



# On $\mu$ -L-closed, $q$ -compact and $q$ -Lindelöf spaces in generalized topological spaces

Eman Almuhur<sup>a,\*</sup>, Manal Al-labadi<sup>b</sup>, Amani Shatarah<sup>c</sup>, Eman A. AbuHijleh<sup>d</sup>, Sarah Khamis<sup>e</sup>

<sup>a</sup>Department of Basic Science and Humanities, Faculty of Arts and Science, Applied Science Private University, Amman, Jordan

<sup>b</sup>Department of Mathematics, Faculty of Arts and Sciences, University of Petra, Amman, Jordan

<sup>c</sup>Department of Mathematics, Faculty of Science and Arts, Taibah University, Al-Madinah, Saudi Arabia

<sup>d</sup>Basic Sciences Department, Al-Zarqa University College, Al-Balqa Applied University, Jordan

<sup>e</sup>College of Commerce and Business, Lusail University, Doha, Qatar

(Communicated by Ali Jabbari)

---

## Abstract

The research paper's main goal is to propose the notions of  $\mu$ -L-closed,  $q$ -compact and  $q$ -Lindelöf spaces in generalized topological spaces. A number of properties concerning those new spaces are investigated and the characteristics of mappings are explored. The traditional definitions and attributes of common generalized topological spaces are applied to the newly formed mathematical concept.

*Keywords:*  $\mu$ -L-closed space, GTS,  $q$ -compact,  $\mu$ -Lindelöf,  $(\mu, \nu)$ -continuous function.

---

## 1. Introduction and Preliminaries

The concept of topological spaces on non-empty sets is a long-standing concept that undoubtedly encompasses the entirety of mathematics as well as many other subjects such as science, engineering, pharmacy, and so on. In the closing years of the twentieth century, Cs'asz'ar introduced the concept of generalized topological spaces [5] which have been studied by numerous mathematicians from all over the world. Lots of mathematicians took a new approach as a result of this, attempting to generalize many topological notions to this new arena.

---

\*Corresponding author

Email addresses: [e\\_almuhur@asu.edu.jo](mailto:e_almuhur@asu.edu.jo) (Eman Almuhur), [manal.allabadi@uop.edu.jo](mailto:manal.allabadi@uop.edu.jo) (Manal Al-labadi), [ashatarah@taibahu.edu.sa](mailto:ashatarah@taibahu.edu.sa) (Amani Shatarah), [emanhijleh@bau.edu.jo](mailto:emanhijleh@bau.edu.jo) (Eman A. AbuHijleh), [skhamis@lu.edu.qa](mailto:skhamis@lu.edu.qa) (Sarah Khamis)

The generalized topology is a subset of the power set  $P(X)$  that is closed under arbitrary unions. A non-empty subset of  $P(X)$  satisfying such condition is called generalized topology on  $X$  (denoted by  $\mu$ ) and  $(X, \mu)$  is called a generalized topological space (or briefly GTS). The subsets of  $P(X)$  are known as  $\mu$ -open sets, that is; an open set  $A$  in the generalized space  $\mu$  is called  $\mu$ -closed if  $X - A$  is  $\mu$ -open. A subset  $A$  of  $\mu$  is the union of  $\mu$ -open sets. Generalized topological space provided in this article actually acts modulo small sets wrapped in an ideal, but in a different way that simplifies things. We may readily benefit from it since ideal compliance with generalized topology is now a basic characteristic of generalized space. Separation axioms, which were first formulated to detect non-homeomorphic topological spaces, have been generalized by topologists. If  $X$  and  $Y$  are two topological spaces and  $X$  fulfills a separation axiom while  $Y$  does not, then they are not homeomorphic. These are basic concepts that can be found throughout the study of topological spaces and its applications.

In recent years, generalized topological spaces have been used to investigate the structure and features of certain of these [14]. In generalized topological spaces, Sarsak [21] investigated the separation axioms  $\mu - D_0$ ,  $\mu - D_1$ ,  $\mu - D_2$ ,  $\mu - T_0$ ,  $\mu - T_1$ ,  $\mu - T_2$ ,  $\mu - R_0$  and  $\mu - R_1$ . The last five axioms appear in more extended forms in Császár  $-R_0$  and Császár  $-R_1$ . A subset  $A$  of  $X$  is said to be  $D_\mu$ -set if there exists a  $\mu$ -open proper subset  $U$  and  $\mu$ -open subset  $V$  such  $A = U - V$ , hence each  $\mu$ -open proper subset of  $X$  is  $D_\mu$ -open. The interior of  $A$  in  $X$  is denoted by  $i_\mu(A)$  is the union each  $\mu$ -open subset contained in  $A$ . The closure of  $A$  in  $X$  is denoted by  $C_\mu(A)$  equals the intersection of each  $\mu$ -closed subset containing  $A$ . Now, each of  $i_\mu$  and  $C_\mu$  are monotonic and idempotent.

Tyagi and Choudhary [22] defined  $M_\mu$  as the union of all  $\mu$ -open subsets of  $X$ . A generalized space  $X$  is said to be strong if  $M_\mu = X$ . The generalized topological space  $(X, \mu)$  is called  $\mu - D_0$  if for every distinct  $x \neq y$  in  $X$ , there exists a  $\mu$ -open set containing  $x$  but not  $y$  or a  $\mu$ -open set containing  $y$  but not  $x$ . Now, a  $D_\mu$ -set is contained in  $M_\mu$  and  $X - M_\mu$  is not trivial or has the property  $\mu - D_0$ . In such a class of a GTS  $X$ , the property of  $\mu - D_0$  does not introduce any non trivial partition. There are no  $D_\mu$ -open sets that contain  $X - M_\mu$  points. If a generalized topological space  $X$  is closed under finite intersections, it is called a quasi-topological space [8]. Every topological space is a quasi-topological space, every quasi-topological space is a GTS, and  $X$  is a topological space if and only if it is a topological space. A subset  $A$  of  $X$  that contains  $X - M_\mu$  is called generalized-closed or g-closed [23] in a GTS  $X$  if  $C_\mu(A) \cap M_\mu \subseteq U$  whenever  $A \cap M_\mu \subseteq U \in \mu$ . A GTS  $X$  is  $\mu - T_0$  if for every  $x \neq y$  in  $M_\mu$ , there exists a  $\mu$ -open subset  $U$  containing  $x$  but not  $y$  or  $U$  contains  $y$  but not  $x$ .  $X$  is  $\mu - T_1$  if for each  $x \neq y$  in  $M_\mu$ , there exists disjoint  $\mu$ -open subsets  $U$  and  $V$  such that  $x \in U$ ,  $y \notin U$  and  $y \in V$ ,  $x \notin V$ .  $X$  is a  $\mu - T_2$  space if for each  $x \neq y$  in  $M_\mu$ , there exists two disjoint  $\mu$ -open subsets  $U$  and  $V$  such that  $x \in U$  and  $y \in V$ .  $X$  is  $\mu$ -regular if for each  $\mu$ -closed subset  $F$  of  $X$  and a point  $x \notin F$ ,  $\exists$  disjoint  $\mu$ -open subsets  $U$  and  $V$  such that  $x \in U$  and  $F \cap M_\mu \subseteq V$ .  $X$  is a  $\mu$ -normal if for each disjoint  $\mu$ -closed subsets  $F_1$  and  $F_2$ , there exist two disjoint  $\mu$ -open subsets  $U_1$  and  $U_2$  such that  $F_1 \cap M_\mu \subseteq U_1$  and  $F_2 \cap M_\mu \subseteq U_2$  [24]. 'A. Cs'az'ar also proposed the concepts of continuous functions and related interior and closure operators. Using a closure operator established on generalized neighborhood systems, he examined characterizations for the generalized continuous function ( $=(\psi, \psi')$ -continuous function). A strong generalized neighborhood system (briefly SGNS) generates a strong generalized neighborhood space (briefly GNS). Also, the SGNS induces and generates a structure (the collection of all sg-open sets on an SGNS) which is a topological generalization. Furthermore, 'A. Cs'az'ar introduced the quasi-topology [9, 2] which is considered a generalized topology [7]. Consider the nonempty set  $X$  with the power set  $exp(X)$  and the continuous function  $\psi : X \rightarrow e^{e^X}$ , then  $\psi$  is called generalized neighborhood system if  $x \in V$  and  $V \in \psi(x)$  [15]. Typically, if  $\psi$  satisfies the following conditions, it is termed a

strong generalized neighborhood system on  $X$  [16]:

- i)  $x \in V$  for all  $V \in \psi(x)$ .
- ii) For all  $U, V \in \psi(x)$ ,  $U \cap V \in \psi(x)$ .

$(X, \psi(x))$  is said to be a strong generalized neighborhood space on  $X$  and  $V$  is a strong generalized neighborhood of  $x \in X$ . Every strong generalized neighborhood system is, definitely, a generalized neighborhood system. If  $\psi$  is a generalized neighborhood system on  $X$  and  $U \subseteq X$ , then the interior of  $U$  on  $\psi$  (denoted by  $i_\psi(U)$ ) is defined by the set of all points  $x \in U$ , where there exists an open subset  $V \in \psi(x)$  such that  $V \subseteq U$ . For closure of  $U$  on  $\psi$  (denoted by  $\gamma_\psi(U)$ ) is the set of all points  $x \in U$ , where there exists an open subset  $V \in \psi(x)$  such that  $U \cap V \neq \phi$ . Firmly,  $i_\psi(A) \subseteq A \subseteq \gamma_\psi(A)$  for every subset  $A$  of  $X$ ,  $i_\psi(A \cap B) = i_\psi(A) \cap i_\psi(B)$ ,  $\gamma_\psi(A) = X - i_\psi(X - A)$  and  $i_\psi(A) = X - \gamma_\psi(X - A)$  for every subsets  $A$  and  $B$  of  $2^X$ . In addition,  $\gamma_\psi(A \cup B) = \gamma_\psi(A) \cup \gamma_\psi(B)$ [16]. Now, if  $(X, \psi(x))$  is a SGNS on  $X$  and a subset  $A$  is contained in  $X$ , then the weak interior of a subset  $A$  on  $X$  is  $I(A) = \{x \in A, A \in \psi(x)\}$  and the closure of a subset  $A$  on  $X$   $C(A) = \{x \in A, X - A \notin \psi(x)\}$ [15]. For the non-empty set  $X$ , the subfamily  $q_X$  of  $e^X$  is said to be a quasi-topology if  $\phi \in q_X$ ,  $q_X$  is closed under finite intersection and  $q_X$  is closed under arbitrary union.  $(X, q_X)$  is a  $q$ -space on  $X$ , a subset  $U$  of  $X$  is  $q$ -open and  $X - U$  is  $q$ -closed [17]. The set of all open subsets of  $q_X$  is denoted by  $QO(X)$  and the set of all closed subsets of  $q_X$  is denoted by  $QC(X)$ . The set  $Q(x) = \{V \in QO(X) : x \in V\}$ . Typically, every topological space is a  $q$ -space on  $X$ , but the inverse is not true. The  $q$ -interior of a subset  $A$  is  $qInt(A) = \bigcup \{U \subseteq A : U \in q_X\}$  and the  $q$ -closure of a subset  $A$  is  $qCl(A) = \bigcup \{A \subseteq F : X - F \in q_X\}$ [17]. An element  $x$  of  $X$  is contained in  $qInt(A)$  if and only if there exists  $U \in Q(x)$  such that  $U$  is contained in  $A$ . Furthermore,  $x$  is contained in  $qCl(A)$  if and only if  $A \cap W \neq \phi \forall W \in Q(x)$ .

If each of its Lindelöf subsets is closed, a topological space is called L-closed by Hdeib and Pareek in 1983. They gathered a lot of information and asked two questions. The first is: Is there a standard L-closed space that isn't a P-space? The second question is: If every countable subset in an L-closed space is closed, when is the inverse true? Henriksen and Woods responded to the questions about Tychonoff spaces two years later.

## 2. $\mu - L$ -closed Spaces

If every open cover of a topological space  $X$  has a countable subcover, it is said to be Lindelöf [1]. A subset  $A$  of a space  $X$  is called semi-open if  $A \subseteq \overline{IntA}$  and  $A$  is said to be semi-closed if  $X \setminus A$  is semi-open [12]. Let  $A$  be a GTS  $X$ , then  $A$  is said to be  $\mu - semi - open$  if  $C_\mu(i_\mu(A))$  and it is called  $\mu - semi - closed$  if  $X - A$  is  $\mu - semi - open$  [19]. A subset  $A$  of a GTS  $X$  is said to be  $\mu$ -Lindelöf if every cover of  $A$  by  $\mu$ -open subsets has a countable subcover and a GTS  $X$  is called  $\mu$ -Lindelöf if each cover of  $X$  by  $\mu$ -open sets has a countable subcover. A quasi-topological space  $X$  is closed under finite intersection. A subset  $A$  of  $X$  is said to be semi-Lindelöf relative to  $X$  if each cover of  $A$  by semi-open subsets of  $X$  has a countable subcover [20].

**Definition 2.1.** A GTS  $X$  is said to be  $\mu - L$ -closed GTS if each of its  $\mu$ -Lindelöf subsets is  $\mu$ -closed.

**Proposition 2.2.** If  $Y$  is a subspace of the generalized space  $X$ , if  $A$  is a  $\mu - semi - closed$  subset of  $Y$  and  $Y$  is closed in  $X$ , then  $A$  is  $\mu$ -closed in  $X$ . Furthermore,  $A = \bigcap_{u \subseteq Y} u$  such that  $u$  is  $\mu$ -open subset of  $Y$ .

**Proposition 2.3.** A GTS  $X$  is said to be hereditarily  $\mu$ -Lindelöf if every subspace of  $X$  is  $\mu$ -Lindelöf.

**Proposition 2.4.** *A hereditarily  $\mu$ -Lindelöf space  $X$  is  $\mu$ - $L$ -closed if it is countable and discrete.*

**Proposition 2.5.** *A subset  $A$  of a  $\mu$ - $L$ -closed GTS  $X$  is  $\mu$ -Lindelöf if and only if for each family  $\tilde{F} = \{F_\alpha : \alpha \in \Lambda\}$  consisting of  $\mu$ -closed subsets of  $X$  with the property that for each countable subfamily  $F$  of  $\tilde{F}$  such that  $(\bigcap F) \cap A \neq \phi$ , then  $(\bigcap \tilde{F}) \cap A \neq \phi$ .*

**Proposition 2.6.** *In a  $\mu$ - $L$ -closed GTS  $X$ ,  $X$  is  $\mu$ -Lindelöf if and only if for each family  $\tilde{F} = \{F_\alpha : \alpha \in \Lambda\}$  consisting of  $\mu$ -closed subsets of  $X$  having the property that for each countable subfamily  $F$  of  $\tilde{F}$  such that  $(\bigcap F) \neq \phi$ , we have  $(\bigcap \tilde{F}) \neq \phi$ .*

If  $A$  is a nonempty subset of a GTS  $X$ , the generalized subspace topology on  $A$  is  $\{V \cap A : V \in \mu\}$  is denoted by  $\mu_A$  and  $(A, \mu_A)$  is the generalized [19]. For the GTS  $X$ . A subset  $A$  of  $M_\mu$  is called  $\mu - G_\delta$ - set if  $A$  is the countable intersection of  $\mu$ -open subsets of  $X$ . A  $\mu - F_\sigma$ - set represents the countable union of  $\mu$ -closed subsets of  $X$  [24]. In a GTS  $X$ , a subset  $U$  of  $M_\mu$  is called  $\mu - d_\delta$ -open subset if  $U = \bigcup_{i \in \Lambda} F_i \cap M_\mu$ , such that  $F_i$  is regular  $\mu - F_\sigma$ -subset of  $X \forall i \in \Lambda$ . Furthermore,  $V = \bigcap_{i \in \Lambda} G_i \cup (X - M_\mu)$  is  $\mu - d_\delta$ -closed for all  $G_i$  regular  $\mu - F_\sigma$ -subset of  $X$  [24]. The intersection of  $\mu - d$ -closed subsets is also closed in  $X$ .

**Proposition 2.7.** *If  $A$  is a nonempty subset of a  $\mu$ - $L$ -closed GTS  $X$ , then:*

- (i)  $\mu_A$  is a GTS.
- (ii)  $\mu_A$  is a  $P$ -space.
- (iii)  $B \subseteq A$  is  $\mu_A$ - $L$ -closed subspace iff  $B = F \cap A$  for some  $\mu$ -closed subset  $F$ .

**Proposition 2.8.** *If  $A$  be a nonempty subset of a  $\mu$ - $L$ -closed GTS  $X$  and  $B \subseteq A$ , then  $B$  is  $\mu$ -Lindelöf iff  $B$  is  $\mu_A$ -Lindelöf.*

**Proof .** Assume that  $A = \{A_\alpha : \alpha \in \Lambda\}$  is a cover of  $A$  by  $\mu_B$ -open subsets of  $X$ . So,  $A_\alpha = \{U_\alpha \cap A : \alpha \in \Lambda\}$  for each  $\mu$ -open subset  $U_\alpha$ . Hence,  $\tilde{U} = \{U_\alpha : \alpha \in \Lambda\}$  is a countable cover of  $A$  by  $\mu$ -open subsets of  $X$ , but  $A$  is  $\mu$ -Lindelöf, there exist a countable subset  $\{\alpha_1, \alpha_2, \dots\}$  of  $\Lambda$  h that  $B \subseteq \bigcup_{i=1}^\infty U_{\alpha_i}$ , thus  $A$  is  $\mu_B$ -Lindelöf.

On the other side, we claim that  $B$  is  $\mu_A$ -Lindelöf. Let  $\tilde{U} = \{U_\alpha : \alpha \in \Lambda\}$  be a cover of  $B$  by  $\mu$ -open subsets of  $X$ , so  $B = \{U_\alpha \cap B : \alpha \in \Lambda\}$  is a  $\mu_A$ -open cover of  $A$ . Since  $B$  is, so  $\mu_A$ -Lindelöf, there exists a countable subset  $\{\alpha_1, \alpha_2, \dots\}$  of  $\Lambda$  such that  $B \subseteq \bigcup_{i=1}^\infty U_{\alpha_i}$ . Hence,  $B$  is  $\mu$ -Lindelöf.  $\square$

**Proposition 2.9.** *If  $X$  is a  $\mu$ - $L$ -closed GTS and  $A$  is a regular  $\mu - G_\delta$ -subset, hence  $A \cup (X - M_\mu)$  is  $\mu - d_\delta$ -closed.*

**Proposition 2.10.** *If a GTS  $(X, \mu')$  is finer than  $(X, \mu)$  and  $(X, \mu)$  is  $\mu$ - $L$ -closed then is  $(X, \mu')$  is  $\mu$ - $L$ -closed [3].*

**Proposition 2.11.** *If  $X$  is a  $\mu$ - $L$ -closed GTS and  $A$  is a regular  $\mu - G_\delta$ -subset, then  $A$  is  $\mu - G_\delta$ -set.*

**Proposition 2.12.** *If  $X$  is a  $\mu$ - $L$ -closed GTS and  $A$  is  $\mu - d_\delta$ -closed subset, then  $A$  is  $\mu - d$ -closed and  $\mu$ -closed.*

**Proposition 2.13.** *If  $X$  is a Noetherian topological space, then each subspace of a  $\mu - L$ -closed GTS is  $\mu - L$ -closed GTS.*

**Proposition 2.14.** *If  $X$  is a  $\mu - L$ -closed GTS, then the finite union of Noetherian subspaces of  $X$  is Noetherian.*

**Proposition 2.15.** *Every Hausdorff Noetherian subspace of a  $\mu - L$ -closed GTS is finite, closed and discrete.*

### 3. Product Properties of $\mu - L -$ Closed Generalized topological Spaces

Let  $X$  and  $Y$  be two generalized topological spaces such that  $f : (X, \mu) \rightarrow (Y, \nu)$  is  $(\mu, \nu) -$ continuous, then  $\forall A, B \subseteq X$  the following are equivalent:

- (i)  $f(c_\mu(A)) \subseteq c_\nu(f(A))$ .
- (ii)  $c_\mu f^{-1}(B) \subseteq f^{-1}(c_\nu(B))$ .

If  $X$  and  $Y$  are two generalized topological spaces such that  $f : (X, \mu) \rightarrow (Y, \nu)$  is  $(\mu, \nu) -$ continuous, bijective and  $f^{-1}$  is  $(\mu, \nu) -$ continuous, then  $f$  is said to be a homeomorphism [24].

**Proposition 3.1.** *If  $X$  and  $Y$  are two generalized topological spaces such that  $X$  is  $\mu -$  Lindelöf and  $Y$  is  $\mu -$ regular. If  $f : (X, \mu) \rightarrow (Y, \nu)$  is a  $(\mu, \nu) -$ continuous onto function, then  $Y$  is  $\mu -$ Lindelöf.*

**Proof .** Assume that  $\tilde{U} = \{u_\alpha : \alpha \in \Lambda\}$  is a  $\mu -$ open cover of  $Y$  where  $u_\alpha$  is a  $\mu -$ open subset of  $Y$ , then  $f^{-1}(u_\alpha)$  is a  $\mu -$ open subset of  $X$  since  $f$  is a  $(\mu, \nu) -$ continuous function.

Now  $Y = \bigcup_{\alpha \in \Lambda} u_\alpha$ , hence  $f^{-1}(Y) = f^{-1}(\bigcup_{\alpha \in \Lambda} u_\alpha) = \bigcup_{\alpha \in \Lambda} f^{-1}(u_\alpha) = X$  because  $f$  is onto.

But  $X$  is  $\mu -$ Lindelöf, thus there exists a countable subset  $\{\alpha_1, \alpha_2, \dots\} \subseteq \Lambda$  such that  $X = \bigcup_{i \in \mathbb{N}} f^{-1}(u_{\alpha_i})$  and  $Y = f(X) = \bigcup_{i \in \mathbb{N}} f(f^{-1}(u_{\alpha_i})) \subseteq u_{\alpha_i}$  since  $f$  is onto and  $Y = \bigcup_{i \in \mathbb{N}} u_{\alpha_i}$  Is  $\mu -$  Lindelöf [4].  $\square$

**Lemma 3.2.** *If  $X$  and  $Y$  are two generalized topological spaces such that each open cover of  $X$  consisting of  $\mu -$ open subsets has a countable refinement consisting of  $\mu -$ open subsets and if  $f : (X, \mu) \rightarrow (Y, \nu)$  is a  $(\mu, \nu) -$ continuous onto function, so  $Y$  is  $\mu -$ Lindelöf.*

**Proof .** Suppose that  $\tilde{U} = \{u_\alpha : \alpha \in \Lambda\}$  is an open cover for  $Y$  where  $u_\alpha$  is a  $\mu -$ open subset of  $Y$ .  $\{f^{-1}(u_\alpha) : \alpha \in \Lambda\}$  is an open cover of  $X$  because  $f$  is  $(\mu, \nu) -$ continuous.

Now,  $\{f^{-1}(u_\alpha) : \alpha \in \Lambda\}$  has a refinement consisting of  $\mu -$ open subsets of  $X$ . Thus, there exists a countable subset  $\{\alpha_1, \alpha_2, \dots\}$  of  $\Lambda$  and  $\{f^{-1}(u_{\alpha_i} : i \in \mathbb{N}\}$  is a subcover of  $X$  consisting of  $\mu -$ open subsets.  $\{u_{\alpha_i} : i \in \mathbb{N}\}$  is an open subcover of  $Y$  consisting of  $\mu -$ open subsets of  $Y$ . Therefore,  $Y$  is  $\mu -$ Lindelöf.  $\square$

**Proposition 3.3.** *If  $X$  and  $Y$  are two generalized topological spaces such that  $Y$  is a  $\mu - L -$ closed space. If  $f : (X, \mu) \rightarrow (Y, \nu)$  is a  $(\mu, \nu) -$ continuous bijective function, then  $X$  is a  $\mu - L -$ closed space.*

**Proof .** Assume that  $f : (X, \mu) \rightarrow (Y, \nu)$  is a  $(\mu, \nu) -$ continuous bijective function.

Suppose that  $Y$  is a  $\mu - L -$ closed space, let  $F$  be a  $\mu -$ Lindelöf subset of  $X$ , so  $f(F)$  is  $\mu -$ Lindelöf since  $f$  is a  $(\mu, \nu) -$ continuous function. But  $Y$  is a  $\mu - L -$ closed space, so  $f(F)$  is a  $\mu -$ closed subset of  $Y$  and  $F = f^{-1}(f(F))$  is a  $\mu -$ closed subset of  $X$  since  $f$  is a one to one function. Thus,  $X$  is a  $\mu - L -$ closed space.  $\square$

**Remark 3.4.** *A GTS  $X$  that is  $\mu - L -$ closed space is homeomorphic to itself.*

**Proposition 3.5.** *If  $X$  and  $Y$  are two generalized topological spaces such that  $X$  is  $\mu -$  Lindelöf,  $Y$  is  $\mu - L -$ closed, if  $f : (X, \mu) \rightarrow (Y, \nu)$  is a  $(\mu, \nu) -$ continuous bijective function, then  $f$  is homeomorphism.*

**Proof .** We claim that  $f$  is a  $\mu -$ closed function. Assume that  $\tilde{U}$  is a  $\mu -$ closed family of  $X$  such that  $C \in \tilde{U}$ , hence  $C$  is a  $\mu -$ closed proper subset of  $X$ . Now, since  $X$  is  $\mu -$ Lindelöf, so  $C$  is a  $\mu -$ Lindelöf subset of  $X$ , thus  $f(C)$  is  $\mu -$ Lindelöf since  $f$  is  $(\mu, \nu) -$ continuous. But  $Y$  is a  $\mu - L -$ closed space, so  $f(C)$  is a  $\mu -$ closed subset of  $Y$ . Therefore  $f$  is a closed and homeomorphic function.  $\square$

**Corollary 3.6.** *If a  $(\mu, \nu) -$ continuous function from a  $\mu - T_2 -$ Lindelöf space to  $\mu - L -$ closed space is closed, then every  $(\mu, \nu) -$ continuous bijective function is homeomorphism.*

**Proposition 3.7.** *Being a  $\mu$ - $L$ -closed space is a topological property.*

**Proof.** *If  $X$  is a  $\mu$ - $L$ -closed space and  $Y$  is any space, if  $f : (X, \mu) \rightarrow (Y, \nu)$  is a homeomorphism. If  $A$  is a  $\mu$ -Lindelöf subset of  $X$ , hence  $f(A)$  is  $\mu$ -Lindelöf because  $f$  is  $(\mu, \nu)$ -continuous. Since  $X$  is a  $\mu$ - $L$ -closed space,  $A$  is  $\mu$ -closed, thus  $f(A)$  is closed since  $f$  is a closed function. Therefore,  $Y$  is a  $\mu$ - $L$ -closed space.  $\square$*

For the non-empty index set  $\Lambda$  and  $\tilde{X} = \{(X_i, \mu_i) : i \in \Lambda\}$ . If  $X = \prod_{i \in \Lambda} X_i$ , then the GTS generated by the basis  $\{u_k : u_k \in \mu_k, u_k = M_{\mu_k} \text{ except for finite number of indecies}\}$  is said to be Cs'asz'ar generalized product GT on  $X$  and  $(X, \mu)$  is the Cs'asz'ar generalized product GTS [10].

**Proposition 3.8.** *Let  $X$  and  $Y$  be two generalized topological spaces such that  $X$  is  $\mu$ - $L$ -closed. For a function  $f$  such that  $f : (X, \mu) \rightarrow (Y, \nu)$  and  $\{(x, f(x)) : x \in X\}$  is a  $\mu$ -Lindelöf subset of  $X \times Y$ , hence  $f$  is  $(\mu, \nu)$ -continuous.*

**Proof.** *If  $\pi_x$  and  $\pi_y$  are two projection functions. Assume that  $\pi'_x = \pi_{x|_f}$ ,  $\pi'_x$  and  $\pi_y$  are  $(\mu, \nu)$ -continuous onto functions. Suppose that  $\tilde{C} = \{(x, f(x)) : x \in X\}$ . Since  $\tilde{C}$  is  $\mu$ -Lindelöf, each  $\mu$ -closed subset of  $f$  is  $\mu$ -Lindelöf, thus  $\pi'_x$  is a closed projection function, that is if  $A \subseteq f$  is a  $\mu$ -closed subset, hence  $A$  is  $\mu$ -Lindelöf since  $X$  is  $\mu$ - $L$ -closed space. Now,  $f$  is defined on  $X$  and  $\pi'_x$  is a bijection. Since  $\pi'_x$  is closed projection, then for each  $\mu$ -open set  $V \subseteq f$ , we have  $\pi'_x(V)$  is  $\mu$ -open subset of  $X$ . Thus  $f = \pi_y \circ (\pi'_x)^{-1}$  is  $(\mu, \nu)$ -continuous.  $\square$*

**Theorem 3.9.** *If  $\{(X_\alpha, \mu^\alpha) : \alpha \in \Lambda\}$  is a family of generalized topological spaces and  $(X, \mu)$  is any generalized space such that a  $f : (X, \mu) \rightarrow (\prod_{\alpha \in \Lambda} X_\alpha, \mu^\alpha)$ , so  $f$  is  $(\mu^{\alpha_1}, \mu^{\alpha_2}, \dots)$ -continuous iff its composition with every projection function  $\pi_\alpha : \prod_{\alpha \in \Lambda} X_\alpha \rightarrow X_\alpha$   $(\mu^{\alpha_1}, \mu^{\alpha_2}, \dots)$ -continuous.*

**Proposition 3.10.** *If  $X$  is a Noetherian space and  $Y$  is a  $\mu$ - $L$ -closed subspace of  $X$ , then the continuous image of  $Y$  is  $\mu$ - $L$ -closed.*

#### 4. $q$ - Compact and Space $q$ - Lindelöf Spaces

**Definition 4.1.** *A topological space  $X$  is called quasi-compact (resp. Lindelöf) if each  $q$ -open cover of  $X$  has a finite (countable) subcover.*

**Definition 4.2.** *If  $X$  and  $Y$  are two topological spaces, then the continuous function  $f : X \rightarrow Y$  is called  $q$ -compact if for every  $q$ -open subset  $W$  of  $Y$ ,  $f^{-1}(W) = U$  is  $q$ -compact subset of  $X$ .*

**Definition 4.3.** *If  $X$  is a topological space and  $Y$  is a subspace of  $X$ , then  $Y$  is  $q$ -retrocompact if the canonical injection  $f : Y \rightarrow X$  is  $q$ -compact.*

**Lemma 4.4.** *A composition of  $q$ -compact functions is  $q$ -compact.*

**Lemma 4.5.** *If  $X$  is a  $q$ -compact topological space,  $F$  is a  $q$ -closed subset of  $X$ , then  $F$  is  $q$ -compact.*

**Proof.** *Suppose that  $X$  is a  $q$ -compact space and  $F$  is a  $q$ -closed subset of  $X$ . Let  $\{U_\alpha : \alpha \in \Lambda\}$  be an open cover of  $F$  where  $U_\alpha$  is a  $q$ -open subset of  $X \forall \alpha \in \Lambda$ , there exist a  $q$ -open subset  $V_\alpha$  such that  $U_\alpha = F \cap V_\alpha \forall \alpha \in \Lambda$ . Now,  $\{(X - F) \cup V_\alpha : \alpha \in \Lambda\}$  is an open cover of  $X$ , but  $X$  is  $q$ -compact, there exists a finite subset  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$  of  $\Lambda$  such that  $X = \bigcup_{i \in \mathbb{N}} (X - F) \cup V_{\alpha_i} \forall \alpha \in \Lambda$ . Hence,  $F = \bigcup_{i \in \mathbb{N}} U_{\alpha_i} \forall \alpha \in \Lambda$ . Thus,  $F$  is  $q$ -compact.  $\square$*

**Proposition 4.6.** *If  $X$  is a Hausdorff space and a subset  $F$  of  $X$  is  $q$ -compact, then  $F$  is  $q$ -closed.*

**Proof .** *Suppose that  $x \in X$  and  $F$  is  $q$ -compact in  $X$  such that  $x \notin F$ . If  $U_\alpha$  and  $V_\alpha$  are two disjoint  $q$ -open subsets of  $X$  such that  $x \in V_\alpha$ ,  $x \notin F \forall \alpha \in \Lambda$ . Now,  $F$  is  $q$ -compact, so  $F \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha$ , there exists a finite subset  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$  of  $\Lambda$  such that  $F \subseteq \bigcup_{i \in \mathbb{N}} U_{\alpha_i} \forall \alpha \in \Lambda$ , so  $V = \bigcap_{i \in \mathbb{N}} V_{\alpha_i}$  contains  $x$  and  $V \cap U_{\alpha_i} = \emptyset \forall i \in \mathbb{N}$ . Thus  $F$  is a  $q$ -closed subset of  $X$ .  $\square$*

**Proposition 4.7.**  *$q$ -compactness is invariant under continuous functions.*

**Proof .** *Let  $X$  and  $Y$  be two topological spaces such that  $X$  is  $q$ -compact space, let  $U_\alpha$  be a  $q$ -open subset of  $Y \forall \alpha \in \Lambda$  such that  $Y = \bigcup_{\alpha \in \Lambda} U_\alpha$ . If  $f : X \rightarrow Y$  is a continuous function, then  $X = \bigcup_{\alpha \in \Lambda} f^{-1}(U_\alpha)$ , but  $X$  is  $q$ -compact, thus there exists a finite subset  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$  of  $\Lambda$  such that  $X = \bigcup_{i \in \mathbb{N}} f^{-1}(U_{\alpha_i})$ . By continuity of  $f$ ,  $f(X) = \bigcup_{i \in \mathbb{N}} U_{\alpha_i}$ . Thus  $Y$  is  $q$ -compact.  $\square$*

**Proposition 4.8.**  *$q$ -retrocompactness is invariant under continuous functions.*

## References

- [1] P. Alexandroff and P. Urysohn, *Mémoire sur les espaces topologiques compacts*, Verh Nederl Akad Wetensch Afd Natuurk, Amsterdam, Koninklijke Akademie van wetenschappen, 1929.
- [2] E. Almuhr and H. Hdeib, *On pairwise L-closed spaces*, J. Adv. Math. Comput. Sci. 30(4) (2019) 1–8.
- [3] E. Almuhr and M. Al-labadi, *Pairwise strongly Lindelöf, pairwise nearly, almost and weakly Lindelöf bitopological apaces*, WSEAS Trans. Math. 20(16) (2021) 152–158.
- [4] E. Almuhr, M. Al-Labadi, A. Shatarah, S. Khamis and N. Omar, *Almost weakly and nearly L-closed topological spaces*, Int. Conf. Inf. Technol. Proc. 2021, pp. 80–84.
- [5] A. Császár, *Generalized open sets in generalized topologies*, Acta Math. Hungar. 75 (1997) 65–87.
- [6] A. Császár, *Separation axioms for generalized topologies*, Acta Math. Hungar. 104(12) (2004) 63–69.
- [7] A. Császár, *Generalized open sets in generalized topologies*, Acta Math. Hungar. 106(12) (2005) 53–66.
- [8] A. Császár, *Further remarks on the formula for  $\gamma$ -interior*, Acta Math. Hungar. 113(1) (2006) 325–332.
- [9] A. Császár, *Remark on quasi-topologies*, Acta Math. Hungar. 119(2) (2008) 197–200.
- [10] A. Császár, *Product generalized topologies*, Acta Math. Hungar. 123(1) (2009) 127–132.
- [11] S. Ganguly and C.K. Basu, *Further characterizations of s-closed spaces*, Indian J. Pure Appl. Math. 3(1) (1992) 635–641.
- [12] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Month. 70(1) (1963) 36–41.
- [13] D. Maio and T. Noiri, *On s-closed spaces*, Indian J. Pure Appl. Math. 18(3) (1987) 226–233.
- [14] J. Makai, E. Peyghan and B. Samadi, *Weak and strong structures and the  $T_{3.5}$  properties for generalized topological spaces*, Acta Math. Hungar. 150(1) (2016) 1–35.
- [15] W.K. Min, *Results on strong generalized neighborhood spaces*, J. Korean Soc. Math. Educ. Ser. B, Pure Appl. Math. 15(3) (2008) 221–227.
- [16] W.K. Min, *On strong generalized neighborhood systems and sg-open sets*, Commun. Korean Math. Soc. 23(1) (2008) 125–131.
- [17] W.K. Min, *A note on quasi-topological spaces*, Honam Math. J. 33(1) (2011) 11–17.
- [18] T. Noiri, *Unified characterizations for modifications of  $R_0$  and  $R_1$  topological spaces*, Rend. Circ. Mat. Palermo 55(2) (2006) 29–42.
- [19] M. Sarsak, *On  $\mu$ -compact sets in  $\mu$ -spaces*, Questions Answers Gen. Topology 31(1) (2013) 49–57.
- [20] M. Sarsak, *On semicompact sets and associated properties*, Int. J. Math. Math. Sci. 2009(36) (2009) 1–9.
- [21] M. Sarsak, *Weak separation axioms in generalized closed*, Acta Math. Hungar. 131(1) (2011) 110–121.
- [22] B.K. Tyagi and R. Choudhary, *On generalized closure operator in generalized topological spaces*, Int. J. Comput. Appl. 82(15) (2016) 1–5.
- [23] B.K. Tyagi and H. Chauhan, *On generalized closed sets in generalized topological spaces*, Cubo 18(10) (2016) 27–45.
- [24] B.K. Tyagi and B.K. Chauhan, *On Some Separation Axioms in Generalized Topological Spaces*, Questions Answers Gen. Topology 36(1) (2018) 9–29.