dL_t^H \tag{3.3}$$

$$dx_t = \mu x_t d_t + \sigma x_t d\left[w_t^H + \theta t^{2H} + aL_t\right] \tag{3.4}$$

$$dx_t = \mu x_t d_t + \sigma x_t dw_t^H + \sigma\theta x_t dt^{2H} + a\sigma x_t dL_t \tag{3.5}$$

Assuming that $\mu = 0$ and $\theta = 0$, then

$$\mathrm{d}x_t = \sigma x_t dw_t^H + a\sigma x_t dL_t \quad \ldots\ldots\ldots\ldots(3) \tag{3.6}$$

using wick Ito lemma [4] to solve the equation (3.6). So the solution is as:

$$x_t = x_0 \exp\left[-0.5\sigma^2 t^{2H} - \sigma w_t^H - \lambda\left(\exp^a - 1\right) + \sigma^2 L_t\right] \tag{3.7}$$

3.2. Fractional Geometric Brownian motion

We suggest that the Fractional represents the Fractional Brownian motion.
Therefore, the form of Stochastic Differential Equation founded by Fractional Brownian motion is:-

$$x_t = \mu x_t d_t + \sigma x_t dB_t^H$$

Where
B_t^H is a Fractional Brownian motion with the Hurst parameter, under probability measure P.
We assume a new probability measure Q which is equivalent to P, that is:

$$B_t^{NH} = B_t^H + \theta t^{2H}$$

Hence,

$$dx_t = \mu x_t d_t + \sigma x_t dB_t^{NH} \tag{3.8}$$
$$= \mu x_t d_t + \sigma x_t d\left[B_t^H + \theta t^{2H}\right] \tag{3.9}$$
$$dx_t = \mu x_t d_t + \sigma x_t dB_t^H + \sigma\theta x_t d_t^{2H} \tag{3.10}$$

As we mentioned before, wick I to lemma will be getting the solution of Equation is:

$$x_t = x_0 \exp\left[-0.5\sigma^2 t^{2H} + \sigma w_t^H\right]$$

When $\mu = 0$ and $\theta = 0$.

4. Conclusion

We have presented the change of measure for the Fractional Stochastic Differential Equations (FSDE) such as Fractional Levy process and Fractional Geometric Brownian motion. Obviously, the transform form the original to new probability measure are approximately symmetry and light-tail distribution. Finally our processes have the good performance under the new probability measure.

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