Int. J. Nonlinear Anal. Appl. 13 (2022) 1, 3589-3597
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.6139

The algebra fuzzy norm of the quotient space and pseudo algebra fuzzy normed space

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(Communicated by Madjid Eshaghi Gordji)

Abstract

The principle aim of this article is to use the notion of algebra fuzzy normed space and its basic properties in order to introduce the notion of algebra fuzzy norm of the quotient space. If the quotient space is fuzzy complete then we prove that U is fuzzy complete. Also, we introduce the notion of pseudo algebra fuzzy normed space and investigate the basic properties of this space. In this direction we define a relation $\sim$ on U in order to introduce the space $\widehat{U}$ of classes $\widehat{u} = [u] = \{z \in U : z \sim u\}$. If U is pseudo algebra fuzzy normed space then we prove that $\widehat{U}$ is algebra fuzzy normed space also if U is a fuzzy complete we prove that $\widehat{U}$ is fuzzy complete.

Keywords: Algebra fuzzy absolute space, Algebra fuzzy normed space, Fuzzy complete, Fuzzy bounded linear operator.

1. Introduction

In [1] the fuzzy topological structure of a fuzzy normed space was studied by Sadeqi and Kia. In [2] Kider introduced a fuzzy normed space. Also he proved this fuzzy normed space has a completion in [3]. Again in [4] Kider introduced a new type of fuzzy normed space. In 2013 [5] Bag and Samanta study finite dimensional fuzzy normed linear spaces and proved some basic results

In [6] Kider and Kadhum introduce the fuzzy norm for a fuzzy bounded operator on a fuzzy normed space and proved its basic properties then other properties was proved by Kadhum in [7]. In [8] Ali proved basic properties of complete fuzzy normed algebra. Again in [9] Kider and Ali

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Received: May 2021 *Accepted:* October 2021

introduce the notion of fuzzy absolute value and study some properties of finite dimensional fuzzy normed space.

The concept of general fuzzy normed space was presented by Kider and Gheeab in [10, 11]. Also, they study the general fuzzy normed space GFB(V, U) and proved basic properties of this space. In 2019 [12] Kider and Kadhum introduce the notion fuzzy compact linear operator and proved its basic properties. In [13] Kider introduce the notion fuzzy soft metric space after that he investigated and proved some basic properties of this space again Kider in [14] introduce a new type of fuzzy metric space called algebra fuzzy metric space after that The algebra fuzzy normed space and its basic properties basic properties of this space is proved. Kider and Gheeab in [15] proved basic properties of the adjoint operator of a general fuzzy bounded operator.

In this paper, first we recall the concept of algebra fuzzy absolute value space and its some basic properties that introduced by Khudhair and Kider in [16] again we recall the concept of algebra fuzzy normed space and its some basic properties that introduced by Khudhair and Kider in [16]. The definition of the algebra fuzzy norm of the quotient space is our first aim after that we introduce the basic properties of this space. The second aim is to define the notion of pseudo algebra fuzzy normed space and prove basic properties of this space.

2. The algebra fuzzy normed space and its basic properties

Definition 2.1. *[1] Assume that* $S \neq \emptyset$, *a fuzzy set* $\widetilde{D}$ *in* S *is represented by* $\widetilde{D} = \{(s, \mu_{\widetilde{D}}(s))\colon s \in S, 0 \leq \mu_{\widetilde{D}}(s) \leq 1\}$ *where* $\mu_{\widetilde{D}}(x)\colon S \to I$ *is a membership function where* $I=[0, 1]$.

Definition 2.2. *[14] The binary operation* $\circledcirc\colon I \times I \to I$ *be is said to be continuous t-conorm if it satisfies*

(i) $\circledcirc$ *is continuous function*

(ii) $t \circledcirc [s \circledcirc r] = [t \circledcirc s] \circledcirc r$

(iii) $s \circledcirc r = r \circledcirc s$

(iv) $s \circledcirc 0 = 0$,

(v) $(r \circledcirc z) \leq (s \circledcirc w)$ *whenever* $r \leq s$ *and* $z \leq w$,

for all $r, s, z, w \in I=[0, 1]$.

Lemma 2.3. *[14] If* $\circledcirc$ *is a continuous t-conorm on* $[0, 1]$ *then*

(i) $1 \circledcirc 1=1$,

(ii) $0 \circledcirc 1 = 1 \circledcirc 0 = 1$,

(iii) $0 \circledcirc 0 = 0$,

(iv) $p \circledcirc p \geq p$ *for all* $p \in [0, 1]$.

Remark 2.4. *[14] If* $\circledcirc$ *is a continuous t-conorm then*

(i) for any $p, q \in (0, 1)$ *with* $p > q$ *we have* $w \in (0, 1)$ *whenever* $p > q \circledcirc w$. *In general for any* $p, q \in (0, 1)$ *with* $p > q$ *we can find* $w_1, w_2, \ldots, w_k \in (0, 1)$ *whenever* $p > q \circledcirc w_1 \circledcirc w_2 \circledcirc \ldots \circledcirc w_k$ *where* $k \in \mathbb{N}$.

(ii) For any $r \in (0, 1)$ $\exists$ $s \in (0, 1)$ *such that* $s \circledcirc s \leq r$. *In general for any* $r \in (0, 1)$ *there exists* $w_1, w_2, \ldots, w_k \in (0, 1)$ *such that* $w_1 \circledcirc w_2 \circledcirc \ldots \circledcirc w_k \leq r$ *where* $k \in \mathbb{N}$.

Example 2.5. *[14]The algebra product* $p \circledcirc q = p + q -- pq$ *is a continuous t-conorm for all* $p, q \in [0, 1]$.

Definition 2.6. *[16] Let* $a_{\mathbb{R}}\colon \mathbb{R} \to I$ *be a fuzzy set and* $\circledcirc$ *be a continuous t-conorm then a is called* ***algebra fuzzy absolute value on*** $\mathbb{R}$ *if*

1. $0 < a_{\mathbb{R}}(\alpha) \leq 1$
2. $a_{\mathbb{R}}(\alpha)$=0 *if and only if* $\alpha = 0$
3. $a_{\mathbb{R}}(\alpha\beta) \leq a_{\mathbb{R}}(\alpha).a_{\mathbb{R}}(\beta)$
4. $a_{\mathbb{R}}(\alpha + \beta) \leq a_{\mathbb{R}}(\alpha) \circledcirc a_{\mathbb{R}}(\beta)$. *For all* α, $\beta \in \mathbb{R}$.

Then ($\mathbb{R}$, $a_{\mathbb{R}}$, $\circledcirc$) is called **algebra fuzzy absolute value space.**

Example 2.7. *[16] Let* $|.|$ *be absolute value on* $\mathbb{R}$ *and* $\alpha \circledcirc \beta = \alpha + \beta - \alpha\beta$ *for all* α, $\beta \in I$. *Define*

$$a_{|.|}(r) = \begin{cases} \frac{|r|}{1+|r|} & if\ r \neq 1 \\ 1 & if\ r = 1 \end{cases}$$

For all r $\in \mathbb{R}$. Then ($\mathbb{R}$, $a_{|.|}$, $\circledcirc$) is algebra fuzzy absolute value space. Also $a_{|.|}$ is called the **standard algebra fuzzy absolute value on $\mathbb{R}$.**

Example 2.8. *[16] Define* $a^{|\cdot|}$: $\mathbb{R} \to I$ *by* $a^{|\cdot|}(\alpha) = \begin{cases} \frac{1}{|\alpha|} & if\ \alpha \neq 0 \\ 0 & if\ \ \alpha = 0 \end{cases}$

for all $\alpha \in$ $\mathbb{R}$. If s$\circledcirc$r=s + r $-$sr for all s, r $\in$I then ($\mathbb{R}$, $a^{|\cdot|}$, $\circledcirc$) is algebra fuzzy absolute value space. Then $a^{|\cdot|}$ is called the **algebra fuzzy absolute value space induced by $|.|$.**

Definition 2.9. *[16] Let (*$\mathbb{R}$, $a_{\mathbb{R}}$, $\circledcirc$*) be algebra fuzzy absolute value space also let* $\{p_n\}_{n=1}^{\infty} \in \mathbb{R}$, *we say* $\{p_n\}_{n=1}^{\infty}$ *is fuzzy approaches to the limit p as n approaches to* ∞ *if* $\forall$ *s* $\in$ *(0,1)* $\exists$ $N \in \mathbb{N}$ *s. t.* $a_{\mathbb{R}}(p_n - p) < s$, $\forall$ $n \geq N$. *If* p_n *is fuzzy approaches to the limit p we write* $lim_{n\to\infty} p_n = p$ *or* $p_n \to p$ *as n approaches to* ∞ *or* $lim_{n\to\infty} a_{\mathbb{R}}(p_n - p) = 0$.

Definition 2.10. *[16] Suppose that (*$\mathbb{R}$, $a_{\mathbb{R}}$, $\circledcirc$*) is algebra fuzzy absolute value space also let* $\{p_n\}_{n=1}^{\infty} \in \mathbb{R}$, *we say* $\{p_n\}_{n=1}^{\infty}$ *is fuzzy Cauchy sequence in* $\mathbb{R}$ *if* $\forall$ *s* $\in$ *(0,1)* $\exists$ $N \in \mathbb{N}$ *s. t.* $a_{\mathbb{R}}(p_k - p_m) < s$, $\forall$ *k, m* $\geq N$.

Definition 2.11. *[16] Let (*$\mathbb{R}$, $a_{\mathbb{R}}$, $\circledcirc$*) be algebra fuzzy absolute value space. We say the sequence* $\{q_n\}_{n=1}^{\infty}$ *in* $\mathbb{R}$ *is fuzzy bounded if* $\exists$ *t* $\in$ *(0, 1) s. t.* $a_{\mathbb{R}}(q_n) < t$ $\forall$ $n \in \mathbb{N}$

Theorem 2.12. *[16] Let (*$\mathbb{R}$, $a_{\mathbb{R}}$, $\circledcirc$*) be algebra fuzzy absolute value space. Then* $\{q_n\}_{n=1}^{\infty}$ *is fuzzy bounded if* $\{q_n\}_{n=1}^{\infty}$ *is a fuzzy Cauchy sequence in* $\mathbb{R}$**.**

Definition 2.13. *[16] If every fuzzy Cauchy sequence in* $\mathbb{R}$ *is fuzzy approaches to a real number in* $\mathbb{R}$ *then algebra fuzzy absolute value (*$\mathbb{R}$, $a_{\mathbb{R}}$, $\circledcirc$*) is called fuzzy complete.*

Definition 2.14. *[16] Let U be a vector space over* $\mathbb{R}$ *and let* $\circledcirc$ *be a continuous t-conorm. Let (*$\mathbb{R}$, $a_{\mathbb{R}}$, $\circledcirc$*) be algebra fuzzy absolute value space and n: U$\to$I be a fuzzy set then n is called* ***algebra fuzzy norm on U*** *if*

1. *0<* $n(u) \leq 1$,
2. $n(u) = 0$ *if and only if* u=0,
3. $n(\alpha u) \leq a_{\mathbb{R}}(\alpha)\ n(u)$ *for all* $0 \neq \alpha \in \mathbb{R}$,

4. $n(u + v) \le n(u) \circledcirc n(v)$,

for all $u, v \in U$. *Then* $(U, n, \circledcirc)$ *is called* ***algebra fuzzy normed space***.

Example 2.15. *[16] Let* $U=C[p, b]$, $t \circledcirc s = t + s - ts$ *for all* $t, s \in I$ *and* $(\mathbb{R}, a_{\mathbb{R}}, \circledcirc)$ *is algebra fuzzy absolute space. Define* $n(r) = max_{s\in[p,\ b]}\ a_{\mathbb{R}}[r(s)]$ *for all* $r \in U$. *Then* $(U, n, \circledcirc)$ *is algebra fuzzy normed space*

Lemma 2.16. *[16] If* $(U, n, \circledcirc)$ *is algebra fuzzy normed space then* $n(u-v) = n(v-u)$ *for all* $u, v \in U$.

Definition 2.17. *[16] Let* $(u_k) \in U$ *where* $(U, n, \circledcirc)$ *is algebra fuzzy normed space. Then* (u_k) *is fuzzy converges to the limit u as k approaches to* ∞ *if* $\forall\ s \in (0,1)\ \exists\ N \in \mathbb{N}$ *s. t.* $n(u_k-u) < s$, $\forall\ k \ge N$. *If* (u_k) *is fuzzy approaches to the limit u we write* $lim_{k\to\infty} u_k = u$ *or* $u_k \to u$ *as k approaches to* ∞ *or* $lim_{n\to\infty} n(u_k-u) = 0$.

Definition 2.18. *[16]Suppose that* $(U, n, \circledcirc)$ *is algebra fuzzy normed space . Then* $fb(u, r) = \{v \in U: n(u-v) < t\}$ *and* $fb[u, r] = \{v \in U: n(u-v) \le t\}$ *is known as the* ***open and closed fuzzy ball*** *with the center* $u \in U$ *and radius t, with* $t \in (0, 1)$ *respectively.*

Lemma 2.19. *[16]The function* $u \longmapsto n(u)$ *is a fuzzy continuous function from U into* $\mathbb{R}$ *when* $(U, n, \circledcirc)$ *and* $(\mathbb{R}, n, \circledcirc)$ *are algebra fuzzy normed spaces.*

Definition 2.20. *[16] Let* $(U, n, \circledcirc)$ *be algebra fuzzy normed space and let* (u_k) *be a sequence in U, we say that* (u_k) *is fuzzy Cauchy sequence in U if* $\forall\ s \in (0, 1) \ni N \in \mathbb{N}$ *s. t.* $n(u_k-u_m) < s$, $\forall\ k, m \ge N$.

Definition 2.21. *[16]Suppose that* $(U, n, \circledcirc)$ *is algebra fuzzy normed space and* $W \subseteq U$ *is known as* ***fuzzy open*** *if* $fb(w, j) \subseteq W$ *for any arbitrary* $w \in W$ *and for some* $j \in (0, 1)$. *Also* $D \subseteq U$ *is known as* ***fuzzy closed*** *if* D^C *is fuzzy open. Moreover the* ***fuzzy closure*** *of D,* $\overline{D}$ *is defined to be the smallest fuzzy closed set contains D.*

Definition 2.22. *[16] Suppose that* $(U, n, \circledcirc)$ *is algebra fuzzy normed space then* $D \subseteq U$ *is known as* ***fuzzy dense*** *in U if whenever* $\overline{D} = U$.

Theorem 2.23. *[16]If* $fb(s, j)$ *is an open fuzzy ball in algebra fuzzy normed space* $(U, n, \circledcirc)$ *then it is a fuzzy open set.*

Definition 2.24. *[16] If* (s_k) *is fuzzy Cauchy sequence in U with* $s_k \to s \in U$ *then the algebra fuzzy normed space* $(U, n, \circledcirc)$ *is known as fuzzy complete.*

Theorem 2.25. *[16]In algebra fuzzy normed space* $(U, n, \circledcirc)$ *if* $u_k \to u \in U$ *then* (u_k) *is fuzzy Cauchy.*

Theorem 2.26. *[16] In algebra fuzzy normed space* $(U, n, \circledcirc)$ *when* $D \subset U$ *then* $d \in \overline{D}$ *if and only if there is* $(d_k) \in D$ *with* $d_k \to d$.

3. The algebra fuzzy norm of the quotient space and some of its basic properties

Let U be a vector space over the field F and let D be a closed subspace of U define a relation on U by v∼u if and only if v−u ∈D then the class $\hat{u}$=[u]={z∈U : z∼v} = u+D. Then $\frac{U}{D}$={u+D: u∈U} is a vector space over the field F when equipped with the operations : (v +D) +(u +D) = (v + u) +D and α(u+D) = (αu)+D.

Definition 3.1. *Let (U, n_U , ⊚) be algebra fuzzy normed space and D⊂U is a fuzzy closed in U. Then the algebra fuzzy norm q of quotient space is defined by q[u +D] = $inf_{d\in D} n_U$[u+d] for all u+D ∈ $\frac{U}{D}$.*

Theorem 3.2. *Let (U, n_U , ⊚) be algebra fuzzy normed space and D⊂U is a closed in U. Then the quotient space ($\frac{U}{D}$, q, ⊚) is algebra fuzzy normed space.*

Proof . Now we prove that ($\frac{U}{D}$, q, ⊚) is satisfying all the conditions of algebra fuzzy normed space.

1. since n_U(u+d) ∈I for all d∈D so q[u+D] ∈I.
2. If (u+D)= $\hat{0}$ in $\frac{U}{D}$ then u∈D and so taking d=−u we get q[u+D]= $inf_{d\in D} n_U$[u+d]=0. On the other hand if q[u+D]=0 then we can find (d_k) in D with n_U[u+d_k]→0 as k→∞. Hence $-d_k$→u in U. Since D is fuzzy closed we deduce that u∈D and so (u+D) = $\hat{0}$ in $\frac{U}{D}$ hence q[u+D]=0 ⇔ (u+D)=$\hat{0}$.
3. For any $\alpha \in \mathbb{R}$, $\alpha \neq 0$ and u∈U q[α(u+D)]= $inf_{d\in D} n_U$[α(u+d)] ≤ $inf_{d\in D} a_{\mathbb{R}}(\alpha)$ n_U[α(u+d)] ≤ $a_{\mathbb{R}}(\alpha)$[$inf_{d\in D} n_U$(u+d)]= $a_{\mathbb{R}}(\alpha)$ q[α(u+D)]
4. For any $u, v \in U q[(u+D)+(v+D)] = q[(u+v)+D)] = inf_{d\in D} n_U[u+v+d] \leq inf_{d\in D} n_U[u+d] \circledcirc inf_{d\in D} n_U[v+d] = q[u+D] \circledcirc q[v+D]$

Hence ($\frac{U}{D}$, q, ⊚) is algebra fuzzy normed space. □

Remark 3.3. *If $(U, n_U, \circledcirc)$ be algebra fuzzy normed space and $D \subset U$ is a closed in U. Then*

1. *$\pi : U \to \frac{U}{D}$ is a natural operator defined by $\pi[u] = u + D$.*
2. *$q(u + D) \leq n_U(u)$*

Theorem 3.4. *Let (U, n_U , ⊚) be algebra fuzzy normed space and D⊂U is a fuzzy closed in U. If ($\frac{U}{D}$, q, ⊚) is fuzzy complete then (U, n_U , ⊚) is fuzzy complete*

Proof . Suppose that (u_k) is a fuzzy Cauchy sequence in U then for any t∈ (0, 1) there is N∈ ℕ such that $n_U[u_m - u_j]$ <t for all m, j ≥N. Now q[$(u_m - u_j)$+ D)] ≤ $n_U[u_m - u_j]$ <t for all m, j ≥N by remark 3.3 That is (u_k+D) is a fuzzy Cauchy sequence in $\frac{U}{D}$ but $\frac{U}{D}$ is fuzzy complete so we can find u+D ∈ $\frac{U}{D}$ such that $(u_k + D)$ →(u+D) ∈ $\frac{U}{D}$. Now in particular u_k →u∈U. Hence U is fuzzy complete. □

Proposition 3.5. *Let (U, n_U , ⊚) be algebra fuzzy normed space. Then U is fuzzy complete if and only if (u_k) is any sequence in U with $\sum_{j=1}^{\infty} n_U[u_j]$ <1 then there is u∈U with $\sum_{j=1}^{k} u_j$→u as k→∞ .*

Proof . If U is fuzzy complete then it is clear that (s_k=[$\sum_{1=1}^{k} u_j$]) is a fuzzy Cauchy sequence and hence fuzzy converges.

Conversely, suppose that (u_k) is any sequence in U such that $\sum_{j=1}^{k} n_U[u_j]$ <1 then there is u∈U with $\sum_{j=1}^{k} u_j \rightarrow$u as k→∞ .

Let (y_k) be a fuzzy Cauchy sequence in U. We construct a subsequence as follows. Let k_1 ∈N be such that $n_U[y_{k_1} - y_j]<\frac{1}{2}$ for all j> k_1. Now Let $k_2>k_1$ be such that $n_U[y_{k_2} - y_j\ ] <\frac{1}{4}$ for all j> k_2 after that we have (y_{k_m}) of (y_k) such that $n_U[y_{k_m}- y_{k_{m+1}}] <\frac{1}{2^m}$, m=1, 2, ... Put u_j=($y_{k_j}-y_{k_{j+1}}$) then $\sum_{j=1}^{\infty} n_U[u_j] = \sum_{j=1}^{\infty} n_U[y_{k_j}-y_{k_{j+1}}] < \sum_{j=1}^{\infty}\frac{1}{2^j}$ <1
By hypothesis there is u∈U such that

$$u = \lim_{m\to\infty} \sum_{j=1}^{m} u_j \ = \lim_{m\to\infty}[(y_{k_1}- y_{k_2}) + (y_{k_2}-y_{k_3}) + \cdots + (y_{k_m} - y_{k_{m+1}})] = \lim_{m\to\infty}[y_{k_1}-y_{k_{m+1}}]$$

Thus is (y_{k_m}) fuzzy converges in U [to (y_{n_1}−u)] . But if a subsequence of a Cauchy sequence converges the whole sequence does that is (y_n) fuzzy converges in U. Thus U is fuzzy complete. □

Theorem 3.6. *Let (U, n_U , ⊚) be algebra fuzzy normed space and D⊂U is a fuzzy closed in U. If (U, n_U , ⊚) is fuzzy complete then ($\frac{U}{D}$, q, ⊚) is fuzzy complete.*

Proof . Suppose that (u_k+D) is a sequence in $\frac{U}{D}$ such that $\sum_{k=1}^{\infty} q\,[u_k+D]$ <1. By definition of the infimum for each k there is d_k∈D with

$$n_U[u_k+d_k] < inf_{\,d\in D} n_U[u_k+d] + \frac{1}{2^k} = q[u_k+D] + \frac{1}{2^k}$$

Hence

$$\sum_{k=1}^{\infty} n_U[u_k+d_k] < \sum_{k=1}^{\infty}[q(u_k+D)\ + \frac{1}{2^k}] < \infty$$

Now U is a fuzzy complete so by the pervious proposition it follows we can find y∈U with y=$lim_{m\to\infty} \sum_{k=1}^{m} (u_k+d_k)$.
We claim that $\sum_{k=1}^{n} (u_k+D) \rightarrow (y + D)$ in $\frac{U}{D}$ .Indeed

$$\begin{aligned} q[(y + D)- \sum_{k=1}^{n} (u_k+D)\,] &= \ q[\left(y- \sum_{k=1}^{n} u_k\right) +D] \\ &= inf_{\,d\in D} n_U\left[\ y- \sum_{k=1}^{n} u_k+d\right] \\ &\le n_U[\ y- \ \sum_{k=1}^{n} u_k- \ \sum_{k=1}^{n} d_k] \\ &= n_U[\ y- \ \sum_{k=1}^{n} (u_k+ d_k)\,] \ \rightarrow 0 \text{ as } \ k\rightarrow\infty \end{aligned}$$

Hence $\sum_{k=1}^{n} (u_k+D)\ \rightarrow$ (y+D). So $\frac{U}{D}$ is fuzzy complete by Proposition 3.5 □

4. Pseudo algebra fuzzy normed space and some of its basic properties

Definition 4.1. *Let U be a vector space over $\mathbb{R}$ and let $\circledcirc$ be a continuous t-conorm. Let ($\mathbb{R}$, $a_{\mathbb{R}}$, $\circledcirc$) be algebra fuzzy absolute value space and n: $U \to I$ be a fuzzy set then n is called* ***pseudo algebra fuzzy norm on U*** *if*

1. $0 < n(u) \leq 1$.
2. *if* $u = 0$ *then* $n(u) = 0$
3. $n(\alpha u) \leq a_{\mathbb{R}}(\alpha)\ n(u)$ *for all* $0 \neq \alpha \in \mathbb{R}$.
4. $n(u + v) \leq n(u) \circledcirc n(v)$
   For all $u, v \in U$.

Then $(U, n, \circledcirc)$ *is called* ***pseudo algebra fuzzy normed space****.*

Remark 4.2. *Every fuzzy normed space is pseudo fuzzy normed space but the converse is not true in general as shown in the following example*

Example 4.3. *Let (U, $\|.\|$) be a normed space where U=$\mathbb{R}$ and $\alpha \circledcirc \beta = \alpha + \beta - \alpha\,\beta$ for all α, β $\in I$. Let $D = \{(u_k) \in U : (u_k)$ is converges to $u \in U$ in $(U, \|.\|)\}$. Define*

$$n_D[(u_k)] = \frac{\|lim_{k\to\infty} u_k\ \|}{1 + \|lim_{k\to\infty} u_k\ \|}$$

for all $(u_k) \in\ D$. Then $(D, n_D, \circledcirc)$ is pseudo algebra fuzzy normed space but not algebra fuzzy normed space.

Proof . By simple calculation we see that (D, n_D, $\circledcirc$) is pseudo algebra fuzzy normed space. Now since (u_k)=$\left(\frac{1}{k}\right) \in$ D where k $\in \mathbb{N}$ and $(u_k) \neq 0$ but $n_D[(u_k)]$=0 since $u_k \to 0$. Hence (D, n_D, $\circledcirc$) is not algebra fuzzy normed space. □

Proposition 4.4. *Suppose that (U, n, $\circledcirc$) is pseudo algebra fuzzy normed space. Define $\sim$ on U by $u \sim v \Leftrightarrow n(u-v)=0$. Then $\sim$ is an equivalent relation on U.*

Proof .

1. $\sim$ is reflexive since $n(u - u) = 0$.
2. $\sim$ is symmetric since if u $\sim$ v then v $\sim$ u because n(u−v)=n(v−u)
3. $\sim$ is transitive, assume that u $\sim$ v and v $\sim$z so n(u−v)=0 and n(v−z)=0

Nowt n(u−z)=n(u−v+ v−z) $\leq$ n(u−v) $\circledcirc$ n(v−z)=0$\circledcirc$0=0 It follows that n(u−z)=0 and hence u $\sim$ z. Hence $\sim$ is an equivalent relation on U. □

Notation . Let $\widehat{U}$ be the set of classes $\widehat{u}$=[u]= [z$\in$U: z $\sim$ u}

Lemma 4.5. *Suppose that (U, n, $\circledcirc$) is pseudo algebra fuzzy normed space. Define $\widehat{n}$ on $\widehat{U}$ by $\widehat{n}[\widehat{u}]$=n[u]. Then $\widehat{n}$ does not depends on the representative u.*

Proof . Let u$\sim$ z then n[u$-$z]=0 so [u]=[z] this implies that $\widehat{n}[\widehat{u}]=\widehat{n}[\widehat{z}]$ □

Theorem 4.6. *If (U, n, $\circledcirc$) is pseudo algebra fuzzy normed space then ($\widehat{U}$, $\widehat{n}$, $\circledcirc$) is algebra fuzzy normed space where $\widehat{n}[\widehat{u}]$=n[u] for all $\widehat{u} \in \widehat{U}$.*

Proof . It is clear that ($\widehat{U}$, $\widehat{n}$, $\circledcirc$) is a pseudo algebra fuzzy normed space .Now if $\widehat{n}$([u])=0 then $\widehat{n}$([u]$-$[0])=0 that is [u]=[0]. Hence ($\widehat{U}$, $\widehat{n}$, $\circledcirc$) is algebra fuzzy normed space □

Proposition 4.7. *Define an operator L:U$\to\widehat{U}$ from the pseudo algebra fuzzy normed space (U, n, $\circledcirc$) into the algebra fuzzy normed space ($\widehat{U}$, $\widehat{n}$, $\circledcirc$) by L(u)=[u]= $\widehat{u}$ for all u $\in$U. Then L is fuzzy isometry.*

Proof . First L is well defined since it $v = u$ then n[u$-$ v]=0 so $u\in[v]$ and $u \sim v$ implies $v\in[u\ ]$ it follows that [v]=[u] or L(v)=L(u) since n(u)= $\widehat{n}$([u])= $\widehat{n}$(L(u)).
Hence L is fuzzy isometry. □

Theorem 4.8. *Suppose that (U, n, $\circledcirc$) is pseudo algebra fuzzy normed space and let L be an operator from the pseudo algebra fuzzy normed space (U, n, $\circledcirc$) into the algebra fuzzy normed space ($\widehat{U}$, $\widehat{n}$, $\circledcirc$) then the set of all open fuzzy balls in ($\widehat{U}$, $\widehat{n}$, $\circledcirc$) form a base for the fuzzy topology $T_{\widehat{n}}$*

Proof . Let u$\in$U and s$\in$ (0, 1) first we show that L[fb(u, s)]=fb(L(u), s). t Let v$\in$ L[fb(u, s)] then v$\in$L(d) where d$\in$ fb(u, s). Now n_U(u$-$d)= $\widehat{n}$([u$-$d])= $\widehat{n}$[L(u)$-$L(d)]<s. This implies that L(d) $\in$ fb(L(u), s) , so v$\in$ fb(L(u), s). Thus L[fb(u, s)] $\subset$fb(L(u), s). Similarly we can show that fb(L(u), s) $\subset$ L[fb(u, s)]. Therefore L[fb(u, s)]=fb(L(u), s). Hence L^{-1} [fb(L(u), s)]= L^{-1} L[fb(u, s)]= fb(u, s). Thus the collection of all open fuzzy balls in ($\widehat{U}$, $\widehat{n}$, $\circledcirc$) form a base for the fuzzy topology $T_{\widehat{n}}$. □

Theorem 4.9. *If (U, n, $\circledcirc$) is a fuzzy complete then ($\widehat{U}$, $\widehat{n}$, $\circledcirc$) is fuzzy complete.*

Proof . Suppose that ($\widehat{u_k}$) is a fuzzy Cauchy sequence in ($\widehat{U}$, $\widehat{n}$, $\circledcirc$) where $\widehat{u_j}$ =$[v_j]$
for each j then (u_k) is a sequence in U. Now for given $s \in (0,1)$ there is N$\in \mathbb{N}$ such that $\widehat{n}\,[\widehat{u_m}-\widehat{u_j}]$ < s for all m, j$\geq$N but $\widehat{n}\,[\widehat{u_m}-\widehat{u_j}]$= $n_U\,[u_m\ --u_j]$ hence $n_U\,[u_m\ --u_j]$ <s for all m, j$\geq$N. Therefore (u_k) is a fuzzy Cauchy sequence in U but U is fuzzy complete hence there is u$\in$ U such that $lim_{k\to\infty}\, n_U\,[u_k-u]$ =0. Let $\widehat{u}$=[u] then $\widehat{u}\in\widehat{U}$ and

$$\lim_{k\to\infty} \widehat{n}\,[\widehat{u_k}-\widehat{u}] \ = \lim_{k\to\infty} n_U\,[u_k-u] \ = 0.$$

So $\widehat{u_k}$ fuzzy converge to $\hat{u}$. Hence ($\widehat{U}$, $\widehat{n}$, $\circledcirc$) is fuzzy complete. □

Remark 4.10. *For any a, b, c $\in$(0, 1) if a$\circledcirc$b=c then a$\leq$c and b$\leq$c.*

Theorem 4.11. *Suppose that (U, n_1, $\circledcirc$) is a pseudo algebra fuzzy normed space and let L:U $\to$U be a fuzzy continuous linear operator. Then*

1. *there exists a pseudo general fuzzy norm n_2 which is equivalent to n_1.*
2. *L is uniformly fuzzy continuous*

Proof . For all u $\in$ U define n_2(u)= n_1(u) $\circledcirc$ $n_1[L(u)]$. We now prove that n_2 is a pseudo algebra fuzzy norm on U.

1. since n_1(u)∈(0, 1] and n_1[L(u)])∈ $(0,1]$ so n_2(u) ∈ $(0,1]$.
2. $n_2(u) = 0$ if and only if $n_1(u) \circledcirc n_1[L(u)] = 0$if and only if $n_1(u) = 0$ and $n_1[L(u)] = 0$ if and only if $u = 0$.
3. $n_2(cu) = n_1(cu) \circledcirc n_1[L(cu)] \leq a_{\mathbb{R}}(c)\ n_1(u) \circledcirc a_{\mathbb{R}}(c)\ n_1[L(u)] \leq a_{\mathbb{R}}(c)[\ n_1(u) \circledcirc n_1(L(u))] = a_{\mathbb{R}}(c)\ n_2(u)$, for all $c \neq$0∈ $\mathbb{R}$.
4. $n_2(u+v) = n_1(u+v) \circledcirc n_1[L(u+v)] \leq n_1 u) \circledcirc n_1 v) \circledcirc n_1[L(u)] \circledcirc n_1[L(v)] \leq \{n_1(u) \circledcirc n_1[L(u)]\} \circledcirc \{n_1(v) \circledcirc n_1[L(v)]\} = n_2(u) \circledcirc n_2(v)$

Hence $(U, n_1, \circledcirc)$ is a pseudo algebra fuzzy normed space. To prove n_1 is equivalent to n_2 let (u_k) be a sequence in U and $u \in U$. Suppose that $lim_{k\to\infty}\, n_1\,(u_k - u) = 0$ then $lim_{k\to\infty}\, n_1\,(L\,(u_k) - L\,(u)) = 0$ Since L is fuzzy continuous. Now

$$\underset{k\to\infty}{lim}\, n_2\,[u_k - u] = \underset{k\to\infty}{lim}\, n_1\,(u_k - u) \circledcirc \underset{k\to\infty}{lim}\, n_1\,[L(u_k) - L(u)] = 0 \circledcirc 0 = 0.$$

It follows that $lim_{k\to\infty}\, n_2\,[u_k - u] = 0$.
Conversely suppose that $lim_{k\to\infty}\, n_2\,[u_k - u] = 0$. That is

$$\underset{k\to\infty}{lim}\, n_1\,(u_k - u) \circledcirc \underset{k\to\infty}{lim}\, n_1\,[L(u_k) - L(u)] = 0$$

this implies that $lim_{k\to\infty}\, n_1\,(u_k - u) = 0$. Hence n_1 is equivalent to n_2
(2) By remark 4.10 we see that $n_1\,[L\,(u) - L\,(v)] \leq n_2\,[u - v]$. Now given r∈(0, 1) such that $n_2\,[u - v] < r$ then $n_1\,[L\,(u) - L\,(v)] < r$. Therefore L is uniformly fuzzy continuous □

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