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The radius, diameter and chromatic number of some zero divisor graph

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Abstract

In this work, the radius, diameter and a chromatic number of zero divisor graph of the ring Z_n for some n are been determined. These graphs are $\Gamma\left(Z_{p^2q^2}\right)$, $\Gamma\left(Z_{p^2}\right)$, $\Gamma\left(Z_{pq}\right)$, $\Gamma\left(Z_{p^3}\right)$, $\Gamma\left(Z_{p^2q}\right)$ and $\Gamma\left(Z_{pqr}\right)$. Furthermore, the largest induced subgraph isomorphic to complete subgraph in the graph $\Gamma\left(Z_{p^3}\right)$ and $\mathit{\Gamma}(p^2q)$ are calculated.

Keywords: Zero divisor graph, Radius, Diameter, Chromatic number.
2020 MSC: 05C75

1. Introduction

Most of the sciences have become dependent on communicating their ideas and solutions to their problems on the graph theory $G\left(V,E\right)$, which depends on its design on two sets, namely the vertex set (V) and the edge set (E). Edges are formed by setting certain conditions for the association of vertices between them. The graph in this work is finite, indirect and simple. Graph theory in mathematics deals with most of its fields such as topological graph [2] and [12], soft graph [1], fuzzy graph [15, 13] [20, 21, 22] and , algebraic graph[9, 10], general graph [11], [17, 14] and[16, 18, 19] and others. The relation between algebraic graph theory and group theory is very strong. The weighted graph is a rich branch of graph theory where it taks some numerical values to the vertices or edges. The notion of a zero divisor graph $(ZDG)$of a commutative ring was presented by I. Beck in [2]. Let R be a ring and let $G(R)$ is the graph has vertex set of it is R and two vertices say $v_1$and v_2 are adjacent if v_1v_2= 0. This graph is denoted by $\Gamma\left(R\right)$ and it is called the zero divisor graph. For more details about graphs and domination in graphs, see [3, 4, 5, 6, 7, 8].

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In this work, we will focus our attention on studying the properties of the algebraic graphs. As it is an extension of the work of researchers A. A. Omran and H. Faisil who have studied new properties on the graph $\Gamma(Z_n)$ which is called the zero divisor graph in the ring Z_n [10]. They discussed many properties of this graph such as the isomorphic graph to the graph $\Gamma(Z_n)$ and determined the domination number of it, especially where n is equal to pq, pqr, p^2, p^2q, p^3 and p^2q^2. Now, in this paper, the radius, diameter and chromatic number are discussed to the graph $\Gamma(Z_n)$ where n is equal to pq, pqr, p^2, p^2q, p^3 and p^2q^2. The two concepts radius and diameter depending on the concept the ***eccentricity*** $e(v)$ of a vertex v is the number $max_{u\in V(G)}d(u,v)$ of a connected graph G, where $d(u,v)$ is the distance between two vertices u and v. The ***radius rad*** G of G is the minimum value of eccentricity among all the vertices of the graph G, while the diameter Diam G of G is the maximum eccentricity. If the graph G has no loops, then G is $k-$colourable if each vertex can be taken one colure such that this colure not repeated to all adjacent vertices . Moreover, if G is k-colourable and does not $(k-1)$-colourable, then G is k-chromatic and denoted by $\chi(G) = k$.

Remark 1.1. *[9] 1)* $\chi(K_n) = n$.
2) $\chi(G) = 1$ *if and only if the graph G is isomorphic to the null graph.*
3) $\chi(G) = 2$ *if and only if the graph G is isomorphic to the non-null bipartite graph.*

Proposition 1.2. *[10]* $|V(\Gamma(Z_{p^2q}))| = p^2 + pq - p - 1$ *where p andq, are primes numbers and* $p < q$.

2. Main Results

Some properties of the ring $Z_n; n = pq,\ pqr,\ p^2,\ p^3,\ p^2q^2$.

Proposition 2.1. *Consider* Z_{pq}*, where p and q are prim and* $p < q$*, then*
1) $\text{Rad}\,\Gamma(\Gamma(Z_{pq})) = \left\{ \begin{array}{ll} 1 & \text{if } p = 2 \\ 2 & \text{otherwise} \end{array} \right\}$.
2) $\text{Diam}(\Gamma(Z_{pq})) = 2$.
3) $\chi(\Gamma(Z_{pq})) = 2$.

Proof . 1) From Proposition 1.1, $\Gamma(Z_{pq}) \equiv K_{p-1,q-1}$, then three cases holds:
Case 1. If $p = 2$, then $\Gamma(Z_{pq}) \equiv K_{1,q-1}$ and this graph is isomorphic to the star graph. Thus, $\text{Rad}\,\Gamma(\Gamma(Z_{pq})) = 1$.
Case 2. If $p > 2$, then $\Gamma(Z_{pq}) \equiv K_{p-1,q-1}$, so, $\forall\ v \in \Gamma(Z_{pq})$ the $e(v) = 2$. Thus, $\text{Rad}\,\Gamma(\Gamma(Z_{pq})) = 1$.
2) $\forall\ v \in \Gamma(Z_{pq})$ the $e(v) \leq 2$, then $\text{Diam}(\Gamma(Z_{pq})) = 2$.
3) By Remark 1.1 the required is getting. □

Proposition 2.2. *Consider* Z_{p^2}*, where p is a prime number, then*
1) $\text{Rad}\,\Gamma(\Gamma(Z_{p^2})) = \text{Diam}(\Gamma(Z_{p^2})) = 1$.
2) $\chi(\Gamma(Z_{pq})) = p - 1$.

Proof . From proposition 2.1.2, $\Gamma(Z_{p^2}) \equiv K_{p-1}$, so
1) for all $v \in \Gamma(Z_{pq})$, $e(v) = 1$, then $\text{Rad}\,\Gamma(\Gamma(Z_{p^2})) = Diam(\Gamma(Z_{p^2})) = 1$.
2) By Remark 1.1, the result is obtained. □

Proposition 2.3. *The largest subgraph is isomorphic to complete graph of the graph* $\Gamma(p^2q)$ *is* K_p.

Proof . From proof of proposition 1.2, $V(\Gamma(Z_{p^2q})) = \{p, 2p, \ldots, p(pq-1),\ q, 2q, \ldots, q(p^2-1)\} - \{pq, 2pq, \ldots, pq(p-1)\}$. Let u_1 and u_2 be any two vertices in the graph $\Gamma(p^2q)$, so the vertex u_1 is adjacent to the vertex u_2 if satisfied one of the two cases:

I) If $u_1 = q$ and $u_2 = jp^2,\ j = 1, 2, \ldots, q-1$, then the set of all multiple constitute the set $D_1 = \{q,\ p^2,\ 2p^2, \ldots, (q-1)\,p^2\}$ and $|D| = q$,but this set is not complete, since if we take two vertices such that these vertices multiple of p^2 say $v_1 = ip^2$ and $v_2 = jp^2$, then $v_1.\ v_2 = ijp^2p^2$ and ij is not multiple of q, then $v_1.\ v_2$ is not multiple of p^2q. Thus, v_1 is not adjacent to v_2.

II) If $u_1 = p$ and $u_2 = ipq,\ i = 1, 2, \ldots, p-1$, then the set of all multiple constitute the the set $D = \{p, 2pq, \cdots, (p-1)pq\}$, so for all two vertices in the set D say u and v, $uv = ip^2q^2$, so u and v are adjacent. Thus, the induced subgraph $\langle D\rangle$ is complete. One can be concluded that $\langle D\rangle$ has the maximum cardinality of all complete as subgraph from graph $\Gamma(p^2q)$ (for example, see Figure 1, where $p = 3$ *and* $q = 5$). □

Depending of the above results, the required is getting.

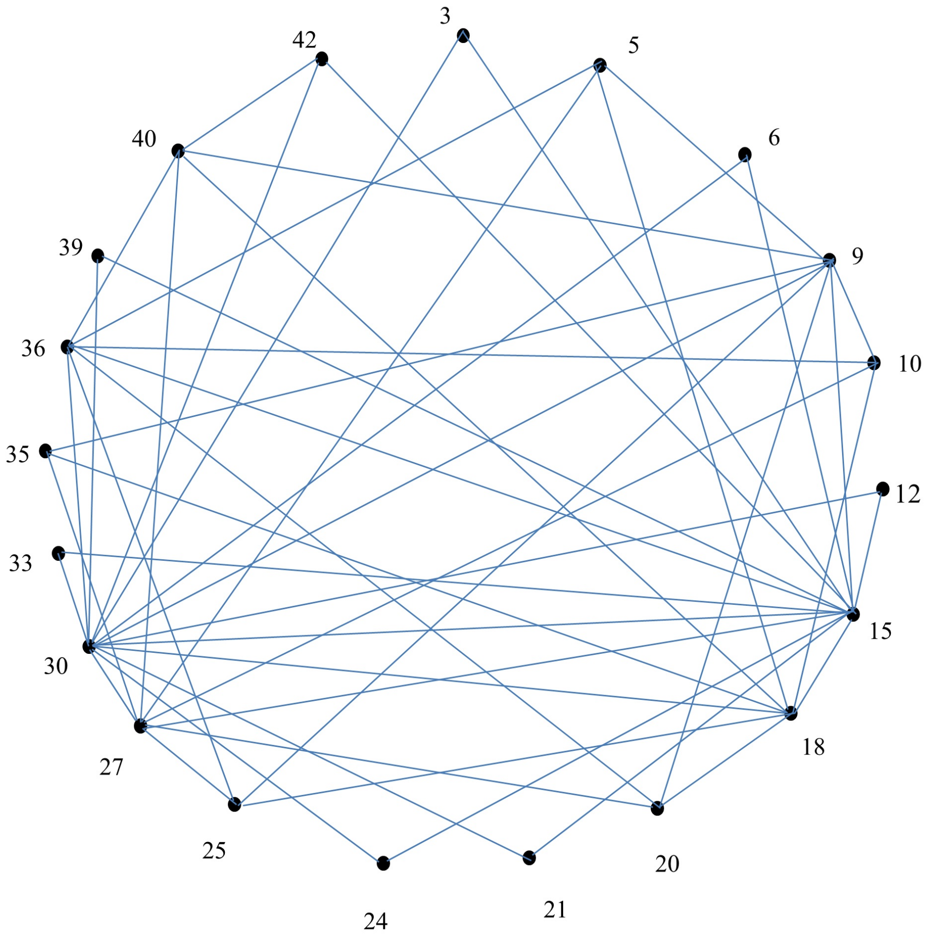

Figure 1: The graph $\Gamma(Z_{45})$.

Proposition 2.4. *Consider Z_{pqr} where p, q and r are prime numbers and $p < q < r$, then*
1) $\mathrm{Diam}(\Gamma(Z_{pqr})) = 3$.
2) $\mathrm{Rad}\,\Gamma(\Gamma(Z_{pqr})) = 2$.
3) $\chi(\Gamma(Z_{pqr})) = 4$.

Proof . The set of vertices $S = \{v_{pq},\ v_{pr},\ v_{qr}\}$ constitute a complete induced subgraph of order three since these vertices are adjacent pairwise.

1) and 2). three cases hold:
Case 1. Let $v \in S$ and $v = v_{pq}$ then $\forall u \neq v$, one of the following holds, the two vertices u *and* v are adjacent or not. If they are adjacent, then $d\,(u, v) = 1$, if not means that they are not adjacent, then the vertex u is adjacent to a vertex on the set S. Thus, $d\,(u, v) = 2$ and so $e\,(v) = 2$.
Case 2. If $v \notin S$, then $\forall u \neq v$, if u is adjacent to v, then $d\,(u, v) = 1$. If they are adjacent to the same vertex in the set S, then $d\,(u, v) = 2$. Finally, if they are adjacent to the difference two vertices in the set S, then $d\,(u, v) = 3$. Thus, $e\,(v) = 3$.
Therefore, according to the two cases above, Diam $(\Gamma\,(Z_{pqr})) = 3$.
and Rad $\Gamma\,(\Gamma\,(Z_{pqr})) = 2$.
3) The subgraph generated by the vertices $S = \{v_{pq},\, v_{pr},\, v_{qr}\}$ is a complete and it is largest complete in the graph $\Gamma\,(Z_{pqr})$. Thus, $\chi\,(\Gamma\,(Z_{pqr})) = |S| + 1 = 3 + 1 = 4$. (for example, see Figure 2, where $p = 2,\; q = 3,\; and\; r = 5$). □

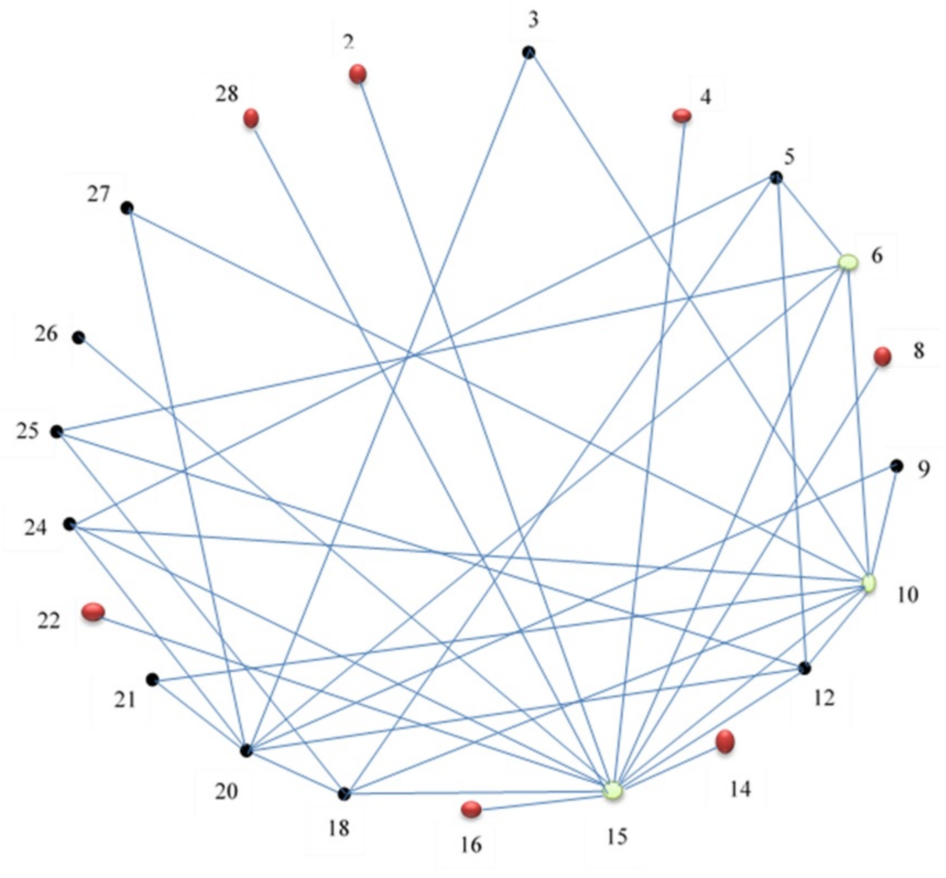

Figure 2: The graph $\Gamma\,(Z_{30})$.

Proposition 2.5. *The largest induced subgraph isomorphic to complete subgraph in the graph* $\Gamma\,(Z_{p^3})$ *is isomorphic to* K_p.

Proof . Assume that $S = \{p,\; p^2,\; 2p^2,\; 3p^2,\; \ldots,\; (p-1)\,p^2\}$. Take arbitrary different two vertices u and v in the set S, then uv is the multiple of p^3, therefore u is adjacent to v. Thus, the induced subgraph generated by the set S is isomorphic to K_p. One can be concluded that the set S is the largest induced subgraph isomorphic to complete subgraph in the graph $\Gamma\,(Z_{p^3})$ is isomorphic to K_p. Thus, the required is getting. □

Proposition 2.6. *Consider Z_{p^3} where p is a prime number, then*
1) $\text{Diam}\left(\Gamma\left(Z_{p^3}\right)\right) = 3$.
2) $\text{Rad}\,\Gamma\left(\Gamma\left(Z_{p^3}\right)\right) = 2$.
3) $\chi\left(\Gamma\left(Z_{p^3}\right)\right) = p + 1$.

Proof . By proof of Proposition 3.2.11, the set $S = \{p,\ p^2,\ 2p^2,\ 3p^2,\ \ldots,\ (p-1)\,p^2\}$ is the largest complete graph as a subgraph of the graph $\Gamma\left(Z_{p^3}\right)$. Thus, in the same manner in the proof of Proposition 3.2.10. the results are obtained □

Proposition 2.7. *Let $Z_{p^2q^2}$ be a ring where p and q are prime numbers, then*
1) $\text{Diam}\left(\Gamma\left(Z_{pqr}\right)\right) = 3$.
2) $\text{Rad}\,\Gamma\left(\Gamma\left(Z_{pqr}\right)\right) = 2$.
3) $\chi\left(\Gamma\left(Z_{pqr}\right)\right) = pq$.

Proof . By proof of the Theorem 2.2.3, the set $S = \{pq,\ 2pq, \ldots,\ (pq-1)\,pq\}$ is the largest a complete graph as a subgraph of the graph $\Gamma\left(Z_{p^2q^2}\right)$. Thus, in the same manner as in the proof of Proposition 3.2.10. the results are obtained. (for example, see Figure 3, where $p = 2,\ q = 3,\ and\ r = 5$). □

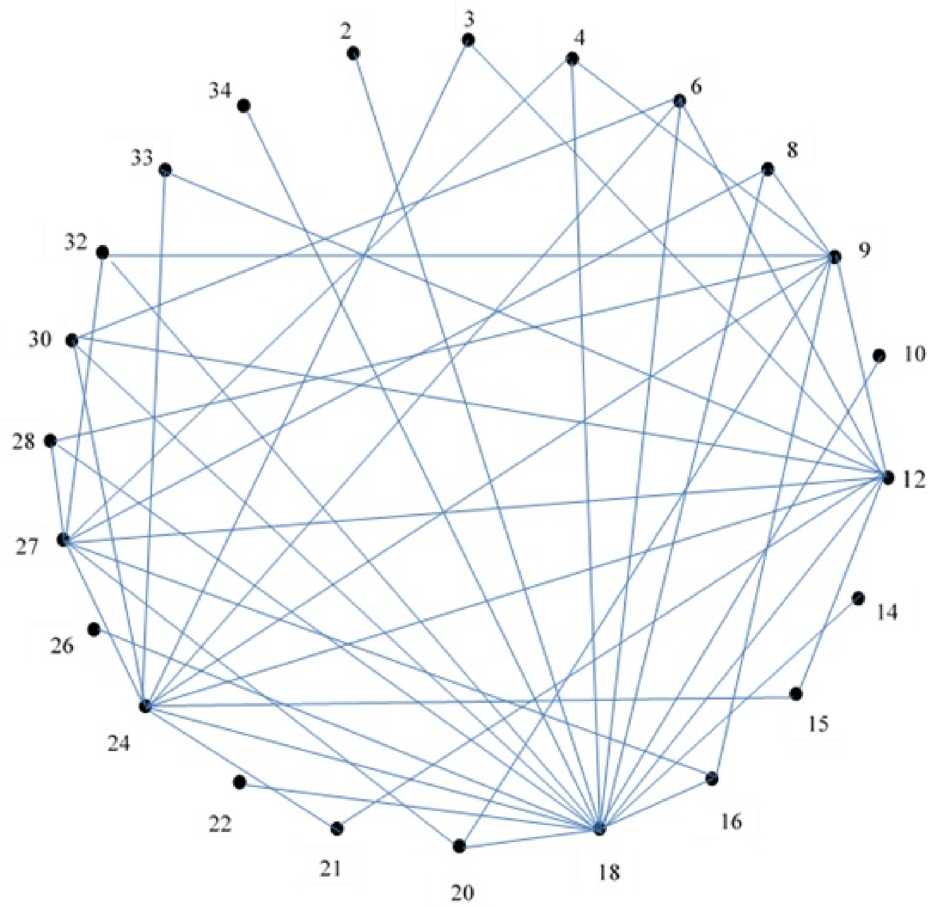

Figure 3: The graph $\Gamma\left(Z_{36}\right)$.

3. Conclusion

Through the foregoing was calculated the radius, diameter and chromatic number of ZDG of Z_n for some n are been determined. These graphs are $\Gamma\left(Z_{p^2q^2}\right)$, $\Gamma\left(Z_{p^2}\right)$, $\Gamma\left(Z_{pq}\right)$, $\Gamma\left(Z_{p^3}\right)$, $\Gamma\left(Z_{p^2q}\right)$ and

$\Gamma(Z_{pqr})$. Moreover, the clique number of two graphs $\Gamma(Z_{p^3})$ and $\Gamma(p^2q)$ are calculated.

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