



Solution of delay differential equations using Tarig-Padé differential transform method with an application to vector-borne diseases

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Abstract

This study presents the differential transform method and its modification proposes a combination of the differential transform method, Tarig transformation and the Padé approximation. This combination may be used as a beneficial strategy to expand the domain of convergence of the approximation solutions. Moreover, the modified differential transform method will be used to solve linear and nonlinear delay differential equations, and this technique will be applied to models of delayed vector-borne diseases and delayed protein degradation.

Keywords: Delay differential equations differential transform method, Modified differential transform method, Tarig transformation, The Padé approximation, TPDTM, Vector-Borne Diseases.

1. Introduction

The differential transform method (DTM) was first applied in the engineering domain [4], which was successfully used by Zhou (1986) to solve linear and nonlinear initial value problems in electric circuit analysis. In recent years, the DTM has been used to solve a one-dimensional planer Bratu problem, differential-in-difference equation, delay differential equations, differential-algebraic

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equation and systems of integro-differential equations (Abdel-Halim, 2007; Arikoglu, 2006; Arikoglu, 2008; Arikoglu, 2006; Ayaz, 2004; Karakoc, 2009; Kurnaz, 2005; Osmanoglu, 1986), [1].

The DTM is a transformation technique based on Taylor’s series expansion and is a beneficial tool to obtain analytical solutions of the differential equations. DTM differs from the high-order Taylor series method because the latter requires several symbolic computations and is thus expensive for large orders [2]. An improved DTM that uses the Laplace transformation and Padé approximation can increase the rate of convergence of the approximation solution to the accurate solution [10]. In this study, we will extend the DTM to solve the n th order differential equations with multiple delays of the form:

$$y^{(n)}(x) = f(x, y(x), y(x - r_1), y(x - r_2), \dots, y(x - r_m)), m \in \mathbb{N} \tag{1.1}$$

where $y : I \rightarrow \mathbb{R}$, f is continuously differentiable real valued function $I \subset \mathbb{R}$ and $r_i > 0$, for $i = 1, 2, \dots, m$.

2. An Overview of DTM

We denote f function as an analytic in a certain domain D and let $t = x_0$ represent any point in D . The function f is then represented by a power series whose centre is located at x_0 . As a definition which is given sequentially, the differential transformation of the function f is given by Eq. (2.1).

$$F(k) = \frac{1}{k!} \left(\frac{d^k f(x)}{dt^k} \right)_{x=x_0} \tag{2.1}$$

where f is the original function and F is the transformed function. The inverse transformation is defined as [2], [3]:

$$f(x) = \sum_{k=0}^{\infty} (x - x_0)^k F(k). \tag{2.2}$$

Substitution of Eq.(2.1) in Eq. (2.2) gives:

$$f(x) = \begin{cases} \sum_{k=0}^{\infty} \frac{(x-x_0)^k}{k!} \left(\frac{d^k f(x)}{dx^k} \right)_{x=x_0}, & x_0 \neq 0 \\ \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(\frac{d^k f(x)}{dx^k} \right)_{x=0}, & x_0 = 0 \end{cases} \tag{2.3}$$

Notably, Eq. (2.3) signifies that the notion of the DTM originates from Taylor’s series expansion. The approximation of the DTM consists of some recursive computations, as summarised in the following steps [4]:

1. The differential transformation of each term in the DDE is computed;
2. The differential transformations of $Y(0), Y(1), Y(2), \dots$ are calculated by the recurrence Eq. (2.1), with a given initial condition;
3. Finally, these values are substituted back into Eq. (2.2).

Amongst the basic properties of the DTM and those given below (for details, see [5, 6, 10]).

1. If $y(x) = c_1 f(x) \pm c_2 g(x)$, then $Y(k) = c_1 F(k) \pm c_2 G(k)$, where $c_1, c_2 \in \mathbb{R}$, $k \in \mathbb{N}$.
2. If $y(x) = g(x) h(x)$, then $Y(k) = \sum_{k_1=0}^k G(k_1) H(k - k_1)$, $k \in \mathbb{N}$.

3. If $y(x) = g(x + a)$, $a \geq 1$, then:

$$Y(k) = \sum_{h_1=k}^N \binom{h_1}{k} a^{h_1-k} G(h_1), \text{ for } N \rightarrow \infty \ a \geq 1$$

where

$$\binom{h_1}{k}$$

stead for the combination of h_1 taken k .

4. If $y(x) = g(x - a)$, $a > 0$, then:

$$Y(k) = \sum_{h_1=k}^N (-1)^{h_1-k} \binom{h_1}{k} a^{h_1-k} G(h_1), \text{ for } N \rightarrow \infty$$

where a may be treated as a constant delay

5. If $y(x) = g_1(x - a_1) g_2(x - a_2)$, provided that $a_1, a_2 > 0$, then:

$$Y(k) = \sum_{h_1=k}^k \sum_{h_2=k-k_1}^N (-1)^{h_1+h_2-k} \binom{h_1}{k} \binom{h_1}{k_1} \binom{h_2}{k-k_1} a_1^{h_1-k} a_2^{h_2-k+k_1} G_1(h_1) G_2(h_2)$$

for $N \rightarrow \infty$, where also a_1 and a_2 are treated as constant delays. It is remarkable that when $a_1 = 0$ and $a_2 = 0$, then property (5) is special case of property(2).

6. If $y(x) = g_1\left(\frac{x}{a_1}\right) g_2\left(\frac{x}{a_2}\right)$, where $a_1, a_2 \geq 1$, then for $N \rightarrow \infty$:

$$Y(k) = \sum_{k_1=0}^k \sum_{h_1=k_1}^N \sum_{h_2=k-k_1}^N (-1)^{h_1+h_2-k} \frac{(a_1 - 1)^{h_1-k_1}}{a_1^{h_1}} \frac{(a_2 - 1)^{h_2-k+k_1}}{a_2^{h_2}},$$

$$\times x_0^{h_1+h_2-k} \binom{h_1}{k_1} \binom{h_2}{k-k_1} G_1(h_1) G_2(h_2).$$

Also, if $a_1 = 1$ and $a_2 = 1$, then property (7) reduces to property (2)

7. If $y(x) = x^n$, then $Y(k) = \delta(k - n)$, where:

$$\delta(k - n) = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}$$

and δ is the Kronecker delta function.

8. If $y(x) = e^{\lambda x}$, then $(k) = \frac{\lambda^k}{k!}$.

9. If $y(x) = \frac{d^n f(x)}{dx^n}$, then $Y(k) = \frac{(k+n)!}{k!} G(k + n)$, $n \in \mathbb{N}$.

10. Suppose that $y(x)$ and $g(x)$ are differentiable functions whose differential transformations are $Y(x)$ and $G(x)$ respectively and $p \in (0, 1)$ is the proportional delay parameter, then $Y(k) = p^k G(k)$ if $y(x) = g(px)$, where $Y(k) [x_0]$ is the k^{th} -component of the differential transform of $y(x)$ at x_0 .

11. Generalization of property (11) starts by assume by that Y, G_1 and G_2 , to be the differential transforms of the functions y, g_1 and g_2 , respectively and $p_1, p_2 \in (0, 1)$, and if $y(x) = g_1(p_1 x) g_2(p_2 x)$, then:

$$Y(k) = \sum_{k_1=0}^k p_1^{k_1} p_2^{k-k_1} G_1(k_1) G_2(k - k_1)$$

12.(a) If $y(x) = \cos x$, then:

$$C(k) = \begin{cases} (-1)^{\frac{k}{2}} \frac{1}{k!} & \text{if } k = 2n, n \in \mathbb{N} \\ 0 & \text{if } k = 2n + 1, n \in \mathbb{N} \end{cases}$$

(b) If $y(x) = \sin x$, then:

$$S(k) = \begin{cases} (-1)^{\frac{k-1}{2}} \frac{1}{k!} & \text{if } k = 2n + 1, n \in \mathbb{N} \\ 0 & \text{if } k = 2n, n \in \mathbb{N} \end{cases}$$

Other properties concerning the differential transforms may be found in literatures (see for example, [4]).

3. Padé Approximation

The best approximation of a function with a specific order is abbreviated as the Padé approximation. Under this technique, the approximation power series corresponds to the power series approximation of the function [10]. The Padé approximation is also the expansion of Taylor polynomial approximation to a rational function. To illustrate this approximation, we suppose that r is a rational function of two polynomials p and q of degrees n and m , respectively, without a common factor, and the degree of r is $N = n + m + 1$ (symbol of orders n and m is denoted by $[n, m]$) of the form:

$$r(x) = \frac{p(x)}{q(x)} = \frac{p_0 + p_1x + \dots + p_nx^n}{q_0 + q_1x + \dots + q_mx^m}, \tag{3.1}$$

which is applied to approximation a function f over a closed interval I contain the origin. To define r at zero, q_0 should not be equal to 0. In fact, we may assume that $q_0 = 1$; otherwise, we simply divide r by q_0 . Consequently, $N + 1$ parameters $q_1, q_2, \dots, q_m, p_0, p_1, \dots, p_n$ are available for the approximation of f by r . The $N + 1$ parameters were chosen so that $f^k(0) = r^k(0)$, for each $k = 0, 1, \dots, N$. When $n = N$ and $m = 0$, the Padé approximation is simply the N^{th} -Maclaurin polynomial.

Consider the difference:

$$\begin{aligned} f(x) - r(x) &= f(x) - \frac{p(x)}{q(x)} \\ &= \frac{f(x)q(x) - p(x)}{q(x)} \\ &= \frac{f(x) \sum_{i=0}^m q_i x^i - \sum_{i=0}^m p_i x^i}{q(x)} \end{aligned}$$

and suppose that f has the Maclaurin series expansion $f(x) = \sum_{i=0}^{\infty} a_i x^i$. Then

$$f(x) - r(x) = \frac{\sum_{i=0}^{\infty} a_i x^i \sum_{i=0}^m q_i x^i - \sum_{i=0}^m p_i x^i}{q(x)}. \tag{3.2}$$

The objective is to choose the constants q_1, q_2, \dots, q_m and p_0, p_1, \dots, p_n so that $f^k(0) - r^k(0) = 0$, for all $k = 0, 1, \dots, N$. This case is equivalent to $f - r$ having a zero of multiplicity $N + 1$ at $x = 0$.

As a result, we select q_1, q_2, \dots, q_m and p_0, p_1, \dots, p_n so that the numerator on the right hand side of Eq. (3.2) which is:

$$(a_0 + a_1x + \dots)(1 + q_1x + \dots + q_mx^m) - (p_0 + p_1x + \dots + p_nx^n) \quad (3.3)$$

has no terms of degree less than or equal to N . For simplified notations, we define $p_{n+1} = p_{n+2} = \dots = p_N = 0$ and $q_{m+1} = q_{m+2} = \dots = q_N = 0$. Then, the coefficient of x^k may be expressed more compactly as $\left(\sum_{i=0}^k a_i q_{k-i}\right) - p_k$ by Eq. (3.3). The rational function for Padé approximation will be obtained from the solution of $N + 1$ linear equations with $N + 1$ unknown q_1, q_2, \dots, q_m and p_0, p_1, \dots, p_n , which is given by [11]:

$$\sum_{i=0}^k a_i q_{k-i} = p_k.$$

The Padé approximation of Eq. (3.1) to f of order $[n, m]$ can be denoted as $\left[\frac{n}{m}\right]_f$. Furthermore, the numerator and denominator have no common factors; those in Eq. (3.1) are constructed so that f and $\left[\frac{n}{m}\right]_f$ and their derivatives agree at $x = 0$ up to $n + m$. That is:

$$f(x) - \left[\frac{n}{m}\right]_f(x) = O(x^{n+m+1}) \quad (3.4)$$

We first calculate all the coefficients q_i , $i = 1, 2, \dots, m$ and then coefficients p_i , $i = 1, 2, \dots, n$. Notably, for a constant value of $n + m + 1$, the error function given by Equation (3.4) is the smallest degree when the numerator and denominator of Eq. (3.1) have the same degree or when the numerator is a degree higher than the denominator [12]. The advantage of Padé approximation is that it predominately gives approximation a preferable than from the truncated series solution from the Taylor series. The reason is that some function may sometimes not have Taylor series expansion and the Padé approximation has the potency to extended the domain of convergence of solutions or comprise finding accurate solutions [10].

4. Tarig Transformation

Given the several interesting approaches applied to solve differential and integral equations, which make visualisation easier, we introduced a new integral transformation method, that is, Tarig transformation. This method is applied to solve differential equations subsequently [7]. The Tarig transformation method is very effective for solve differential and integral equations and subsequently a linear system of differential and integral equations. The Tarig transformation method is based on the Fourier transform, which was introduced by Tarig M. Elzaki (2010), [8]. As defined for functions of exponential order, the Tarig transformation method is defined by the following integral equations:

$$T[f(x)] = F(u) = \frac{1}{u} \int_0^{\infty} f(x) e^{\frac{-x}{u^2}} dx, \quad x \geq 0, \quad u \neq 0$$

The variable u in this transformation is used as the transformation variable. This transformation has a deeper connection with the Laplace transformation [7]. The sufficient conditions for the existence of Tarig transformation are that f be piecewise continuous and of exponential order, indicating that Tarig transformation may or may not exist.

Tarig transformation can certainly treat all problems that are usually treated by the well-known and extensively used Laplace transformation [9]. To connect Tarig and Laplace transformations, some of the most important properties that connect them may be recalled as in Theorems 1-7 given in [7, 9].

Theorem 4.1. *If $T[f(x)] = F(u)$, then:*

1. $T[f'(x)] = \frac{F(u)}{u^2} - \frac{1}{u}f(0)$.
2. $T[f''(x)] = \frac{F(u)}{u^4} - \frac{1}{u^3}f(0) - \frac{1}{u}f'(0)$.
3. $T[f^n(x)] = \frac{F(u)}{u^{2n}} - \sum_{i=1}^n u^{2(i-n)-1} f^{(i-1)}(0)$.

Theorem 4.2. *If $T[f(x)] = F(u)$, then $T\left[\int_0^t f(w) dw\right] = u^2F(u)$.*

Theorem 4.3. *If $T[f(x)] = F(u)$, then:*

1. $T[xf(x)] = \frac{1}{2}\left[u^3 \frac{d}{du}F(u) + u^2F(u)\right]$.
2. $T[xf'(x)] = \frac{u^3}{2} \frac{d}{du}\left[\frac{F(u)}{u^2} - \frac{1}{u}f(0)\right] + \frac{u^2}{2}\left[\frac{F(u)}{u^2} - \frac{1}{u}f(0)\right]$.
3. $T[xf''(x)] = \frac{u^3}{2} \frac{d}{du}\left[\frac{F(u)}{u^4} - \frac{1}{u^3}f(0) - \frac{1}{u}f'(0)\right] + \frac{u^2}{2}\left[\frac{F(u)}{u^4} - \frac{1}{u^3}f(0) - \frac{1}{u}f'(0)\right]$.

Properties (iii) given in Theorem (4.3) may be generalized using mathematical induction to be written as follows

$$T[xf^{(n)}(x)] = \frac{u^3}{2} \frac{d}{du}\left[\frac{F(u)}{u^{2(n)}} - \sum_{i=1}^n u^{2(i-n)-1} f^{(i-1)}(0)\right] + \frac{u^2}{2}\left[\frac{F(u)}{u^{2n}} - \sum_{i=1}^n u^{2(i-n)-1} f^{(i-1)}(0)\right].$$

Theorem 4.4 (Convolution). *Let $f(x)$ and $g(x)$ having Laplace transforms $F(s)$ and $G(s)$ and Tarig transform $M(u)$ and $N(u)$, respectively, then:*

$$T[(f * g)(x)] = uM(u) N(u)$$

where $*$ is the convolution operation.

The most important property that relates between Tarig and Laplace transformations is given in the next theorem:

Theorem 4.5 (Relation between Laplace and Tarig Transformation). *If $T[f(x)] = G(u)$ and $L[f(x)] = F(s)$, then :*

$$G(u) = \frac{F\left(\frac{1}{u^2}\right)}{u}$$

Additional properties concerning Tarig transformation are given in the next theorems.

Theorem 4.6. *If $T[f(x)] = F(u)$, then $T[f(ax)] = \frac{1}{a}F(au)$.*

Theorem 4.7. *If $a, b \in \mathbb{R}$, and f, g are two functions of exponential order, then:*

$$T[af(x) + bf(x)] = aT(f(x)) + aT(g(x))$$

5. Modified Differential Transformation Method

Further studies can use some results that are stated and proven to solve delay differential equations through the proposed approach, that is, the modified DTM (MDTM).

Theorem 5.1. *If $y(x) = g(x - a) \frac{d^n}{dx^n} [h(x)]$, then:*

$$Y(k) = \frac{(k+n)!}{k!} \sum_{k_1=0}^k \sum_{h_1=k_1}^N (-1)^{h_1-k_1} \binom{h_1}{k_1} a^{h_1-k_1} G(h_1) H(k - k_1 + n)$$

Proof . Let the differential transform of $g(x - a)$ and $\frac{d^n}{dx^n} [h(x)]$ at $x = x_0$ be $G(k)$ and $H(k)$, respectively. By using Properties (2), we have the differential transform of $y(x)$ is as follows:

$$Y(k) = \sum_{k_1=0}^k G(k_1) H(k - k_1). \tag{5.1}$$

By Properties (4), we get:

$$G(k) = \sum_{h_1=k}^N (-1)^{h_1-k} \binom{h_1}{k} a^{h_1-k} G(h_1), \quad N \rightarrow \infty$$

and also from Properties(9), we get

$$H(k) = \frac{(k+n)!}{k!} H(k+n)$$

Substituting these values into Eq.(5.1), implies to:

$$Y(k) = \frac{(k+n)!}{k!} \sum_{k_1=0}^k \sum_{h_1=k_1}^N (-1)^{h_1-k_1} \binom{h_1}{k_1} a^{h_1-k_1} G(h_1) H(k - k_1 + n)$$

for $N \rightarrow \infty$. \square

Theorem 5.2. *If $y(x) = (g(x - a))^2$, then:*

$$Y(k) = \sum_{k_1=0}^k \sum_{h_1=k_1}^N (-1)^{h_1-k} \binom{h_1}{k_1} a^{h_1-k_1} G(h_1) G_2(k - k_1)$$

Proof: Let the differential transform of $g(x - a)$ and at $x = x_0$ be $G(k)$. By using Properties (2), we have the differential transform of $y(x)$ as:

$$Y(k) = \sum_{k_1=0}^k G_1(k_1) G_2(k - k_1)$$

Accordingly, from Properties (4), we get:

$$G_1(k) = \sum_{h_1=k}^N (-1)^{h_1-k} \binom{h_1}{k} a^{h_1-k} G_1(h_1), \quad N \rightarrow \infty$$

Substituting last equation into the differential transform $Y(k)$, implies to:

$$Y(k) = \sum_{k_1=0}^k \sum_{h_1=k_1}^N (-1)^{h_1-k_1} \binom{h_1}{k_1} a^{h_1-k_1} G_1(h_1) G_2(k - k_1) \blacksquare$$

Theorem 5.3. *If $y(x) = g(x - a)h(x)$, then:*

$$Y(k) = \sum_{k_1=0}^k \sum_{h_1=k_1}^N (-1)^{h_1-k_1} \binom{h_1}{k_1} a^{h_1-k_1} G(h_1) H(k - k_1)$$

Proof . Let the differential transform of $g(x - a)$ and $h(x)$ at $x = x_0$ be $G(k)$ and $H(k)$, respectively. By using Properties(2) and (2.3), we get:

$$Y(k) = \sum_{k_1=0}^k \sum_{h_1=k_1}^N (-1)^{h_1-k_1} \binom{h_1}{k_1} a^{h_1-k_1} G(h_1) H(k - k_1)$$

for $N \rightarrow \infty$. \square

Now, the MDTM will be presented as a hybrid approach to improve the DTM’s convergence by using Tarig transformation and Padé approximation. The solution series obtained by the DTM, although they contain numerous terms, may converge in a limited area. Therefore, the domain of convergence of the truncated power series expands the Tarig-Padé DTM (TPDTM) and predominantly result to the accurate solution. To improvement the solution of convergent series which was get through the DTM, we initially apply Tarig transformation and then evaluate its Padé approximation, transforming converted series into a meromorphic function. Finally, we take inverse Tarig transformation for the function obtained using Padé approximation to get the analytical solution. Therefore, based on the aforementioned method, we may obtain an accurate solution for the linear and nonlinear delay differential equations.

The TPDTM algorithm may be summarised as follows:

Step 1: Apply the DTM to the initial conditions and the system of differential equations to get a repetition system for the unknown $Y(0), Y(1), Y(2), \dots$; use the transformed initial conditions and replace the system calculations to determine the unknown $Y(0), Y(1), Y(2), \dots$.

Step 2: Use the differential inverse transform formula in Eq. (2.2) to get an approximate solution for the initial value problem.

Step 3: Apply Tarig transform with respect to x for the power series obtained in Step (2).

Step 4: Calculate the Padé approximation $[\frac{n}{m}]_f$ for the transformed series; n and m are arbitrarily chosen, but they should be smaller than the order of the power series. In this step, the Padé approximation extends the domain of the truncated series solution to obtain a higher accuracy while ignoring convergence.

Step 5: Eventually, applied the inverse Tarig transformation in Step (4), we get the accurate or approximate solution of the problem under consideration.

6. Illustrative Examples

As an illustration of the above approach, some examples are considered.

Example 6.1. *Consider the delay differential equation given in [10]:*

$$y'(x) = -y(x) + \frac{1}{10}y\left(\frac{x}{5}\right) - \frac{1}{10}e^{-\frac{x}{5}}, \quad 0 \leq x \leq 1 \tag{6.1}$$

subject to the initial condition:

$$y(0) = 0 \tag{6.2}$$

To solve this problem by using properties for DTM transformation, the following recurrence relations is used first:

$$\frac{(k + 1)}{k!} Y(k) = -Y(k) + \frac{1}{10} \left(\frac{1}{5}\right)^k Y(k) - \frac{1}{10} \frac{\left(-\frac{1}{5}\right)^k}{k!} \tag{6.3}$$

and form the initial condition Eq.(6.2), we get $Y(0) = 1$

Substituting $Y(0)$ in Eq.(6.3) recursively, the following transforms is obtained:

$$Y(1) = -1, \quad Y(2) = -\frac{1}{2}, \quad Y(3) = -\frac{1}{3!}, \quad y(4) = \frac{1}{4!}, \quad \dots \tag{6.4}$$

Therefore, according Eq.(2.2), it is obtained that:

$$y(x) = \sum_{k=0}^{\infty} Y(k) x^k = 1 - x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots \tag{6.5}$$

Now, applying the TPDTM for the third-order approximate solution

$$y_3(x) = \sum_{k=0}^3 Y(k) x^k = 1 - x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 \tag{6.6}$$

Applying the Tarig transformation with respect to $y(x)$ yields to:

$$\begin{aligned} T[y(u)] &= u - u^3 + \frac{1}{2!}2!u^5 - \frac{1}{3!}3!u^7 \\ &= u - u^3 + u^5 - u^7 \end{aligned} \tag{6.7}$$

For $n \geq 1, m \geq 1$ and $m + n \leq 7$, all $\left[\frac{n}{m}\right]_y(u)$ -Padé approximants of Eq.(6.7) with $n = 4, m = 3$, is given by:

$$\begin{aligned} T(u) - r(u) &= T(u) - \frac{p(u)}{q(u)} \\ &= T(u) q(u) - p(u) \\ &= (u - u^3 + u^5 - u^7) (1 + q_1u + q_2u^2 + q_3u^3) - (p_0 + p_1u + p_2u^2 + p_3u^3 + p_4u^4) \end{aligned}$$

Expanding and equating the like coefficients of the same power of u produces:

$$\begin{aligned} u^0 : 0 &= p_0 ; u^1 : 1 = p_1 ; u^2 : -q_1 = p_2 ; u^3 : q_2 - 1 = p_3 ; \\ u^4 : q_3 - q_1 &= p_4 ; u^5 : 1 - q_2 = 0 ; u^6 : q_1 - q_6 = 0 ; u^7 : q_2 - 1 = 0 \end{aligned}$$

Solving the last system of algebraic equations numerically will give:

$$p_0 = 0, \quad p_1 = 1, \quad p_2 = 0, \quad p_3 = 0, \quad p_4 = 0, \quad q_1 = 0, \quad q_2 = 1, \quad q_3 = 0$$

Then, substituting p_i 's and q_i 's, gives:

$$p(u) = u, \quad q(u) = 1 + u^2$$

So the Padé approximant is:

$$\left[\frac{n}{m}\right](u) = \frac{p(u)}{q(u)} = \frac{u}{1 + u^2} \tag{6.8}$$

Eventually, applying the inverse Tarig transform for the last Padé approximation, we reaches at the improved solution that is corresponding to the accurate solution $y(x) = e^{-x}$; which is also the exact solution of the delay differential equation [10].

The next example is given in [10], which will be solved here using the Laplace-Padé differential transform method. We found that the method of solution followed by [10] consist so many wrong step. We give the solution using the modified method proposed in this work.

Example 6.2. Consider the next one nonlinear delay differential equation:

$$y''' \left(\frac{x}{2} \right) - 2y'' \left(\frac{x}{2} \right) y' \left(\frac{x}{2} \right) + \frac{1}{4}y(t) = \frac{1}{4}\cosh \left(\frac{x}{2} \right), \quad 0 \leq t \leq 1 \tag{6.9}$$

subject to the initial conditions:

$$y(0) = 0, \quad y'(0) = 2, \quad y''(0) = 0 \tag{6.10}$$

The solution approach starts by using the differential transform DTM results to get next one repetition relations:

$$\begin{aligned} & \left(\frac{1}{2} \right)^{k+3} \frac{(k+1)(k+2)(k+3)}{k!} Y(k+3) \\ & - 2 \sum_{r=0}^k \left(\frac{1}{2} \right)^{k+2} \left(\frac{1}{2} \right)^{k-r+1} \frac{(r+2)!(k-r+1)!}{r!(k-r)!} Y(r+2) Y(k-r+1) + \frac{1}{4} Y(k) = \frac{1}{4} \frac{1}{2} \left(\frac{\left(\frac{1}{2}\right)^k + \left(-\frac{1}{2}\right)^k}{k!} \right) \end{aligned}$$

$$\begin{aligned} & \left(\frac{1}{2} \right)^{k+3} \frac{(k+1)(k+2)(k+3)}{k!} Y(k+3) \tag{6.11} \\ & - \left(\frac{1}{2} \right)^{2k+2} \sum_{r=0}^k 2^r (r+2)(k-r+1) Y(r+2) Y(k-r+1) + \frac{1}{4} Y(k) = \frac{1}{8} \left(\frac{\left(\frac{1}{2}\right)^k + \left(-\frac{1}{2}\right)^k}{k!} \right) \end{aligned}$$

While the differential transform of the initial conditions, are

$$Y(0) = 0, \quad Y(1) = 2, \quad Y(2) = 0 \tag{6.12}$$

Hence, substituting Eq.(6.11) in Eq.(6.12) recursively, we get the following results for the solution of the differential equation transform:

For $k = 0$: $\frac{Y(0)}{4} + \frac{3 Y(3)}{4} - \frac{Y(1) Y(2)}{2} - \frac{1}{4}, \quad Y(3) = \frac{1}{3}$
 For $k = 1$: $\frac{Y(1)}{4} + \frac{3 Y(4)}{2} - \frac{Y(2)^2}{4} - \frac{3 Y(1)Y(3)}{4}, \quad Y(4) = 0$
 For $k = 2$: $\frac{Y(2)}{4} + \frac{15Y(5)}{16} - \frac{3 Y(1)Y(4)}{4} - \frac{15Y(2)Y(3)}{32} - \frac{1}{32}, \quad Y(5) = \frac{1}{30}$
 For $k = 3$: $\frac{Y(5)}{4} + \frac{5 Y(6)}{16} - \frac{9 Y(3)^2}{64} - \frac{5 Y(1)Y(5)}{8} - \frac{13 Y(2)Y(4)}{32}, \quad Y(6) = -\frac{1}{12}$

Therefore, according to Eq.(2.2), we have:

$$y(x) = \sum_{k=0}^{\infty} Y(k) x^k = 2x + \frac{1}{3}x^3 + \frac{1}{30}x^5 - \frac{1}{12}x^6 + \dots \tag{6.13}$$

Now, using five-order TPDTM approximate solution:

$$y_5(x) = \sum_{k=0}^5 Y(k) x^k = 2x + \frac{1}{3}x^3 + \frac{1}{30}x^5 \tag{6.14}$$

Applying the Tarig transformation with respect to x for $y(x)$ yields to:

$$\begin{aligned} T[y(t)] &= 2u^3 + \frac{1}{3}3!u^7 + \frac{5!}{30}x^{11} \\ &= 2u^3 + 2 u^7 + \frac{5!}{30}x^{11} \end{aligned} \tag{6.15}$$

For $n \geq 1, m \geq 1$ and $n + m \leq 11, \forall \left[\frac{n}{m} \right]_y(u)$ -Padé approximants of Eq.(6.16), such that $n = 6, m = 5$, implies to:

$$\begin{aligned} T(u) - r(u) &= T(u) - \frac{p(u)}{q(u)} \\ &= T(u) q(u) - p(u) \\ &= \left(2u^3 + 2 u^7 + \frac{5!}{30}x^{11} \right) (1 + q_1u + q_2u^2 + q_3u^3 + q_4u^4 + q_5u^5) \\ &\quad - (p_0 + p_1u + p_2u^2 + p_3u^3 + p_4u^4 + p_5u^5 + p_6u^6) \end{aligned}$$

Carrying out some simplifications and calculations of the last equation and solving the resulting system obtained by equating the like power of u , to get:

$$\begin{aligned} p_0 = 0, p_1 = 0, p_2 = 0, p_3 = 2, p_4 = 0, p_5 = 0, p_6 = 0 \\ q_1 = 0, q_2 = 0, q_3 = 0, q_4 = -1, q_5 = 0 \end{aligned}$$

Hence:

$$p(u) = 2u^3, q(u) = 1 - u^4$$

So the Padé approximant is:

$$\left[\frac{n}{m} \right]_y(u) = \frac{p(u)}{q(u)} = \frac{2u^3}{1 - u^4} \tag{6.16}$$

Eventually, applying the inverse Tarig transform of the Padé approximant Eq.(6.16), we reaches to the improved solution, which agree to the approximate solution:

$$y(x) = 2\sinh(x) = 2 \left(\frac{e^x - e^{-x}}{2} \right) = e^x - e^{-x}$$

which is the same solution obtained in [10] as the exact solution.

7. Application of TPDTM for Solving Vector-Borne Diseases with Delays

In this section, the diseases biological problem of the vector-borne diseases with time delay will be solved. The governing equation for the proportion of humans diseases is reduced in [13] to:

$$\dot{x}(t) = \frac{pa^2qme^{-\alpha\tau}e^{-\mu\tau}x(t-\tau)}{pae^{-\mu\tau}x(t-\tau) + \mu} (1 - x(t-\tau)) - \alpha x(t) \tag{7.1}$$

with premasters $P = pae^{-\mu\tau} = 13.9498, Q = Ppaqme^{-\alpha\tau} = 225.816, \tau = 1, \alpha = 2.74768, \mu = 12$ and the initial conditions $x_1(0) = 0.73$

To solve this equations, start from Eq.(7.1), to get:

$$\begin{aligned}
 (\dot{x}(t) + \alpha x(t))(Px(t - \tau) + \mu) &= Qx(t - \tau)(1 - x(t - \tau)) P\dot{x}(t) x(t - \tau) \\
 + \alpha P x(t) x(t - \tau) + \mu\dot{x}(t) + \alpha\mu x(t) - Qx(t - \tau) + Q(x(t - \tau))^2 &= 0
 \end{aligned}
 \tag{7.2}$$

Apply properties and theorems of DTM, to get:

$$\begin{aligned}
 P \frac{(k+1)!}{k!} \sum_{k_1=0}^k \sum_{h_1=k_1}^N (-1)^{h_1-k_1} \binom{h_1}{k_1} a^{h_1-k_1} X(h_1) X(k-k_1+1) + \\
 \alpha P \sum_{k_1=0}^k \sum_{h_1=k_1}^N (-1)^{h_1-k_1} \binom{h_1}{k_1} a^{h_1-k_1} X(h_1) X(k-k_1) + \mu \frac{(k+1)!}{k!} X(k+1) + \\
 \alpha\mu X(K) - Q \sum_{h_1=k}^N (-1)^{h_1-k} \binom{h_1}{k} a^{h_1-k} X(h_1) + \\
 Q \sum_{k_1=0}^k \sum_{h_1=k_1}^N (-1)^{h_1-k_1} \binom{h_1}{k_1} a^{h_1-k_1} X(h_1) X(k-k_1)
 \end{aligned}
 \tag{7.3}$$

Consider now the differential transform of $y(x)$ at $x_0 = 0$, with the transformed initial condition $X(0) = 0.73$, substituting in Eq.(7.3) recursively we arrive at the following results:

$$X(1) = -4.576 \times 10^{-5}, \quad X(2) = 3.464 \times 10^{-5}, \quad Y(3) = -5.053 \times 10^{-5}, \quad y(4) = 6.592 \times 10^{-4}, \dots
 \tag{7.4}$$

Therefore, according to Eq.(2.2):

$$y(x) = \sum_{k=0}^{\infty} Y(k) x^k = 0.73 - 4.576 \times 10^{-5}x + 3.464 \times 10^{-5}x^2 - 5.053 \times 10^{-5}x^3 + 6.592 \times 10^{-4}x^4
 \tag{7.5}$$

while using the TPDTM for the second-order approximate solution

$$y_2(x) = \sum_{k=0}^2 Y(k) x^k = 0.73 - 4.576 \times 10^{-5}x + 3.464 \times 10^{-5}x^2
 \tag{7.6}$$

Applying Tarig transform with respect to for $y(x)$ productivity to:

$$T[y(u)] = 0.73u - 4.576 \times 10^{-5}u^3 + 2!3.464 \times 10^{-5}u^5
 \tag{7.7}$$

For $n \geq 1, m \geq 1$ and $m+n \leq 5, \forall \left[\frac{n}{m}\right]_y(u)$ -Padé approximants of Eq.(7.7) such take $n = 3, m = 2$, give:

$$\begin{aligned}
 T(u) - r(u) &= T(u) - \frac{p(u)}{q(u)} = T(u)q(u) - p(u) \\
 &= \left(0.73u - 4.576 \times 10^{-5}u^3 + 2!3.464 \times 10^{-5}u^5\right) (1 + q_1u + q_2u^2) - (p_0 + p_1u + p_2u^2 + p_3u^3)
 \end{aligned}$$

Expanding and collecting the like terms of u produces:

$$u^0 : 0 = p_0 ; u^1 : 0.73 = p_1 ; u^2 : 0.73 q_1 = p_2 ; u^3 : 0.73q_2 - 0.00004576 = p_3 ;$$

$$u^4 : -0.00004576q_1 = 0 ; u^5 = 0.00006928 - 0.00004576 q_2 = 0 ; u^6 : 0.00006928q_1 = 0$$

The last resulted system may be solved numerically to get:

$$p_0 = 0, \quad p_1 = \frac{73}{100}, \quad p_2 = 0, \quad p_3 = \frac{2932}{2653}, \quad q_1 = 0, \quad q_2 = \frac{433}{286}$$

Hence:

$$p(u) = \frac{73}{100}u + \frac{2932}{2653}u^3, \quad q(u) = 1 + \frac{433}{286}u^2$$

So the Padé approximation is:

$$\begin{aligned} \left[\frac{n}{m} \right] (u) &= \frac{p(u)}{q(u)} = \frac{\frac{73}{100}u + \frac{2932}{2653}u^3}{1 + \frac{433}{286}u^2} = \frac{838552 u}{1148749} - \frac{497211 u}{57437450 (433u^2 + 286)} \\ &= \frac{838552}{1148749}u - \frac{497211 u}{57437450 \times 286 \left(\frac{433}{286}u^2 + 1\right)} \\ &= \frac{838552}{1148749}u - \frac{1}{33038} \frac{u}{\left(1 + \frac{433}{286}u^2\right)} \end{aligned} \tag{7.8}$$

Eventually, applying the inverse Tarig transform of the Padé approximant Eq.(7.8),we reaches to the improved solution which agree to the accurate solution:

$$y(x) = \frac{838552}{1148749} - \frac{1}{33038} e^{-\frac{433}{286} x}$$

Table 1 presents the some discrete results and the residue error, while Figures 1 and 2 present the solutions and the residue errors, respectively.

Table 1: Solution of the vector-borne diseases problem and its residue error using the DTM and the TPDTM

x	$y(x)$ using TPDTM	Residue error for (TPDTM)	$y(x)$ using DTM	Residue error for (DTM)
0	0.729939	0.037397	0.73	0.090488
10	0.72997	0.018604	7.274476	6.413069×10^3
20	0.72997	0.018604	105.810701	2.041504×10^6
30	0.72997	0.018604	533.347493	5.888445×10^7
40	0.72997	0.018604	1.685102e3	6.25302×10^8
50	0.72997	0.018604	4.114498e3	3.86746×10^9
60	0.72997	0.018604	8.533169e3	1.704497×10^{10}

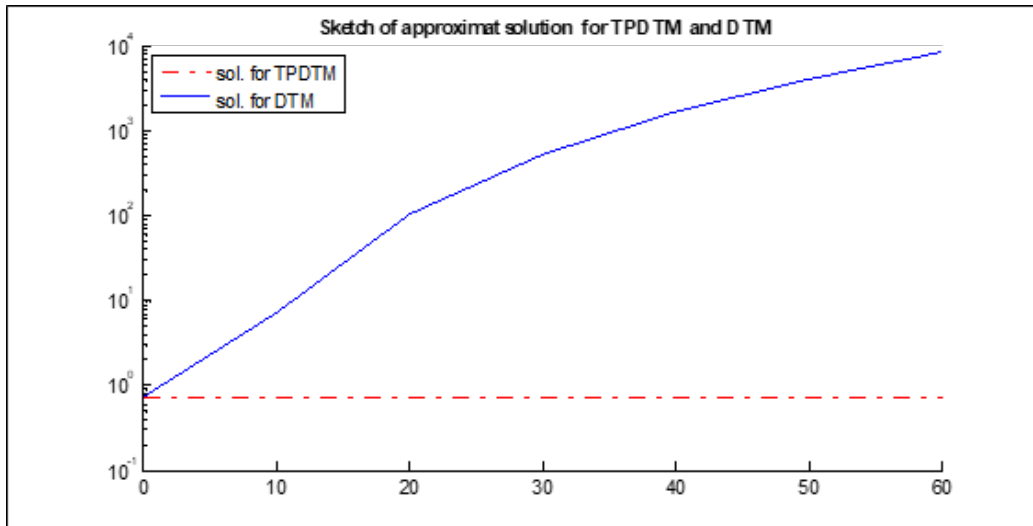


Figure 1: The solution of the vector-borne diseases problem using the DTM and the TPDTM.

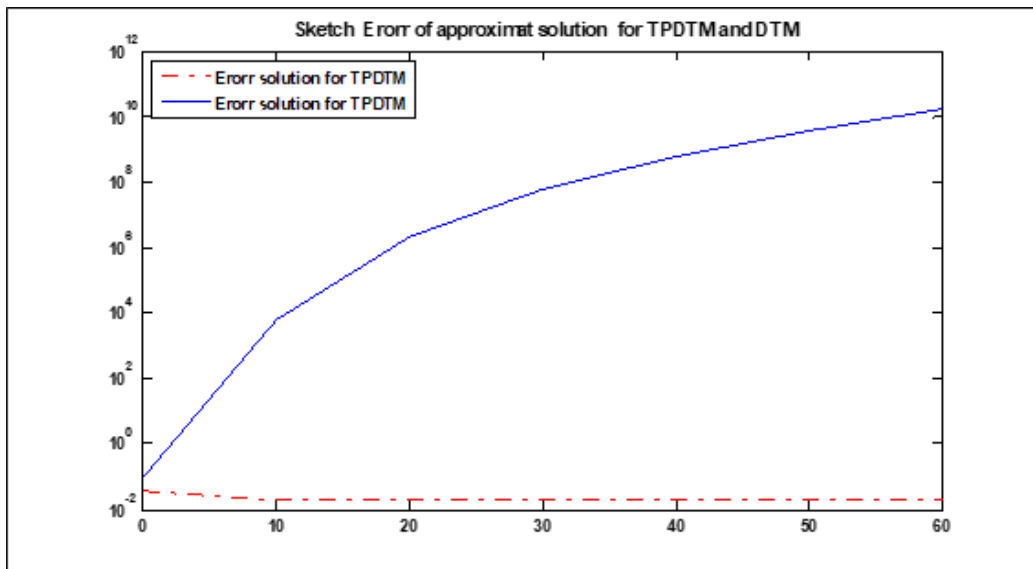


Figure 2: The residue error of the vector-borne diseases problem using the DTM and the TPDTM.

8. Conclusions

In this paper, we present Differential Transformation Method (DTM) and proved some new theorems using to solve linear and nonlinear DDEs. Also we present the Modified Differential Transformation Method (MDTM) as a hybrid approach to improve the DTM’s convergence by combine form of the differential transform method with Tarig transformation, and Padé approximation (TPDTM). The main advantage for TPDTM method is its ability to collect the two strongest method to finding a fast convergent series solution of DDEs and This technique is faster than Laplace- Padé differential transform method (LPDTM), we can found solution by this method followed by [10] consist many wrong step which is solve using the Laplace-Padé differential transform method, we saw this modified is effectively used to find accurate solution of linear and nonlinear of delay differential transform.

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