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Estimating parameters of Marshall Olkin Topp Leon exponential distribution via grey wolf optimization and conjugate gradient with application

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Abstract

A new probability model for positively skewed datasets like economic data, medical, engineering and other sciences was developed in this paper. The new distribution is named the Marshall Olkin Topp Leon Exponential distribution and it was generated using the Marshall Olkin Topp Leon -G family of distributions. It has three parameters and it is very flexible in fitting several and different datasets. Its basic mathematical properties were studied and two methods like maximum likelihood estimation via Gray Wolf optimization and Conjugate Gradient used for the estimation of model parameters. A real-life dataset was used to illustrate the flexibility of the distribution and it was found that the new model provides a better fit to real-life datasets than other distributions.

Keywords: Marshall Olkin distribution, Topp Leon-G family of distributions, Mathematical properties, Exponential distribution.

1. Introduction

The purpose of generating new compound distributions from known families of distributions is to extend the baseline distributions and develop them by adding one or more shape parameters. These compound distributions have however been found to be better than the parent distribution in terms of flexibility and modeling capability. There are several families of distributions that can

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be used for this purpose but in this research, the interest is on the Marshall Olkin Topp Leone -G family of distributions. The Marshall Olkin Topp Leon -G family of distributions is relatively new, and it has only two extra parameters [11]. By this, the resulting composite distribution will only have three parameters; one from the baseline distribution (Exponential distribution) and two from the Marshall Olkin Topp Leon -G family of distributions. We motivate the use of this distribution in some ways. First, we extend exponential distribution by proposing the MOTLE by adding two shape parameters to give the distribution more flexibility for application in real data. Further, we find and study some mathematical properties of the new distribution. In 2019, Abdullah et al., [2] estimating the parameters for extension of Burr Type X distribution by Using Conjugate Gradient. As well as Abdullah et al., [1] Modified new conjugate gradient method for Unconstrained Optimization. Exponential distribution is one of the most important probability models in the theory of statistics and other fields of sciences. The model is appropriate for modeling real life phenomena with monotonous failure rates [3]. It has many applications in statistics as well as life testing, circuit studies and reliability analysis [14, 15]. This paper has been structured to develop the Marshall Olkin Topp Leone exponential distribution, establish its various properties, estimate its unknown parameters and to demonstrate its strength using a real life application.

2. Marshall-Olkin Topp-Leone-G (MOTL-G) Family of Distribution

For a given baseline distribution with cdf, W(x), and pdf, w(x), Marshall and Olkin (1997) suggested a new flexible family of probability distributions based on defined a cumulative function through presenting an additional parameter, $\propto > 0$, known as the shape parameter. The cumulative function and probability density function of the Marshall-Olkin (MO) family are defined by [9, 10], respectively:

$$F_{MO}{-}G{=}\frac{M(x)}{1{-}\overline{\propto}\overline{M}\,(x)} \quad , \qquad \overline{M}\,(x) = 1{-}M\,(x)\,, \ \ \overline{\propto}{=}\ 1{-}\propto \tag{2.1}$$

$$f_{MO}{-}G{=}\frac{\propto m(x)}{\left[1{-}\overline{\propto}\overline{M}\,(x)\,\right]^2} \tag{2.2}$$

A natural way of generating families of distributions on some other support from on the interval [0, 1] a simple starting parent distribution with probability density function is to apply the cumulative function to a family of distributions. The cdf of the Topp and Leone distribution is given by

$$M_{TL}(\mathrm{x}) = [1-(1{-}G\,(x))^2]^b \tag{2.3}$$

The PDF of Topp-Leone family as follows [4, 16]:

$$m_{TL}\,(x) = 2b\,(1{-}G\,(x))\,.g\,(x)\,.\{\,1-(1{-}G\,(x))^2\}^{b-1}\ , \qquad b>0 \tag{2.4}$$

Compound Eq. (2.3) in Eq. (2.1) we get the cdf of new family, by driven the Eq. (2.5) we have the pdf of new family

$$F_{MOTL}{-}G\,(x) = \frac{\{\,1-[1{-}G(x)]^2\}^b}{1{-}\overline{\propto}\left[\{\,1-[1{-}G(x)]^2\}^b\right]} \tag{2.5}$$

$$f_{MOTL}{-}G\,(x) = \frac{2\propto b(1{-}G\,(x)\,).g(x).\{\,1-(1{-}G\,(x))^2\}^{b-1}}{(1{-}\overline{\propto}\left[\{\,1-[1{-}G(x)]^2\}^b\right]\,)^2} \tag{2.6}$$

where $\propto$, b are the shape parameters [13].

3. MOTL-E Distribution

The random variable X is said to have an exponential (E) distribution with the scale parameter. It is perhaps the most widely applied statistical distribution for many problems in reliability in many fields. In this point, we propose a generalization of E distribution with the hope it will attract wider applicability in real life phenomenones. let the cdf and the pdf of E distribution as follows

$$G(x) = 1-e^{-\delta x} \tag{3.1}$$

$$g(x) = \partial e^{-\delta x} \tag{3.2}$$

inserting Eq. (3.2) in Eq. (2.6) we have the cdf of new distribution as follows:

$$F_{MOTLE}(x) = \frac{\{\,1-(1-e^{-\delta x})^2\}^b}{1-\overline{\propto}\left[1-\left\{\,1-(e^{-\delta x})^2\right\}^b\right]} \tag{3.3}$$

The pdf and hazard function corresponding to Eq. (3.3) can be written as respectively

$$f_{MOTLE}(x) = \frac{2\propto b e^{-\delta x} e^{-\delta x}\{1-\, e^{-2\delta x}\}^{b-1}}{(1-\overline{\propto}\left[\{\,1-[1-e^{-2\delta x}\}^b\right])^2} \tag{3.4}$$

And hazard function can be found as follows [5]:

$$H_{MOTLE}(x) = \frac{\frac{2\propto b e^{-\delta x} e^{-\delta x}\{1-\, e^{-2\delta x}\}^{b-1}}{(1-\overline{\propto}\left[\{\,1-[1-e^{-2\delta x}\}^b\right])^2}}{1-\frac{\{\,1-(1-e^{-\delta x})^2\}^b}{1-\overline{\propto}\left[1-\left\{\,1-(e^{-\delta x})^2\right\}^b\right]}} \tag{3.5}$$

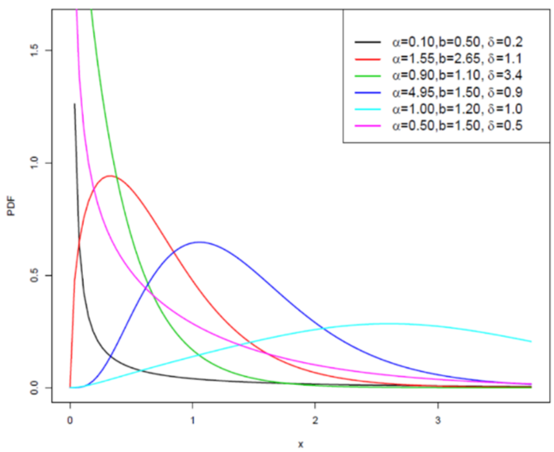

Figure 1: Plot for the pdf of Marshall Olkin Topp Leon Exponential distribution

Figures 1 illustrate some possible shapes of the density function for selected parameter values. The density of MOTLE distribution as we see is more flexible than E distribution. The density of MOTLE can be skewed from right, skewed from the left and also symmetric (see Figure 1). We can see that the additional shape parameters $\propto$, and b, allows for a higher degree of flexibility. The survival function can be obtained for the Marshall Olkin Topp Leone Exponential distribution using the following relationship:

$$S_{MOTLE}(x) = 1 - \frac{\left\{ 1-(1-e^{-\delta x})^2 \right\}^b}{1-\overline{\propto}\left[1-\left\{ 1-(e^{-\delta x})^2 \right\}^b\right]} \tag{3.6}$$

4. Properties of new Distribution

4.1. Expansion for the probability density function

Expanding the pdf of a probability model is very useful in developing and studying some mathematical properties which otherwise would have required prolonged algebraic processes. Recall from Equation (3.5) that:

$$f_{MOTLE}(x) = \frac{2\propto b e^{-\delta x} e^{-\delta x} \{1- e^{-2\delta x}\}^{b-1}}{(1-\overline{\propto}\left[\{ 1-[1-e^{-2\delta x}\}^b\right])^2}$$

By using binomial expansion on the denominator of pdf as follows

$$\begin{aligned}(1-\overline{\propto}\left[\{ 1-[1-e^{-2\delta x}\}^b\right])^{-2} &= \sum_{j=0}^{\infty}(j+1)\ (\overline{\propto}\ \left[\{ 1-[1-e^{-2\delta x}]\}^b\right])^j \\ &= \sum_{j=0}^{\infty}(j+1)\ \overline{\propto}^j\ [\{ 1-[1-e^{-2\delta x}]\}^b]^j\end{aligned}$$

By using binomial expansion again as follows:

$$(1-z)^{-k} = \sum_{j=0}^{\infty}\frac{\Gamma(k+j)}{\Gamma k.j!}.z^j \tag{4.1}$$

$$\begin{aligned}(1-z)^{b-1} &= \sum_{i=0}^{\infty}(-1)^i \begin{pmatrix} b-1 \\ i \end{pmatrix} z^i \qquad , \begin{pmatrix} b-1 \\ i \end{pmatrix} \Longrightarrow \frac{\Gamma(b+1-j)}{\Gamma(b-1.j!} \\ &= \sum_{j=0}^{\infty}(j+1)\ \overline{\propto}^j \sum_{i=0}^{\infty}(-1)^j\frac{\Gamma(j+1)}{\Gamma j.j!}\{ 1-e^{-2\delta x}\}^{bi}\end{aligned}$$

We can rewrite the pdf as follows:

$$f_{MOTLE}(x) = 2\propto b\ \delta\ e^{-2\delta x}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(j+1)\overline{\propto}^j\,(-1)^i \begin{pmatrix} j \\ i \end{pmatrix}\ \{ 1-e^{-2\delta x}\}^{b(i+1)-1}$$

Let $\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(j+1)\,\overline{\propto}^j\,(-1)^i \begin{pmatrix} j \\ i \end{pmatrix} = W$ then

$$f_{MOTLE}(x) = 2\propto b\ \delta\ e^{-2\delta x}\ W\ \{ 1-e^{-2\partial x}\}^{b(i+1)-1}$$

$$f_{MOTLE}(x) = 2\propto b\ \delta\ e^{-2\delta x} w \sum_{k=0}^{\infty}(-1)^k\overline{\propto}^j\,(-1)^k \begin{pmatrix} b\,(1+1)-1 \\ k \end{pmatrix}\ (e^{-2\delta x})^k)$$

$$f_{MOTLE}(x) = 2\propto b\ \delta\ e^{-2\delta x}\ .W_{i,j,k}\ e^{-2\delta k x} \tag{4.2}$$

where

$$W_{i,j,k} = \sum_{j,i,k=0}^{\infty} (-1)^{k+1} \overline{\propto}^{j} (-1)^{k} (j+1) \begin{pmatrix} j \\ i \end{pmatrix} \begin{pmatrix} b(1+1)-1 \\ k \end{pmatrix}.$$

Equation (4.2) is the main equation in this subsection and it will use later to find some properties.

4.2. Quantile Function

This can easily be obtained as the inverse of the cdf as follows:
Let.

$$1 - e^{-2\delta x} = K \tag{4.3}$$

This way, the quantile function for the Marshall Olkin Topp Leone Exponential distribution is obtained as follows:

$$\begin{aligned}
&u = \frac{(K)^b}{(1-\overline{\propto}[1-(K)^b]} \\
&(K)^b = u - u\overline{\propto}[1-(K)^b] \\
&(K)^b = u - u\overline{\propto} + u\overline{\propto}(K)^b \\
&(K)^b - u\overline{\propto}(K)^b = u(1-\overline{\propto}) \\
&(K)^b(1-u\overline{\propto}) = u(1-\overline{\propto}) \Longrightarrow (K)^b = \frac{u(1-\overline{\propto})}{1-\overline{\propto}u} \\
&\left((1-e^{-2\delta x})^b\right)^{1/b} = \left(\frac{u(1-\overline{\propto})}{1-u\overline{\propto}}\right)^{1/b} \\
&1-e^{-2\delta x} = \left(\frac{u(1-\overline{\propto})}{1-u\overline{\propto}}\right)^{1/b} \\
&e^{-2\delta x} = 1 - \left(\frac{u(1-\overline{\propto})}{1-u\overline{\propto}}\right)^{1/b} \\
&-2\delta x = -\ln[1 - \left(\frac{u(1-\overline{\propto})}{1-u\overline{\propto}}\right)^{\frac{1}{b}}]
\end{aligned}$$

To take ln two sides

$$x = -\frac{-\ln[1 - \left(\frac{u(1-\overline{\propto})}{1-u\overline{\propto}}\right)^{\frac{1}{b}}]}{2\delta}$$

When $u = 0.5$, we find the median

$$x_{0.5} = \frac{-\ln[1 - \left(\frac{0.5\ \overline{\propto}}{1-0.5\overline{\propto}}\right)^{\frac{1}{b}}]}{2\delta}$$

$$x = \frac{-\ln[1 - \left(\frac{U\ \overline{\propto}}{1-U\overline{\propto}}\right)^{\frac{1}{b}}]}{2\partial} \qquad U \sim Uniform(0,1) \tag{4.4}$$

4.3. Moments

Moments play important roles in finding and measuring some statistical characteristics such as finding flattening and spacing, coefficient of variation, standard deviation and other characteristics. The rth moment of the Marshall Olkin Topp Leone Exponential distribution can be obtained using the following relationship;

$$M_r(t) = \int_0^\infty x^r f(x)\ dx = 2\propto b\ \delta\ W_{i,j,k} \int_0^\infty x^r e^{-2\delta x(1+k)} dx$$

Let. $y = 2b\delta x(1+\mathrm{k}) \Longrightarrow x = \frac{\mathrm{y}}{2\partial(1+\mathrm{k})}$ and $dx = \frac{\mathrm{dy}}{2\delta(1+\mathrm{k})}$ then

$$\int_0^\infty x^r e^{-2\delta x(1+k)} dx = \int_0^\infty \frac{y^r}{2\delta(1+\mathrm{k})^r}.e^{-y}\frac{\mathrm{dy}}{2\delta(1+\mathrm{k})} = \frac{1}{(2\delta(1+\mathrm{k}))^{r+1}} \int_0^\infty y^{r+1-1}.e^{-y}\mathrm{dy} = \frac{\Gamma(r+1)}{[2\delta(1+\mathrm{k})]^{r+1}}$$

$$\boldsymbol{M_r}(\boldsymbol{t}) = \frac{2\propto b\delta W_{i,j,k}\Gamma(r+1)}{(2^{r+1}\delta^{r+1}(k+1)^{r+1}} = \frac{2\propto bW_{i,j,k}\Gamma(r+1)}{(2\delta)^r(k+1)^{r+1}} \tag{4.5}$$

4.4. Moment generating function

For a random variable X, the moment generating function (mgf) is given by the relationship. By expressing e^{xt} in form of a sequence:

$$\begin{aligned}
\mathrm{E}(e^{xt}) &= E(\sum_{x=r}^\infty \frac{t^r x^r}{r!}) \\
\mathrm{E}(e^{xt}) &= \sum_{x=r}^\infty \frac{t^r}{r!} E(x^r) \\
&= \frac{2\propto bW_{i,j,k}\sum_{r=0}^\infty \frac{t^r}{r!}\Gamma(r+1)}{(2\delta)^r(k+1)^{r+1}} \\
&= \sum_{r=0}^\infty \frac{2\propto bW_{i,j,k.}t^r\Gamma(r+1)}{(2\delta)^r(k+1)^{r+1}}
\end{aligned} \tag{4.6}$$

4.5. Order Statistics

If $\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n$ denote random samples from the densities of a Marshall Olkin Topp Leone Exponential distribution as defined in (3.3) and Equation (3.4) respectively; the pdf of the kth order statistics of the Marshall Olkin Topp Leone Exponential distribution is obtained as follows:

$$g_{(r,n)} = \frac{n!}{(r-1)!(n-r)!}\ [\ F_{MOTLE}(x)]^{r-1}[1-\ F_{MOTLE}(x)]^{n-r}\ f_{MOTLE}(x)$$

When r = 1 the distribution of minimum order statistics for the Marshall Olkin Topp Leone Exponential distribution is therefore given as:

$$\begin{aligned}
g_{(1,n)} &= \frac{n!}{(1-1)!(n-1)!}[1-\ F_{MOTLE}(x)]^{n-1}\ f_{MOTLE}(x) \\
&= n[1-\ F_{MOTLE}(x)]^{n-1}.\ f_{MOTLE}(x)
\end{aligned}$$

And when r = n the distribution of maximum order statistics for the Marshall Olkin Topp Leone Exponential distribution is therefore given as:

$$g_{(n,n)} = \frac{n!}{(n-1)!\,(n-n)!}[1- \ F_{MoTLE}\,(x)]^{n-1}\ f_{MoTLE}\,(x)$$
$$= n[F_{MoTLE.}\ \ (x)]^{n-1}.\ f_{MoTLE}\,(x)$$

By substituting eq (3.3) and (3.4) in equation above we have

$$g_{(n,n)} = n\left[\frac{\{\ 1-e^{-2\partial x}\}^{b}}{\left[1-\overline{\propto}\{\ 1-e^{-2\partial x}\}^{b}\right]}\right]\ .[\frac{2\propto b\partial e^{-2\partial x}(\ 1-e^{-2\partial x})^{b-1}}{-((1-\overline{\propto})[1-\ \ \{1-e^{-2\partial x}\}^{b}])^{2}}]$$

5. Metaheuristic Algorithm

Metaheuristic optimization algorithms have taken a great role in solving engineering application problems, because they are easy to implement, have good ability to escape stagnation in a local optimum, and due to their flexibility; therefore, they give a good solution to a range of issues in various disciplines [8].

5.1. Gray wolf algorithm

The gray wolf optimization algorithm is one of the optimization algorithms. It has good ability to find competing solutions with other best optimization algorithms, in which proposed by Mirjalili et al., 2016 [8, 12, 18]. They Simulated the social behavior of gray wolves in search of food, where they live in groups of 5-12 wolves, by means of applying the hierarchy of leadership strategy. The latter means the hunting process These Strategy involves three steps such as a social hierarchy stratification, un encircling the victim and assaulting it.

5.2. Social life

Wolves live in groups of 5-12 wolves to support each other in order to obtain food, by applying the hierarchy of command strategy, in which group members are classified into four categories such as alphas, betas, deltas and omega. The group is led by alphas, with decision-making powers such as where to sleep, hunt, and details of group leadership and direction, and a key role in producing new solutions. In the second level comes the role of the betas wolves, which remain next to the alphas wolves, and through them the alphas can make a decision, which in turn provides a quick response to the decisions of the leader alphas, while the secondary wolves, which belong to the categories of seniors, guards, hunters, scouts and so-called deltas wolves are located at the third level. The Figure (2.2) shows the leadership hierarchy in the wolf pack

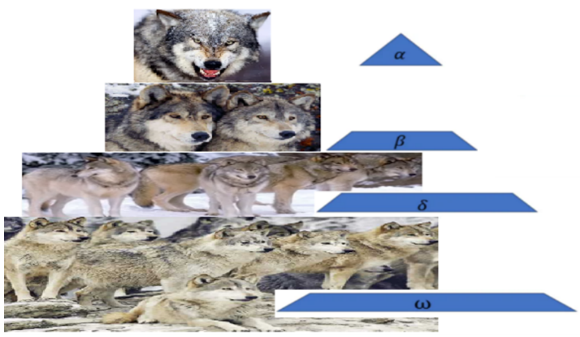

Figure 2: Hierarchy of the social life of wolves

5.3. Surround and Attack

The hunting mechanism begins with gray wolves encircling prey; mathematically, the encirclement behavior is modeled as follows:

$$\mathrm{Y}\,(\mathrm{t}+1) = \mathrm{Y_p}\,(\mathrm{t}) - \mathrm{Z}*\left|\mathrm{B}*\mathrm{Y_p}\,(\mathrm{t}) - \mathrm{Y}\,(\mathrm{t})\right| \tag{5.1}$$

$$\mathrm{Z} = 2\mathrm{z}*\mathrm{r}_1 - \mathrm{z} \tag{5.2}$$

$$\mathrm{L} = 2*\mathrm{r}_2 \tag{5.3}$$

$$z = 2 - 2\frac{t}{max - iter.} \tag{5.4}$$

Where,
Y refers to the position vector of the gray wolf.
Y_p refers to the position vectors of prey,
t refers to the current iteration,
Z and B refers to the coefficient vectors,
r_1 and r_2 refer to random vectors in [0, 1] $\wedge$n, this vector is distance control parameter, where its value decreases linearly from 2 to 0 over the course of iterations,
$max - iter.$ is the maximum iterations
Now, the wolves finish Encircling the Prey. After completing the briefing phase, the wolves start preparing for the attack phase. Gray wolves can identify the location of potential prey well, and the search task is mainly entrusted with the guidance of α and the rest of the wolves. The algorithm operates in an iterative fashion. The three best wolves are selected in each cycle. (α, β, δ) in the current population are selected for retention, then the positions of the other wolves (agents of search) are revised depending on their prior location data. The process of attacking the prey in sport was modelled as follows:

$$Y_1 = Y_\alpha - Z_1 * |L_1 * Y_\alpha - Y| \tag{5.5}$$

$$Y_2 = Y_\beta - Z_2 * |L_2 * Y_\beta - Y| \tag{5.6}$$

$$Y_3 = Y_\delta - Z_3 * |L_3 * Y_\delta - Y| \tag{5.7}$$

$$Y\,(t+1) = \frac{Y_1\,(t) + Y_2\,(t) + Y_3\,(t)}{3} \tag{5.8}$$

Where the above equations represent the values,(Y_α, Y_β, and Y_δ), which refer to the wolf position vectors (α, β and δ), respectively. The values of the calculations for (Z_1, Z_2, and Z_3) are similar to Z , also, the calculations for (L_1, L_2, and L_3) are similar to **L**.
In fact, they are used Extracted **(L)** To update their prey and agency locations, use the following numbers to indicate the distance between current candidate wolves and the top three wolves [18]:

$$H_\alpha = L_1 * Y_\alpha - Y \tag{5.9}$$

$$H_\beta = L_2 * Y_\beta - Y \tag{5.10}$$

$$H_\delta = L_3 * Y_\delta - Y \tag{5.11}$$

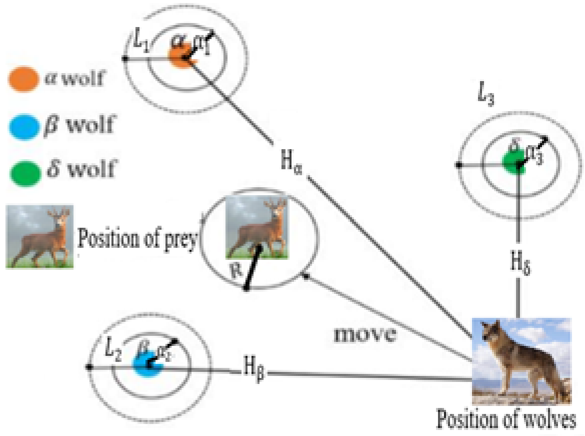

Figure 3: Position Update of Prey Location for Grey Wolves Group

Figure 3 shows the steps for the candidate solution to finally fall within the positions circle by the group of wolves (α, β, and δ) according to their role. Note that, the role of the second group (β) in the hierarchy, whose mission is to follow, encircle, and harass prey until it stops moving. The next stage is the orientation of the best three wolves in a relation to the location of the prey. The rest of the groups updates it around the clock, as each group begins to search for information about the location of prey in a scattered manner. In the harmony with the position of the leader of the group and his assistants in a relation to the position of the prey, the positions update, and then he begins to pounce on the prey to overthrow it. It is important to $\beta$and sometimes the prey can escape intelligently. One can understand the working mechanism of this algorithm through the following flowchart in figure (2.4) [6]:

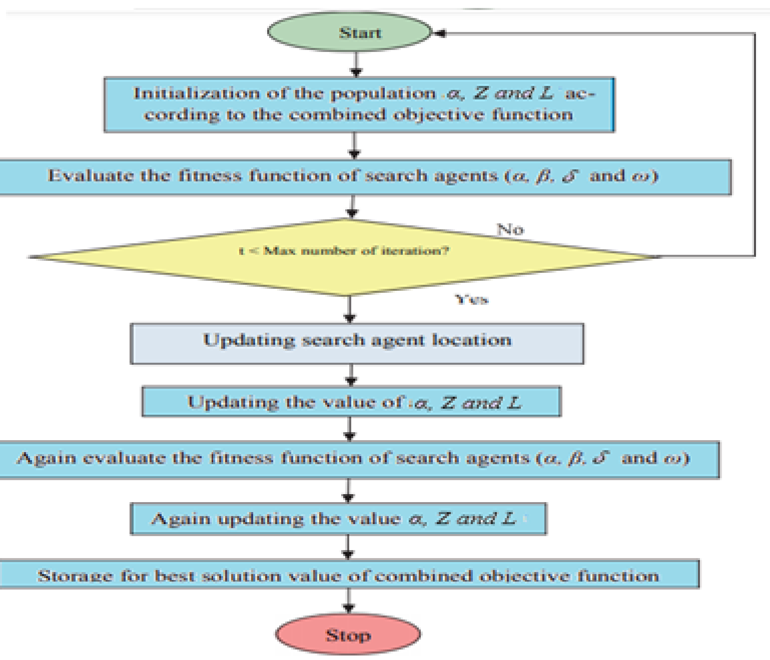

Figure 4: Flowchart for Grey Wolves Algorithm

6. Conjugate Gradient Methods

The method of Conjugate Gradient start with initial point $x_k$and direction $d_k$to generate converge of iterations until get the minimum of the function $f$by use the following equation, where α_k represent the step size [7].

$$x_{k+1} = x_k + \alpha_k d_k \tag{6.1}$$
$$d_{k+1} = -g_{k+1} + \beta_k d_k \tag{6.2}$$

where g_{k+1} represented the gradient of the function f

7. BFGS Method

The BFGS algorithm is one specific way for updating the calculation of the inverse Hessian, instead of recalculating it every iteration. It, or its extensions, may be one of the most popular Quasi-Newton or even second-order optimization algorithms used for numerical optimization [17]. $d_k = -B_k^{-1}\nabla f(x_k)$, where BFGS update as follow

$$B_{k+1}{=}B_k{+}\frac{y_k^T y_k}{y_k^T s_k}{-}\frac{B_k s_k^T s_k B_k}{s_k^T B_k s_k} \tag{7.1}$$

where

$$s_k{=}x_{k+1}{-}x_k, \quad y_k{=}g_{k+1}{-}g_k$$

The update of inverse

$$B_{k+1}^{-1}{=}\left(I{-}\frac{s_k y_k^T}{y_k^T s_k}\right) B_k^{-1}\left(I{-}\frac{y_k s_k^T}{y_k^T s_k}\right){+}\frac{s_k s_k^T}{y_k^T s_k} \tag{7.2}$$

8. Maximum Likelihood Estimation

Here, the parameters of the Marshall Olkin Topp Leone Exponential distribution are estimated using the method of maximum likelihood. Let $x_1, x_2, \ldots, x_n$ be random samples distributed according to the Marshall Olkin Topp Leone Exponential distribution, the likelihood function is obtained by the relationship. Using the expression in Equation (3.4) then;

$$\mathrm{L} = \prod_{i=0}^{n} \frac{2\propto b e^{-2\delta x_i}\{1-\ e^{-2\delta x_i}\}^{b-1}}{(1{-}\overline{\propto}\left[\{\ 1-[1{-}e^{-2\delta x_i}\}^{b}\right]\ )^2} \tag{8.1}$$

By taking the natural logarithm, the log-likelihood function is obtained as;

$$\begin{aligned}\mathrm{Log}\,(\mathrm{L}) =&\mathrm{nLog}\,(2) + \mathrm{nLog}\,(\propto) + \mathrm{nLog}\,(b) - \sum_{i=0}^{n} 2\delta x_i + (\mathrm{b}-1)\sum_{i=0}^{n} Log\left(\{1-\ e^{-2\delta x_i}\}\right) \\ &- 2\sum_{i=0}^{n} Log\left(\left(1{-}\overline{\propto}\left[\{\ 1-[1{-}e^{-2\delta x_i}\}^{b}\right]\right)\right)\end{aligned} \tag{8.2}$$

The estimate of each of the parameters can therefore be obtained when the first partial derivative of the log-likelihood function for each of the parameters is taken, equated to zero and solved simultaneously. It is good to note that the solution cannot be obtained in closed form. This can however be resolved by solving numerically using available software like R and other sophisticated software.

9. Application

A real life application is presented in this section to demonstrate the usefulness of the Marshall Olkin Topp Leone Exponential distribution. Comparisons are made with the Marshall Olkin Topp Leone Exponential BFGS, Marshall Olkin Topp Leone Exponential CG, Marshall Olkin Topp Leone Exponential GWO Methods with respect to their negative log-likelihood (NLL), Akaike Information Criteria (AIC), Consistent Akaike Information Criteria (CAIC), Bayesian Information Criteria (BIC) values.

The data represents the waiting time (mins) of 100 bank customers before service is being rendered. The dataset has previously been analyzed by *Oguntunde et al.*, (2017) [14] to predict the failure times of these engines. The result is presented in Table 1.

Table 1: Table of result

Distributions	NLL	AIC	CAIC	BIC
MOTLEBFGS	317.014	640.029	640.279	647.844
MOTLECG	338.78	683.765	684.015	691.580
MOTLEGWO	317.083	640.166	640.416	647.9815

The newly developed MOTLEBFGS displays a very good potential in Table 1 as it has the lowest values for the NLL, AIC, CAIC, BIC, statistic. The maximum likelihood estimates for the parameters are provided in Table 2.

Table 2: Parameter estimates

Distributions	Estimates
MOTLEBFGS	$\widehat{\alpha} = 2.3048,\ \hat{b} = 0.7621,\ \widehat{\delta} = 0.073$
MOTLECG	$\widehat{\alpha} = 0.8044,\ \hat{b} = 0.8383,\ \widehat{\delta} = 0.0408$
MOTLEGWO	$\widehat{\alpha} = 2.1945,\ \hat{b} = 1.022,\ \widehat{\delta} = 0.079$

To further validate the results obtained, the histogram plot of the dataset with the distributions compared is presented in Figure 5.

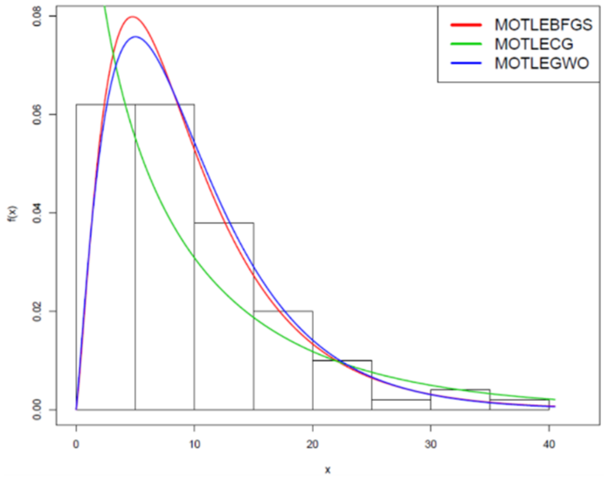

Figure 5: Histogram plot of the dataset with the compared methods

The corresponding empirical cdf plot is presented in Figure 6. Figures 5 and 6 show that the MOTLEBFGS method fits the dataset better than the other methods.

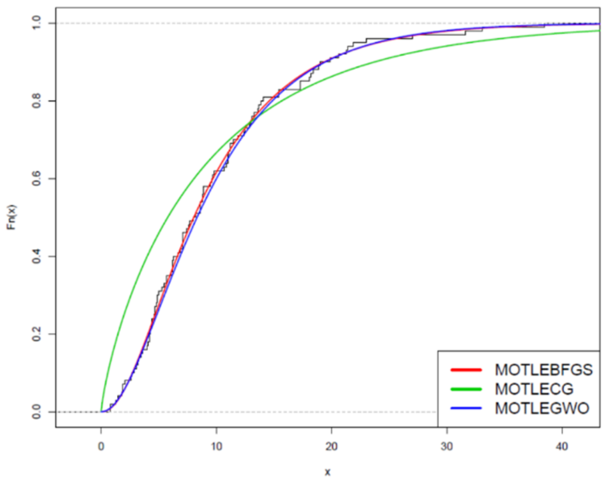

Figure 6: Empirical cdf of the dataset with the compared methods

10. Conclusions

The Marshall Olkin Topp Leone Exponential distribution has been successfully defined and studied in this paper. The model is positively skewed and unimodal in shape, its various statistical properties were also obtained. The model is characterized by high flexibility. A parameter estimation was using MLE to estimate the unknown model parameter. An application to real life dataset reveals that the method of BFGS is a strong competitor then the other methods GC and GWO. The model can also be applied to other real life datasets.

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