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New results of modern concept on the fourth-Hankel determinant of a certain subclass of analytic functions

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Abstract

A form for the fourth Hankel determinant is given in this paper as

$$H_4(1) = \begin{vmatrix} 1 & \mathfrak{a}_2 & \mathfrak{a}_3 & \mathfrak{a}_4 \\ \mathfrak{a}_2 & \mathfrak{a}_3 & \mathfrak{a}_4 & \mathfrak{a}_5 \\ \mathfrak{a}_3 & \mathfrak{a}_4 & \mathfrak{a}_5 & \mathfrak{a}_6 \\ \mathfrak{a}_4 & \mathfrak{a}_5 & \mathfrak{a}_6 & \mathfrak{a}_7 \end{vmatrix}$$

The modern concept of the fourth Hankel determinant is studied for the subclass of analytic functions $\mu(\beta, \lambda, t)$ defined here using the concept of subordination. Bounds on the coefficients $|a_n|$ with n = 2,3,4, 5,6,7 for the functions in this newly introduced class are given and the upper bound of the fourth Hankel determinant for this class is obtained. Lemmas used by the authors of this paper improve the results from a previously published paper. Interesting particular cases are given in the corollaries of the main theorems.

Keywords: Subordination, Analytic Function, Fourth Hankel Determinant, Coefficient Bounds.
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1. Introduction and Definition

Assign μ refers to the experience of functions f analytic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, and normalized by conditions $f(0) = 0, f'(0) - 1 = 0$.

An analytic function $f \in \mu$ has Taylor series expansion of the form:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad (z \in U). \tag{1.1}$$

If a Schwarz function exists, w be analytic such that $f(z) = g(w(z))$ with $w(0) = 0$ and $|w(z)| \leq 1$, $(z \in U)$, the function f is said to be subordinate to g in U and expressed as $f(z) \prec g(z)$.

Using the method of subordination, Ma and Minda [9] developed a class of starlike functions and examined classes $S^*(\Phi)$ and $G^*(\Phi)$ which are defined by

$$S^*(\Phi) = \{f \in \mu : \frac{zf'(z)}{f(z)} \prec \Phi(z),\ z \in U\},$$

and

$$G^*(\Phi) = \{f \in \mu : 1 + \frac{zf''(z)}{f(z)} \prec \Phi(z),\ z \in U\}$$

Mediratta et al. [10] introduced the family $S_e^* := S^*(e^z)$, which is defined as:

$$S_e^* = \{f \in \mu : \frac{zf'(z)}{f(z)} \prec e^z,\ z \in U\} \tag{1.2}$$

In the same way, using the Alexandar type connection in [10], we get:

$$G_e^* = \{f \in \mu : 1 + \frac{zf''(z)}{f(z)} \prec e^z,\ z \in U\} \tag{1.3}$$

We can deduce from the preceding explanation that the families S_e^* and G_e^* examined in this idea are symmetric around the real axis.

Pommerenke [12, 13] defined the Hankel determinant $H_q(n)$ as a function $f \in \mu$ of the type (1.1) for given parameters $q, n \in \mathbb{N} = \{1, 2, ...\}$, as follows:

$$H_q(n) = \begin{vmatrix} \mathfrak{a}_n & \mathfrak{a}_{n+1} & \cdots & \mathfrak{a}_{n+q-1} \\ \mathfrak{a}_{n+1} & \mathfrak{a}_{n+2} & \cdots & \mathfrak{a}_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ \mathfrak{a}_{n+q-1} & \mathfrak{a}_{n+q} & \cdots & \mathfrak{a}_{n+2q-2} \end{vmatrix},\ (a_1 = 1). \tag{1.4}$$

In the theory of singularities [5] and the investigation of power series with integral coefficients, the Hankel determinant is very important. The reader is encouraged to read [3] and [11] for more information. For several subfamilies of univalent functions, the growth of $H_q(n)$ has been explored. We know that the function $H_2(1) = a_3 - a_2^2$ for $q = 2$ and $n = 1$ is a well recognized Fekete-s̈zego functional. For the bi-convex and bi-starlike classes, the second Hankel determinant $H_2(2)$ is given by $H_2(2) = a_2 a_4 - a_3^2$ ([1, 4, 7]). On the other hand, Krishna et al. [8] provided a sharp estimate of $H_2(2)$ for the set of Bazilevic functions. Srivastava and colleagues [17, 16, 15] recently discovered the symmetric q-derivative operator is used to estimate the second Hankel determinant for bi-univalent functions.

For functions, the third-order Hankel determinant of form (1.1) is written as:

$$H_3(1) = \begin{vmatrix} 1 & \mathfrak{a}_2 & \mathfrak{a}_3 \\ \mathfrak{a}_2 & \mathfrak{a}_3 & \mathfrak{a}_4 \\ \mathfrak{a}_3 & \mathfrak{a}_4 & \mathfrak{a}_5 \end{vmatrix} = -a_5 a_2^2 + 2a_2 a_3 a_4 - a_3^3 + a_3 a_5 - a_4^2.$$

Babalola [2] presented the first work on $H_3(1)$ in 2010.

The third Hankel determinant was studied in [4] for a certain subclass of starlike functions and this form was used in that paper for computing the fourth Hankel determinant for the same subclass. The third Hankel determinant was not used in the present paper as mentioned in the cited paper [4]. A definition for fourth Hankel determinant is introduced in our paper written as the determinant of a matrix. A new subclass of analytic functions is also introduced by means of subordination and estimate are given for the upper bound of the fourth Hankel determinant for functions in this subclass.

We now establish a concept based on the fourth Hankel determinant as follows:

$$H_4(1) = \begin{vmatrix} 1 & \mathfrak{a}_2 & \mathfrak{a}_3 & \mathfrak{a}_4 \\ \mathfrak{a}_2 & \mathfrak{a}_3 & \mathfrak{a}_4 & \mathfrak{a}_5 \\ \mathfrak{a}_3 & \mathfrak{a}_4 & \mathfrak{a}_5 & \mathfrak{a}_6 \\ \mathfrak{a}_4 & \mathfrak{a}_5 & \mathfrak{a}_6 & \mathfrak{a}_7 \end{vmatrix} = -a_4\mathcal{S}_1 + a_5\mathcal{S}_2 - a_6\mathcal{S}_3 + a_7\mathcal{S}_4, \tag{1.5}$$

where

$$\begin{aligned} |\mathcal{S}_1| &= |a_2||a_4a_6 - a_5^2| + |a_3||a_3a_6 - a_4a_5| - |a_4||a_3a_5 - a_4^2|, \\ |\mathcal{S}_2| &= |a_4a_6 - a_5^2| - |a_2||a_3a_6 - a_4a_5| + |a_3||a_3a_5 - a_4^2|, \\ |\mathcal{S}_3| &= |a_3a_6 - a_4a_5| + |a_2||a_2a_6 - a_3a_5| - |a_4||a_2a_4 - a_3^2|, \\ |\mathcal{S}_4| &= |a_3||a_2a_4 - a_2^2| - |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|, \end{aligned} \tag{1.6}$$

The first and second kinds of Chebyshev polynomials are the most well-known and defined in the situation of a real variable x on $(-1, 1)$ by

$$T_n(x) = \cos(n \ \arccos \ x),$$

$$U_n(x) = \frac{\sin[(n+1)\arccos \ x]}{\sin(\arccos \ x)} = \frac{\sin[(n+1)\arccos \ x]}{\sqrt{1-x^2}}, \text{ respectively.}$$

We think about the function.

$$H(t,z) = \frac{1}{1-2t+z^2}, \ t \in \left(\frac{1}{2}, 1\right), \ z \in U.$$

It is common knowledge that if $t = \cos\alpha$, $\alpha \in \left(0, \frac{\pi}{3}\right)$, then

$$H(t,z) = 1+\sum_{n=1}^{\infty} \frac{\sin[(n+1)\alpha]}{\sin\alpha} z^n = 1+2\cos\alpha\, z+(3\cos^2\alpha-\sin^2\alpha)z^2+(8\cos^3\alpha-4\cos\alpha)z^3+..., z \in U,$$

that is

$$H(t,z) = 1 + U_1(t)z + U_2(t)z^2 + U_3(t)z^3 + U_4(t)z^4 + ...,\ t \in \left(\frac{1}{2}, 1\right),\ z \in U.$$

where $U_n(t) = \frac{\sin[(n+1)\arccos\ t]}{\sqrt{1-t^2}},\ n \in \mathbb{N},$

The second class Chebyshev polynomials are represented by $U_n(t)$. The second kind Chebyshev polynomials notion explains that

$$U_{n+1}(t) = 2tU_n(t) - U_{n-2}(t).$$

We get that

$$U_1(t) = 2t,\ U_2(t) = 4t^2 - 1,\ U_3(t) = 8t^3 - 4t,\ (\text{for each } n \in \mathbb{N}).$$

The first goal of this paper is to look into the estimate of $|H_4(1)|$ for the above-mentioned classes S_e^* and G_e^*. Moreover.

Lemma 1.1. *[6] If P be a class of all analytic functions $p(z)$ of the form:*

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{1.7}$$

with $p(0) = 1$ and $\Re e\{p(z)\} > 0$ for all $z \in U$. Then $|p_n| \leq 2$, for every $(n = 1, 2, 3, ...)$. This disparity is sharp for each n.

Mendiratta et al. recently developed the S_1^* subclass of analytic functions related with exponential functions in [10].

Lemma 1.2. *[10] If the function $f \in S_1^*$ and of the kind 1.1, then*

$$|a_2| \leq 1,\ |a_3| \leq \frac{3}{4},\ |a_4| \leq \frac{17}{36},\ |a_5| \leq 1.$$

where S_1^* denote the class of analytic functions to third Hankel determinant.

Lemma 1.3. *[14] If the function $f \in S_e^*$ and of the form 1.1, then*

$$|a_2| \leq 1,\ |a_3| \leq \frac{3}{4},\ |a_4| \leq \frac{1}{18},\ |a_5| \leq \frac{1}{96}\ |a_6| \leq \frac{1}{600}, \textit{and}\ \ |a_7| \leq \frac{2401}{3600}.$$

Lemma 1.4. *[14] If the function $f \in G_e^*$ and of the kind 1.1, then*

$$|a_2| \leq \frac{1}{2},\ |a_3| \leq \frac{3}{12},\ |a_4| \leq \frac{1}{72},\ |a_5| \leq \frac{1}{480},\ |a_6| \leq \frac{1}{3600}, \textit{and}\ \ |a_7| \leq \frac{343}{3600}.$$

2. The Subclass $\boldsymbol{\mu(\beta, \lambda, t)}$

We're going to add a new subclass now, of analytic functions using the concept of subordination.

Definition 2.1. *A function $f \in \mu$ given by (1.1) is said to be in the class $\mu(\beta, \lambda, t)$ if the following conditions holds:*

$$1 + [(1-\lambda)z^3 f''''(z) + \beta z^2 f'''(z) + \beta z f'''(z)] - 1 \prec H(t, z), \tag{2.1}$$

where $(\lambda \geq 0, \beta \geq 0)$, and

$$H(t, z) = 1 + U_1(t)z + U_2(t)z^2 + U_3(t)z^3 + U_4(t)z^4 + ..., \ t \in \left(\frac{1}{2}, 1\right), \ z \in U.$$

We'll start by calculating estimates for the coefficients $|a_n|$ and $n = 2, 3, 4, 5, 6, 7$, for functions in the class $\mu(\beta, \lambda, t)$.

The primary findings of our current inquiry are now stated and proven. related to this subclass.

In the following Theorem, we find the coefficients $|a_2|, |a_3|, |a_4|, |a_5|, |a_6|$ and $|a_7|$.

Theorem 2.2. *If the function f is given by (1.1) belongs to the subclass $\mu(\beta, \lambda, t)$, then*

$$|a_2| \leq 2, \ |a_3| \leq \frac{2}{(1+\beta)}, \ |a_4| \leq \frac{3}{2(1-\lambda) + 2\beta + 1}, \ |a_5| \leq \frac{27}{30(1-\lambda) + 15\beta + 5},$$

$$|a_6| \leq \frac{54}{60(1-\lambda) + 20\beta + 5}, \textit{ and } \ |a_7| \leq \frac{162}{140(1-\lambda) + 35\beta + 7}.$$

Proof . If $f \in \mu(\beta, \lambda, t)$, then there exist an analytic function ϑ in U with $|\vartheta(z)| \leq 1$, we have

$$1 + [(1-\lambda)z^3 f''''(z) + \beta z^2 f'''(z) + \beta z f'''(z)] - 1 = H(t, \mathcal{T}(z)).$$

Then, according to the concept of subordination, $\mathcal{T}$ is a Schwarz function of the form:

$$\mathcal{T}(z) = \sum_{n=1}^{\infty} p_n z^n, \ (z \in U),$$

$$\begin{aligned} F(z) = \frac{1+\mathcal{T}(z)}{1-\mathcal{T}(z)} &= 1 + 2p_1 z + 2(p_2 + p_1^2)z^2 + 2[p_3 + p_1(2p_2 + p_1^2)]z^3 + 2[p_4 + p_2^2 + p_1^2(3p_2 + p_1^2) + 2p_1p_3]z^4 \\ &+ 2[p_5 + 2p_2(p_3 + 2p_1^3) + 3p_1(p_1p_3 + p_2^2) + (p_1(2p_4 + p_1^4))]z^5 \\ &+ 2[p_6 + p_1^3(4p_3 + p_1^3) + p_1^2(3p_4 + 5p_2p_1^2) + 2p_1(p_5 + 3p_2p_3) + p_3^2 + p_2^2(p_2 + 6p_1^2) + 2p_2p_4]z^6 + ... \end{aligned} \tag{2.2}$$

Since $f \in \mu$ has the Maclurian series given by (1.1), we have

$$\begin{aligned} &1 + [(1-\lambda)z^3 f''''(z) + \beta z^2 f'''(z) + \beta z f'''(z)] - 1 = \\ &\big[2a_2 z + 6(\beta+1)a_3 z^2 + [24(1-\lambda) + 24\beta + 12]a_4 z^3 + [120(1-\lambda) + 60\beta + 20]a_5 z^4 \\ &+[360(1-\lambda) + 120\beta + 30]a_6 z^5 + [840(1-\lambda) + 210\beta + 42]a_7 z^6 + ...\big]. \end{aligned} \tag{2.3}$$

Using (2.2) with (2.3), comparing the coefficients of

$$2a_2 = 2p_1,$$

$$6(\beta+1)a_3 = 2(p_2+p_1^2),$$

$$[24(1-\lambda)+24\beta+12]a_4 = 2[p_3+p_1(2p_2+p_1^2)],$$

$$[120(1-\lambda)+60\beta+20]a_5 = 2[p_4+p_2^2+p_1^2(3p_2+p_1^2)+2p_1p_3],$$

$$[360(1-\lambda)+120\beta+30]a_6 = 2[p_5+2p_2(p_3+2p_1^3)+3p_1(p_1p_3+p_2^2)+(p_1(2p_4+p_1^4))],$$

$$[840(1-\lambda)+210\beta+42]a_7 = 2[p_6+p_1^3(4p_3+p_1^3)+p_1^2(3p_4+5p_2p_1^2)+2p_1(p_5+3p_2p_3)+p_3^2+p_2^2(p_2+6p_1^2)+2p_2p_4].$$

By Lemma 1.1 ($|p_n| \leq 2$ for the coefficients $|a_n|$ and $n = 2,3,4,5,6,7$), we obtain

$$|a_2| \leq 2, \tag{2.4}$$

$$|a_3| \leq \frac{2}{(1+\beta)}, \tag{2.5}$$

$$|a_4| \leq \frac{3}{2(1-\lambda)+2\beta+1}, \tag{2.6}$$

$$|a_5| \leq \frac{27}{30(1-\lambda)+15\beta+5}, \tag{2.7}$$

$$|a_6| \leq \frac{54}{60(1-\lambda)+20\beta+5}, \tag{2.8}$$

$$\text{and } |a_7| \leq \frac{162}{140(1-\lambda)+35\beta+7}, \tag{2.9}$$

The proof is complete. □

We can find $|H_4(1)|$ in the following Theorem.

Theorem 2.3. *If a function f of form* (1.1) *belongs to the subclass $\mu(\beta,\lambda,t)$, then*

$$|H_4(1)| \leq \frac{-\delta^4(1-\lambda+2\beta)(1-\lambda+5\beta)(1-\lambda)[105(1-\lambda+3\beta)(K_1(t,\varpi))-189(K_2(t,\varpi))]}{K(t,\varpi)} +$$
$$\frac{\delta^3(1-\lambda+2\beta)^2(1-\lambda+5\beta)[8450(1-\lambda+3\beta)(1-\lambda+4\beta)(K_4(t,\varpi))-546(1-\lambda+2\beta)(1-\lambda)^2(K_3(t,\varpi))]}{K(t,\varpi)},$$

where

$$K(t,\varpi) = (1+\beta)^2(2(1-\lambda)+2\beta+1)^3(30(1-\lambda)+15\beta+5)^3(60(1-\lambda)+20\beta+5)^2(140(1+-\lambda)+35\beta+7),$$

$$\begin{aligned}K_1(t,\varpi) &= 324(30(1-\lambda)+15\beta+5)^2(2(1-\lambda)+2\beta+1)\\&- 1458(2(1-\lambda)+2\beta+1)^2(60(1-\lambda)+20\beta+5)\\&+ 216(2(1-\lambda)+2\beta+1)^2(30(1-\lambda)+15\beta+5)^2\\&- 162(1+\beta)(60(1-\lambda)+20\beta+5)(2(1-\lambda)+2\beta+1)\\&- (2(1-\lambda)+2\beta+1)(1+\beta)(30(1-\lambda)+15\beta+5)^2(60(1-\lambda)+20\beta+5)\\&+ (1+\beta)^2(30(1-\lambda)+15\beta+5)^2(60(1-\lambda)+20\beta+5),\end{aligned}$$

$$\begin{aligned}K_2(t,\varpi) &= 162(1+\beta)(30(1-\lambda)+15\beta+5)^2\\&- 729(1+\beta)(2(1-\lambda)+2\beta+1)(60(1-\lambda)+20\beta+5)\\&- 216(2(1-\lambda)+2\beta+1)(30(1-\lambda)+15\beta+5)^2\\&+ 162(1+\beta)(60(1-\lambda)+20\beta+5)(30(1-\lambda)+15\beta+5)\\&+ 108(1+\beta)(2(1-\lambda)+2\beta+1)(30(1-\lambda)+15\beta+5)(60(1-\lambda)+20\beta+5)\\&- 18(1+\beta)(30(1-\lambda)+15\beta+5)^2(60(1-\lambda)+20\beta+5),\end{aligned}$$

$$\begin{aligned}K_3(t,\varpi) &= 108(2(1-\lambda)+2\beta+1)(30(1-\lambda)+15\beta+5)\\&- 81(1+\beta)^2(2(1-\lambda)+2\beta+1)(60(1-\lambda)+20\beta+5)\\&+ 216(1+\beta)(30(1-\lambda)+15\beta+5) - 108(60(1-\lambda)+20\beta+5)(1+\beta)(2(1-\lambda)+2\beta+1)^2\\&- 18(1+\beta)^2(30(1-\lambda)+15\beta+5)(60(1-\lambda)+20\beta+5)\\&- 12(2(1-\lambda)+2\beta+1)(30(1-\lambda)+15\beta+5)^2(60(1-\lambda)+20\beta+5),\end{aligned}$$

$$\begin{aligned}K_4(t,\varpi) &= [12-8(2(1-\lambda)+2\beta+1)](2(1-\lambda)+2\beta+1)\\&- [9(1+\beta)-12(2(1-\lambda)+2\beta+1)](30(1-\lambda)+15\beta+5)\\&+ [54-108(1+\beta)](2(1-\lambda)+2\beta+1)^2\end{aligned}$$

Proof . From (1.1), fourth Hankel determinant can be written as:

$$|H_4(1)| = |-a_4\mathcal{S}_1 + a_5\mathcal{S}_2 + a_6\mathcal{S}_3 + a_7\mathcal{S}_4|,$$

where

$$|\mathcal{S}_1| = |a_2||a_4a_6 - a_5^2| + |a_3||a_3a_6 - a_4a_5| - |a_4||a_3a_5 - a_4^2|,$$

$$|\mathcal{S}_2| = |a_4a_6 - a_5^2| - |a_2||a_3a_6 - a_4a_5| + |a_3||a_3a_5 - a_4^2|,$$

$$|\mathcal{S}_3| = |a_3a_6 - a_4a_5| + |a_2||a_2a_6 - a_3a_5| - |a_4||a_2a_4 - a_3^2|,$$

$$|\mathcal{S}_4| = |a_3||a_2a_4 - a_2^2| - |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|.$$

Inserting (2.4),(2.5),(2.6),(2.7),(2.8),(2.9) in (1.6), we get

$$|\mathcal{S}_1| = \frac{K_1(t,\varpi)}{D_1(t,\varpi)}, \tag{2.10}$$

$$|\mathcal{S}_2| = \frac{K_2(t,\varpi)}{D_2(t,\varpi)}, \tag{2.11}$$

$$|\mathcal{S}_3| = \frac{K_3(t,\varpi)}{D_3(t,\varpi)}, \tag{2.12}$$

$$|\mathcal{S}_4| = \frac{K_4(t,\varpi)}{D_4(t,\varpi)}, \tag{2.13}$$

where

$$\begin{aligned}
D_1(t,\varpi) &= (1+\beta)^2(2(1-\lambda)+2\beta+1)^2(30(1-\lambda)+15\beta+5)^2(60(1-\lambda)+20\beta+5),\\
D_2(t,\varpi) &= (1+\beta)(2(1-\lambda)+2\beta+1)(30(1-\lambda)+15\beta+5)^2(60(1-\lambda)+20\beta+5),\\
D_3(t,\varpi) &= (1+\beta)^2(2(1-\lambda)+2\beta+1)^2(30(1-\lambda)+15\beta+5)(60(1-\lambda)+20\beta+5),\\
D_4(t,\varpi) &= (1+\beta)(2(1-\lambda)+2\beta+1)^2(30(1-\lambda)+15\beta+5).
\end{aligned}$$

Using (2.10),(2.11),(2.12),(2.13) in (1.5), we get

$$|H_4(1)| \le \frac{(60(1-\lambda)+20\beta+5)(140(1-\lambda)+35\beta+7)[-3(30(1-\lambda)+15\beta+5)(K_1(t,\varpi))+27(1+\beta)(2(1-\lambda)+2\beta+1)(K_2(t,\varpi))]}{K(t,\varpi)}$$
$$+\frac{(2(1-\lambda)+2\beta+1)(30(1-\lambda)+15\beta+5)^2(140(1-\lambda)+35\beta+7)[162(1+\beta)(K_4(t,\varpi))-54(2(1-\lambda)+2\beta+1)(K_3(t,\varpi))]}{K(t,\varpi)}$$

where

$$K(t,\varpi) = (1+\beta)^2(2(1-\lambda)+2\beta+1)^3(30(1-\lambda)+15\beta+5)^3(60(1-\lambda)+20\beta+5)^2(140(1-\lambda)+35\beta+7),$$

$$\begin{aligned}
K_1(t,\varpi) &= 324(30(1-\lambda)+15\beta+5)^2(2(1-\lambda)+2\beta+1)\\
&\quad - 1458(2(1-\lambda)+2\beta+1)^2(60(1-\lambda)+20\beta+5)\\
&\quad + 216(2(1-\lambda)+2\beta+1)^2(30(1-\lambda)+15\beta+5)^2\\
&\quad - 162(1+\beta)(60(1-\lambda)+20\beta+5)(2(1-\lambda)+2\beta+1)\\
&\quad - (2(1-\lambda)+2\beta+1)(1+\beta)(30(1-\lambda)+15\beta+5)^2(60(1-\lambda)+20\beta+5)\\
&\quad + (1+\beta)^2(30(1-\lambda)+15\beta+5)^2(60(1-\lambda)+20\beta+5),
\end{aligned}$$

$$\begin{aligned}
K_2(t,\varpi) &= 162(1+\beta)(30(1-\lambda)+15\beta+5)^2\\
&\quad - 729(1+\beta)(2(1-\lambda)+2\beta+1)(60(1-\lambda)+20\beta+5)\\
&\quad - 216(2(1-\lambda)+2\beta+1)(30(1-\lambda)+15\beta+5)^2\\
&\quad + 162(1+\beta)(60(1-\lambda)+20\beta+5)(30(1-\lambda)+15\beta+5)\\
&\quad + 108(1+\beta)(2(1-\lambda)+2\beta+1)(30(1-\lambda)+15\beta+5)(60(1-\lambda)+20\beta+5)\\
&\quad - 18(1+\beta)(30(1-\lambda)+15\beta+5)^2(60(1-\lambda)+20\beta+5),
\end{aligned}$$

$$\begin{aligned}K_3(t,\varpi) &= 108(2(1-\lambda)+2\beta+1)(30(1-\lambda)+15\beta+5)\\&- 81(1+\beta)^2(2(1-\lambda)+2\beta+1)(60(1-\lambda)+20\beta+5)\\&+ 216(1+\beta)(30(1-\lambda)+15\beta+5) - 108(60(1-\lambda)+20\beta+5)(1+\beta)(2(1-\lambda)+2\beta+1)^2\\&- 18(1+\beta)^2(30(1-\lambda)+15\beta+5)(60(1-\lambda)+20\beta+5)\\&- 12(2(1-\lambda)+2\beta+1)(30(1-\lambda)+15\beta+5)^2(60(1-\lambda)+20\beta+5),\end{aligned}$$

$$\begin{aligned}K_4(t,\varpi) &= [12-8(2(1-\lambda)+2\beta+1)](2(1-\lambda)+2\beta+1)\\&- [9(1+\beta)-12(2(1-\lambda)+2\beta+1)](30(1-\lambda)+15\beta+5)\\&+ [54-108(1+\beta)](2(1-\lambda)+2\beta+1)^2.\end{aligned}$$

□

In the scenario when β is equal to zero, The following is a Corollary:

Corollary 2.4. *Let* (1.1) *denote that* $f(z)$ *belongs to the class* $\mu(0,\lambda,t)$. *Then there's,*

$$|H_4(1)| \leq \frac{(60(1-\lambda)+5)(140(1-\lambda)+7)[-3(30(1-\lambda)+5)(\Lambda_1(t,\varpi))+27(2(1-\lambda)+1)(\Lambda_2(t,\varpi))]}{\Lambda(t,\varpi)} + \frac{(2(1-\lambda)+1)(30(1-\lambda)+5)^2(140(1-\lambda)+7)[162(\Lambda_4(t,\varpi))-54(2(1-\lambda)+1)(\Lambda_3(t,\varpi))]}{\Lambda(t,\varpi)}$$

where

$$\Lambda(t,\varpi) = (2(1-\lambda)+1)^3(30(1-\lambda)+5)^3(60(1-\lambda)+5)^2(140(1-\lambda)+7),$$

$$\begin{aligned}\Lambda_1(t,\varpi) &= 324(30(1-\lambda)+5)^2(2(1-\lambda)+1) - 1458(2(1-\lambda)+1)^2(60(1-\lambda)+5)\\&+ 216(2(1-\lambda)+1)^2(30(1-\lambda)+5)^2 - 162(60(1-\lambda)+5)(2(1-\lambda)+1)\\&- (2(1-\lambda)+1)(30(1-\lambda)+5)^2(60(1-\lambda)+5) + (30(1-\lambda)+5)^2(60(1-\lambda)+5),\end{aligned}$$

$$\begin{aligned}\Lambda_2(t,\varpi) &= 162(30(1-\lambda)+5)^2 - 729(2(1-\lambda)+1)(60(1-\lambda)+5)\\&- 216(2(1-\lambda)+1)(30(1-\lambda)+5)^2 + 162(60(1-\lambda)+5)(30(1-\lambda)+5)\\&+ 108(2(1-\lambda)+1)(30(1-\lambda)+5)(60(1-\lambda)+5) - 18(30(1-\lambda)+5)^2(60(1-\lambda)+5),\end{aligned}$$

$$\begin{aligned}\Lambda_3(t,\varpi) &= 108(2(1-\lambda)+1)(30(1-\lambda)+5) - 81(2(1-\lambda)+1)(60(1-\lambda)+5)\\&+ 216(30(1-\lambda)+5) - 108(60(1-\lambda)+5)(2(1-\lambda)+1)^2 - 18(30(1-\lambda)+5)(60(1-\lambda)+5)\\&- 12(2(1-\lambda)+1)(30(1-\lambda)+5)^2(60(1-\lambda)+5),\end{aligned}$$

$$\Lambda_4(t,\varpi) = [12-8(2(1-\lambda)+1)](2(1-\lambda)+1) - [9-12(2(1-\lambda)+1)](30(1-\lambda)+5) - 54(2(1-\lambda)+1)^2.$$

As a result, The following is a Corollary if $\lambda = 0$:

Corollary 2.5. *Let $f(z)$ is given by* (1.1) *belongs to the class $\mu(0, \beta, t)$. Then, we have,*

$$|H_4(1)| \leq \frac{(65+20\beta)(147+35\beta)[-3(35+15\beta)(\mathcal{C}_1(t,\varpi))+27(1+\beta)(3+2\beta)(\mathcal{C}_2(t,\varpi))]}{\mathcal{C}(t,\varpi)}$$
$$+\frac{(3+2\beta)(35+15\beta)^2(147+35\beta)[162(1+\beta)(\mathcal{C}_4(t,\varpi))-54(3+2\beta)(\mathcal{C}_3(t,\varpi))]}{\mathcal{C}(t,\varpi)}$$

where

$$\mathcal{C}(t,\varpi) = (1+\beta)^2(3+2\beta)^3(35+15\beta)^3(65+20\beta)^2(147+35\beta),$$

$$\begin{aligned}\mathcal{C}_1(t,\varpi) &= 324(35+15\beta)^2(3+2\beta) - 1458(3+2\beta)^2(65+20\beta) \\ &+ 216(3+2\beta)^2(35+15\beta)^2 - 162(65+20\beta)(3+2\beta) \\ &- (3+2\beta)(1+\beta)(35+15\beta)^2(65+20\beta) + (1+\beta)^2(35+15\beta)^2(65+20\beta),\end{aligned}$$

$$\begin{aligned}\mathcal{C}_2(t,\varpi) &= 162(1+\beta)(30+15\beta+5)^2 - 729(1+\beta)(2+2\beta+1)(60+20\beta+5) \\ &- 216(2+2\beta+1)(30+15\beta+5)^2 + 162(1+\beta)(65+20\beta)(35+15\beta) \\ &+ 108(1+\beta)(2+2\beta+1)(35+15\beta)(65+20\beta) - 18(1+\beta)(35+15\beta)^2(65+20\beta),\end{aligned}$$

$$\begin{aligned}\mathcal{C}_3(t,\varpi) &= 108(3+2\beta)(35+15\beta) - 81(1+\beta)^2(3+2\beta)(65+20\beta) \\ &+ 216(1+\beta)(35+15\beta) - 108(65+20\beta)(1+\beta)(3+2\beta)^2 - 18(1+\beta)^2(35+15\beta)(65+20\beta) \\ &- 12(3+2\beta)(35+15\beta)^2(65+20\beta),\end{aligned}$$

$$\mathcal{C}_4(t,\varpi) = [12 - 8(3+2\beta)](3+2\beta) - [9(1+\beta) - 12(3+2\beta)](35+15\beta) + [54 - 108(1+\beta)](3+2\beta)^2.$$

In case, $\lambda = 0$ and $\beta = 0$ As a result, The following is a Corollary:

Corollary 2.6. *Let $f(z)$ is given by* (1.1) *belongs to the class $\mu(0, 0, t)$. Then, we have,*

$|H_4(1)| \leq\approx 212.414.$

In case applying Lemma (1.2) and by applying $|a_6|, |a_7|$ from Theorem 2.2 in fourth Hankel determinant (1.6), As a result, we have the following Corollary:

Theorem 2.7. *If the function $f \in \mu(\beta, \lambda, t)$ and is of the form* (1.1)*, then we have*

$$|H_4(1)| \leq -\Delta_1(t,\varpi) + \Delta_2(t,\varpi) - \Delta_3(t,\varpi) + \Delta_4(t,\varpi),$$

where

$$\Delta_1(t,\varpi) = \frac{-17}{36}\left(\frac{55875}{(60(1-\lambda)+20\beta+5)} - \frac{57077}{46656}\right),$$

$$\Delta_2(t,\varpi) = \left(\frac{15}{(60(1-\lambda)+20\beta+5)} - \frac{129708}{186624}\right),$$

$$\Delta_3(t,\varpi) = \frac{-54}{(60(1-\lambda)+20\beta+5)}\left(\frac{-27}{2(60(1-\lambda)+20\beta+5)} - \frac{12816}{2304}\right),$$

$$\Delta_4(t,\varpi) = \frac{52908}{746496(140(1-\lambda)+35\beta+7)}.$$

Proof . Let $f \in \mu(\beta,\lambda,t)$. Then, we can be write fourth Hankel determinant as :

$$|H_4(1)| = -a_4\mathcal{S}_1 + a_5\mathcal{S}_2 - a_6\mathcal{S}_3 + a_7\mathcal{S}_4,$$

where

$$|\mathcal{S}_1| = |a_2||a_4a_6 - a_5^2| + |a_3||a_3a_6 - a_4a_5| - |a_4||a_3a_5 - a_4^2|,$$

$$|\mathcal{S}_2| = |a_4a_6 - a_5^2| - |a_2||a_3a_6 - a_4a_5| + |a_3||a_3a_5 - a_4^2|,$$

$$|\mathcal{S}_3| = |a_3a_6 - a_4a_5| + |a_2||a_2a_6 - a_3a_5| - |a_4||a_2a_4 - a_3^2|,$$

$$|\mathcal{S}_4| = |a_3||a_2a_4 - a_2^2| - |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|.$$

By applying Lemma 1.2 and $|a_6|, |a_7|$ from Theorem 2.2 in fourth Hankel determinant (1.6),we get

$$|\mathcal{S}_1| = |a_2||a_4a_6 - a_5^2| + |a_3||a_3a_6 - a_4a_5| - |a_4||a_3a_5 - a_4^2| = \frac{55875}{(60(1-\lambda)+20\beta+5)} - \frac{57077}{46656}, \quad (2.14)$$

$$|\mathcal{S}_2| = |a_4a_6 - a_5^2| - |a_2||a_3a_6 - a_4a_5| + |a_3||a_3a_5 - a_4^2| = \frac{15}{(60(1-\lambda)+20\beta+5)} - \frac{129708}{186624}, \quad (2.15)$$

$$|\mathcal{S}_3| = |a_3a_6 - a_4a_5| + |a_2||a_2a_6 - a_3a_5| - |a_4||a_2a_4 - a_3^2| = \frac{-27}{2(60(1-\lambda)+20\beta+5)} - \frac{12816}{2304}, \quad (2.16)$$

$$|\mathcal{S}_4| = |a_3||a_2a_4 - a_2^2| - |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2| = \frac{326592}{746496}. \quad (2.17)$$

Inserting values (2.14),(2.15),(2.16),(2.17) in (1.5), then we have

$$|H_4(1)| \le -\Delta_1(t,\varpi) + \Delta_2(t,\varpi) - \Delta_3(t,\varpi) + \Delta_4(t,\varpi),$$

where

$$\Delta_1(t,\varpi) = \frac{-17}{36}\left(\frac{55875}{(60(1-\lambda)+20\beta+5)} - \frac{57077}{46656}\right),$$

$$\Delta_2(t,\varpi) = \left(\frac{15}{(60(1-\lambda)+20\beta+5)} - \frac{129708}{186624}\right),$$

$$\Delta_3(t,\varpi) = \frac{-54}{(60(1-\lambda)+20\beta+5)}\left(\frac{-27}{2(60(1-\lambda)+20\beta+5)} - \frac{12816}{2304}\right),$$

$$\Delta_4(t,\varpi) = \frac{52908}{746496(140(1-\lambda)+35\beta+7)}.$$

□

In case $\beta = 0$, As a result, The following is a Corollary:

Corollary 2.8. *Let $f(z)$ is given by* (1.1) *be in the class $\mu(0,\lambda,t)$. Then*

$$|H_4(1)| \leq -\Upsilon_1(t,\varpi) + \Upsilon_2(t,\varpi) - \Upsilon_3(t,\varpi) + \Upsilon_4(t,\varpi),$$

where

$$\Upsilon_1(t,\varpi) = \frac{-17}{36}\left(\frac{55875}{(60(1-\lambda)+5)} - \frac{57077}{46656}\right),$$

$$\Upsilon_2(t,\varpi) = \left(\frac{15}{(60(1-\lambda)+5)} - \frac{129708}{186624}\right),$$

$$\Upsilon_3(t,\varpi) = \frac{-54}{(60(1-\lambda)+5)}\left(\frac{-27}{2(60(1-\lambda)+5)} - \frac{12816}{2304}\right),$$

$$\Upsilon_4(t,\varpi) = \frac{52908}{746496(140(1-\lambda)+7)}.$$

In case $\lambda = 0$ As a result, The following is a Corollary:

Corollary 2.9. *Let $f(z)$ given by* (1.1) *be in the class $\mu(\beta,0,t)$. Then*

$$|H_4(1)| \leq -S_1(t,\varpi) + S_2(t,\varpi) - S_3(t,\varpi) + S_4(t,\varpi),$$

where

$$S_1(t,\varpi) = \frac{-17}{36}\left(\frac{55875}{(65+20\beta} - \frac{57077}{46656}\right),$$

$$S_2(t,\varpi) = \left(\frac{15}{(65+20\beta} - \frac{129708}{186624}\right),$$

$$S_3(t,\varpi) = \frac{-54}{(65+20\beta)}\left(\frac{-27}{2(65+20\beta)} - \frac{12816}{2304}\right),$$

$$S_4(t,\varpi) = \frac{52908}{746496(147+35\beta)}.$$

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