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Properties of measures of association within an extended FGM

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Abstract

In this paper, we propose an extension to the bivariate FGM copula within a polynomial function of degree one. The desired extension depends on the modification that was shown by Sriboonchitta-Kreinovich [13]. We also illustrate a general form of such extension with degree n. We examine various necessary and sufficient conditions which prove that the illustrated function within the extension is the copula. Eventually, we present several calculations of the most popular dependencies within the proposed FGM copula of degree one.

Keywords: Copulas, Distribution functions, FGM copula family, Nonparametric measures

1. Introduction

The proposed extension depends on the Sriboonchitta-Kreinovich model of the FGM copula which was published in 2017 to answer some questions associated with the FGM copula as the most successful copula among other copulas. In fact, this type of extension is an equivalent approach to several extensions that have been presented in [4, 8, 9, 10].

We firstly substitute a polynomial function of degree one $\theta(0,y) = \theta(y) = a_1 y + a_o, a_o, a_1 \in R$,into the standard Sriboonchitta-Kreinovich FGM copula $C(x,y) = xy + \theta(x,y)x(1-x)y(1-y)$ to obtain the function (denoted by $C_{(a_o,a_1)}$), $C_{(a_o,a_1)}(x,y) = xy + \theta(y)x(1-x)y(1-y) = xy + (a_1 y + a_o)x(1-x)(1-y)$.

The main objective is to examine the function $C_{(a_o,a_1)}(x,y)$ for being copula. This can be performed by proving the essential three conditions of copula. Also, we have to prove some necessary

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and sufficient conditions associated with the 2-increasing condition so that we can verify that such function is copula.

In fact, the claim in the previous paragraph is not enough to decide that the proposed function is a copula. There are more transparent solutions that we need to prove.

The main solution is related to the necessary and sufficient conditions, respectively, which are associated with the coefficients of the proposed polynomial of degree one. This is related to convexity, which means that we have to show all the solutions of convex set of (a_o, a_1) that belong to R.

Finally, we refer to the organization of this paper. In the next part, we recall some preliminaries, and basic concepts of copulas, their properties, and some essential concepts of the modified Sriboonchitta-Kreinovich FGM copula. In section three, we show our main proposal, and some conditions related to the proof of the proposed extension. As application, we present several calculations of the measures of association like Spearman rank correlation, Kendall's tau, Gini's gamma, and Blomqvist β via the function $C_{(a_o,a_1)}(x, y)$. Last part is devoted to present some conclusions, and future works.

2. Preliminaries and basic concepts

In this part, we review some basic concepts related to copulas, the FGM copula family formula, and the Sriboonchitta-Kreinovich model of FGM copula. We also show the basic formulas of classical dependences that are associated with the proposed extension. Of course, it is important to mention that copula concepts and its name were firstly invented by Sklar [12] through his central theorem.

Definition 2.1. *[9, 12] A two-dimensional function $C : [0, 1]^2 \to [0, 1]$ is called bivariate copula, if it holds the following conditions*

C1. C is grounded for all $x \in [0, 1]$;

C2. For all $x \in [0, 1], C(x, 1) = x = C(1, x)$;

C3. C is 2-increasing for all $x_1, x_2, y_1, y_2 \in [0, 1]$ such that $x_1 \leq x_2$, and $y_1 \leq y_2$.

Through Sklar's theorem we can easily figure out that copula is the bridge between the joint distribution function to its margins within copula in such a way that makes the description of the dependence structure of bivariate or multivariate random variables deepest and more effective, see [12].

$$Pr(X \leq x, Y \leq y) = C(Pr(X \leq x), Pr(Y \leq y)) \tag{2.1}$$

where X, Y are random variables of bivariate probability distribution function with their margins, respectively. In [4, 9], it is mentioned that when $Pr(X \leq x), Pr(Y \leq y)$ are continuous, then this yields a unique bivariate copula C.

Moreover, when $Pr(X \leq x, Y \leq y) = H(x, y), Pr(X \leq x) = H_1(x)$ and $Pr(Y \leq y) = H_2(y)$. We can see that with respect to inverse transformation technique (quasi inverses), Sklar's has shown an equivalent formula to the one in (2.1). Let H_1^{-1}, H_2^{-1} be the inverses of the margins H_1, H_2, respectively. Then

$$H(H_1^{-1}(x), H_2^{-1}(y)) = C(x, y)$$

Further, we review the general form of the FGM copula family. This copula and its family were named by the scientists Farlie [4], Gumbel [**?**], and Morgenstern [2], who have firstly derived and discussed it in detail. The general form is derived by assuming that C is a symmetric function with quadratic section whether in x or in y. We can review that copula by the following way

$$For all x, y \in [0, 1], C(x, y) = xy + kx(1 - x)y(1 - y) \tag{2.2}$$

where $k \in [-1, 1]$. Indeed, the interval $[-1, 1]$ is the only interval that can verify that C is copula. This is an equivalent version of copula with polynomial of degree zero.

In 1996, Nelsen has shown a very rich interpretations related to the family of copula in (2.2), and its aspects with symmetry and asymmetry properties. Moreover, in the 2017, Sriboonchitta-Kreinovich have explained a model of FGM copula that has the following form

$$Forall x, y \in [0, 1], C(x, y) = xy + \theta(x, y)x(1 - x)y(1 - y) \tag{2.3}$$

where $\theta(x, y)$ is a polynomial of degree n. They have shown a proof of the copula in (2.3) within two different approaches. The first one depends on computational complexity technique, while the second proof was shown by some fuzzy logic techniques, see [13, 15].

Eventually, we recall the standard formulas of measures of association, see [9]. Each dependence structure of any population or sample can be described by nonparametric measures like Spearman's rho, Kendall's tau, Gini's gamma, and Blomqvist beta when the data of that population is nonelliptical (does not follow normal distribution). In association with copulas the correlation coefficients have by the following forms, see [8].

$$\rho_c = 12 \int_1^0 \int_0^1 [C(x, y) - xy] dy dx \tag{2.4}$$

$$\tau_c = 4 \int_1^0 \int_0^1 C(x, y) \frac{\partial^2 C(x, y)}{\partial x \partial y} dy dx - 1 \tag{2.5}$$

$$\gamma_c = \left[\int_0^1 C(x, 1 - x) dx - \int_1^0 [x - C(x, x)] dx \right] \tag{2.6}$$

$$\beta_c(x, y) = 4C(\frac{1}{2}, \frac{1}{2}) - 1 \tag{2.7}$$

All the measures of association above depend on concordance and discordance concepts. In the next part, we show some calculations of them via the extended FGM copula.

3. Extension of FGM copula via polynomial function

3.1. Methodology and derivations

polynomial copula with quadratic horizontal section has the form, see [4, 8, 10].

$$C(x, y) = a(y)x^2 + b(y)x + c(y) \tag{3.1}$$

where $c(y) = 0, a(0) = a(1) = 0, and b(y) = y - a(y)$.

The function in (3.1) with the conditions above is copula, see [8].

On the other hand, the FGM copula that has been presented by Sriboonchitta-Kreinovich has the following formula, see [13].

$$C(x, y) = xy + \theta(x, y)(x - x^2)(y - y^2) \tag{3.2}$$

Note that, $\theta(x, y)$ is a polynomial function that can be defined with respect to horizontal section y as $\theta(y)$. Also, it is important to mention that equations (3.1), and (3.2) are equivalent. Mathematically speaking

$$C(x, y) = xy + \theta(\theta, y)x(1 - x)y(1 - y) = \theta(y)y(1 - y)x^2 + (y + \theta(y)y(1 - y))x \tag{3.3}$$

where,$a(y) = \theta(y)y(1-y), b(y) = y - a(y) and c(y) = 0$.

A function $\theta(y)$ can be extended to a polynomial of degree n, that is $\theta(y) = a_n y^n + \cdots + a_1 y + a_o$. Substituting the right-hand side of $\theta(y)$ in equation (3.2) leads to the following function.

$$C_{(a_o,a_1,\ldots,a_n)}(x,y) = xy + (a_n y^n + \cdots + a_1 y + a_o)(x - x^2)(y - y^2) \tag{3.4}$$

Indeed, the function in (3.4) can be rewritten as shown in equation (3.5).

$$C_{(a_o,a_1,\ldots,a_n)}(x,y) = xy + x(1-x)(a_o y + (a_1 - a_o)y^2 + \cdots + (a_n - a_{n-1})y^{n+1} - a_n y^{n+2}) \tag{3.5}$$

In particular, when the degree of $\theta(y) = 0$, we have the standard FGM, see [9].

In particular, we only investigate the polynomial of degree one ($\theta(y) = a_1 y + a_o$), where $a_o, a_1 \in R$. It is obvious that the coefficients of the polynomial $\theta(y)$ are linearly increasing. From equation (3.5), the FGM copula with respect to the polynomial of degree one can be written in the following form

$$C_{(a_o,a_1)}(x,y) = xy + (x - x^2)(a_o y + (a_1 - a_o)y^2 - a_1 y^3) \tag{3.6}$$

Notice that, the polynomial $\theta(x,y)$ with respect to vertical section, which is equivalent to $\theta(x) = a_1 x + a_o$ has the following form

$$C_{(a_o,a_1)}(x,y) = xy + (a_o x + (a_1 - a_o)x^2 - a_1 x^3)(y - y^2) \tag{3.7}$$

It is clear that the function $C_{(a_o,a_1)}$ holds the boundary conditions of copula. Mathematically speaking, $C_{(a_o,a_1)}(x,0) = C_{(a_o,a_1)}(0,y) = 0$, and $C_{(a_o,a_1)}(x,1) = x(C_{(a_o,a_1)}(1,y) = y)$. So, the only remaining condition that we need to verify is the 2-increasing condition. In [4, 8, 9, 10], it has shown that such condition can be provided by showing that

1. $C_{(a_o,a_1)}(x,y) \geq 0$ for all $x, y \in [0,1]$;
2. $C_{(a_o,a_1)}(x,y) = \int_0^x \int_0^y C_{(a_o,a_1)}(s,t)dsdt$

where $\dfrac{\partial^2 C(a_o,a_1)}{\partial x \partial y}$, and it is the joint copula density. By differentiating the function in (3.6), we obtain the joint density in (3.8), that is

$$C_{(a_o,a_1)}(x,y) = 1 + (1-2x)(a_o + 2(a_1 - a_o)y - 3a_1 y^2) \tag{3.8}$$

To show that the function in (3.8) is non-negative, it is sufficient to test the non-negativity of the extreme points of all $x, y \in [0,1]$. In other words, we have to find the two-dimensional domain of $(a_o, a_1) \in T$, that satisfies a convex subset of a set $T \in R^2$.

First part of the solution can be assigned by the following lemma.

Lemma 3.1. *From equation (3.8), suppose that $f_1(x) = 1-2x$, and $f_2(y) = a_o + 2(a_1 - a_o)y - 3a_1 y^2$, such that f_1, f_2 are independent functions. Then $C_{(a_o,a_1)}$ is non-negative, if and only if, $-1 \leq a_o \leq 1$, and $-1 \leq a_1 + a_o \leq 1$.*

Proof . It is given that f_1, f_2 are independent. Then this means that $Ran(f_1, f_2) = Ran(f_1)Ran(f_2)$. Thus

1. For $a_1 = 0$, and $a_o \in [-1,1]$, this is nothing more than the standard FGM copula, and the proof over that is verified and trivial.

2. For $-1 \leq a_1 + a_o \leq 1$, where $a_o \in [-1,1]$, we have $\forall x \in [0,1], Ran[f_1] = [-1,1], Ran[f_2] = [-|a_1 + a_o|, |a_1 + a_o|]$.

Thus, the range of the joint density of in (3.8) can be verified by the following way

$$Ran(1 + [-1,1][-|a_1 + a_o|, |a_1 + a_o|]) = [1 - |a_1 + a_o|, 1 + |a_1 + a_o|]$$

Therefore, for all$x, y \in [0,1], C_{(a_o,a_1)}(x,y) \geq 0$, if and only, if $a_o \in [-1,1]$, and $|a_1 + a_o| \leq 1$.

In fact, the proof of 3.1 shows that the only valid solutions lie in the region of the four vertices of the convex set over the subset $T \in R^2$. These vertices are $(a_o, a_1) = (-1,0), (1,0), (-1,2)$, and (1,-2), respectively.

Graphically, the convex set within the obtained vertices of the constants $a_o \in [-1,1], a_1 \in [-2,2]$ has the following shape Once again, we emphasize that the first two vertices $(-1,0), (1,0)$ yield the

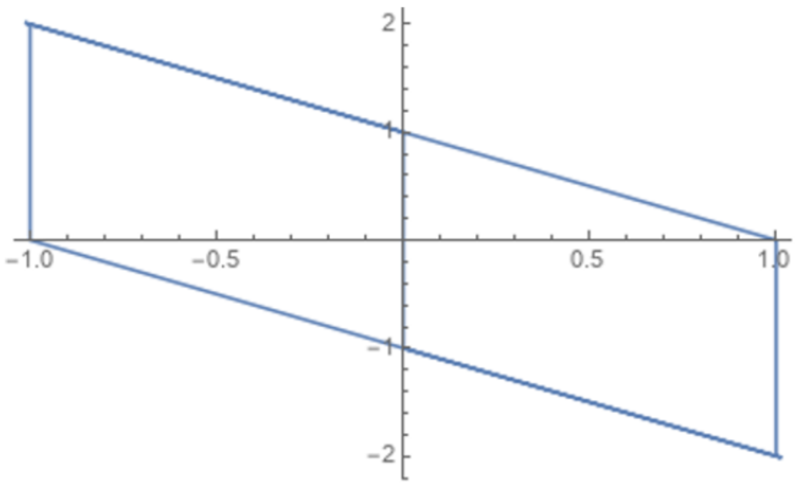

Figure 1: Convex set of the four chosen vertices

standard FGM copula, see [4]. While the second two vertices (-1,2),(1,-2) yield the extension of the FGM copula. In terms of these vertices we obtain the following forms of the function in (3.6).

$$C_{(-1,2)}(x,y) = xy + (x - x^2)(-y + 3y^2 - 2y^3) \tag{3.9}$$
$$C_{(1,-2)}(x,y) = xy + (x - x^2)(y - 3y^2 + 2y^3) \tag{3.10}$$

So, we are obliged to show the necessary and the sufficient conditions of the copulas whether in (3.9), or (3.10). In advance, let us present a simple counter example that prove such conditions when $|a_o + a_1| \geq 1$, which leads to a function that does not hold the nonnegative condition. Suppose that $(a_o, a_1) = (-1,2), (1,-2)$, and let $a_o = -0.9, a_1 = 2$, implies that $a_o + a_1 = 1.1$ (or, $a_o = 0.9, a_1 = -2 \rightarrow a_o + a_1 = -1.1$), which is clear that neither $a_o + a_1 \leq 1$ nor $a_o + a_1 \geq -1$. Therefore, we can find that at least the ordered pair $(x,y) = (0,1)$, implies that $C_{(-0.9,2)}(0,1) = 1 + +(1)(-0.9 + 5.8 - 6) = -0.1$. This yields a negative value of the joint density and this is unlike the condition that we have proposed.

Similarly, we obtain that $C_{(a_o,a_1)}(x,y) < 0$ for at least $(x,y) = (1,1)$ with $a_o = 0.9$, and $a_1 = -2$. Therefore, we only have a density function $C_{(a_o,a_1)}$ is non-negative if and only if $a_o = -1$, and $0 \leq a_1 \leq 2$, or $a_o = 1$, and $-2 \leq a_1 \leq 0$. $\square$

Remark 3.2. *For any ordered pair* (a_o, a_1) *that belongs to the intervals* $[0, -1]$, *and* $[0, 2]$ *or the intervals* $[1, 0]$, *and* $[-2, 0]$, *for example the pairs* $(-1, 1)$, *or* $(1, -1)$, *we can see that the function* $C_{(a_o,a_1)}(x, y)$ *has the following forms*

$$C_{(-1,1)}(x, y) == xy + (x - x^2)(-y + 2y^2 - y^3) \tag{3.11}$$
$$C_{(1,-1)}(x, y) = xy + (x - x^2)(y - 2y^2 + y^3) \tag{3.12}$$

and these functions can also be examined for being copulas that belong to the extension of FGM copula.

Now, let us return to prove the necessary and the sufficient conditions of the function in (3.6). Necessity of $a_o \in [-1, 1]$ when $a_1 = 0$ generates copula (the standard FGM copula), see [10]. It is not difficult to see that a convex combination of parameters above leads to convex combination of copulas, this of densities too.

In particular, substituting the vertex $(-1, 2)$, in equation (3.9) leads to the following result.

Theorem 3.3. *For all* $x, y \in [0, 1]$, *the function* $C_{(-1,2)}(x, y)$ *is copula, if and only if,* $Ran(C_{(-1,2)})(x, y)) = [0, 2]$.

Proof . Recall the function in (3.9) and differentiate it partially with respect to x, and y. We obtain a joint density, that is $C_{(-1,2)}(x, y) = 1 + (1 - 2x)(-1 + 6y - 6y^2)$.

Thus, $Ran[1 - 2x] = [-1, 1]$, and $Ran[-1 + 6y - 6y^2] = [-1, \frac{1}{2}]$, Hence, $Ran[(1 - 2x)(-1 + 6y - 6y^2)] = Ran[1 - 2x]Ran[-1 + 6y - 6y^2]$.

Therefore, the range of the density of that function is $Ran[1 + [-1, 1][-1, 1/2]] = [0, 2]$.

Showing that, we have obtained a copula.

For the vertex $(1, -2)$, we obtain the same result via the range of the joint density of the copula in (3.8).

To show sufficiency of the copulas in (3.8), and (3.9), we construct the following theorem: □

Theorem 3.4. *Let* $C_{(a_o,a_1)}(x, y)$ *be a function that satisfies the boundary conditions of copula* $C_{(a_o,a_1)}(x, 0) = C_{(a_o,a_1)}(0, y) = 0, C_{(a_o,a_1)}(x, 1) = x = C_{(a_o,a_1)}(1, x)$. *Then the function* $C_{(a_o,a_1)}(x, y)$ *is copula, if and only if, that function is 2-increasing, if and only if,* $V_{C([x_1,x_2]\times[y_1,y_2])} \geq 0$.

Proof . It suffices to show that $C_{(a_o,a_1)}$ in (3.6) holds the 2-increasing property. Thus

$$\begin{aligned} V_{C([x_1,x_2]\times[y_1,y_2])} &= (x_2 - x_1)(y_2 - y_1) + [a_o + a_1(y_2 - y_1)][(x_2 - x_1)(1 - x_2 - x_2)][(y_2 - y_1)(1 - y_2 - y_1)] \\ &= (x_2 - x_1)(y_2 - y_1)[1 + (a_o + a_1(y_2 - y_1))(1 - x_2 - x_1)(1 - y_2 - y_1)] \geq 0 \end{aligned}$$

It is clear that $(x_2 - x_1)(y_2 - y_1) \geq 0$.

Consequently, for all $x_1, x_2, y_1, y_2 \in [0, 1], Ran[a_o + a_1(y_2 - y_1)](1 - x_2 - x_1)(1 - y_2 - y_1) = [-1, 1]$.

Therefore, $Ran(1 + [-1, 1]) = [0, 2]$.

It follows that $C_{(a_o,a_1)}$ is 2-increasing, and copula, if and only if (a_o, a_1) are whether $(-1, 2)$, or $(1, -2)$.

This completes the proof. □ Similarly, we can also show that the function in (3.7) has also the 2-increasing property.

Furthermore, we need to prove and discuss more sufficient conditions in order to verify the 2-increasing property for the function $C_{(a_o,a_1)}$ over the vertices $(a_o, a_1) = (-1, 2), (1, -2)$. Let's firstly rewrite the function in (3.6) by the following way:

$$C_{(-1,2)}(x, y) = xy + f_1(y)(x - x^2) \tag{3.13}$$
$$C_{(1,-2)}(x, y) = xy + f_2(y)(x - x^2) \tag{3.14}$$

where, $f_1(y) = (-y + 3y^2 - 2y^3)$, and $f_2(y) = (y - 3y^2 + 2y^3)$. Then according to (Theorem 3.2.4, and Corollary 2.3.5, Nelsen [9]), one could illustrate the following corollary.

Corollary 3.5. *A function $C_{(a_o,a_1)}$ be a copula, if and only if $f_1(y)$, or $f_2(y)$ satisfy the following conditions*

1. $f_1(0) = f_1(1) = f_2(0) = f_2(1) = 0$;
2. *For all $y_1, y_2 \in [0,1], y_1 \leq y_2, |f_1(y_2) - f_1(y_1)| \leq |y_2 - y_1|$ (Lipschitz condition); Similarly, $f_2(y)$ holds this condition*
3. *$f_1(y)$, or $f_2(y)$ are absolutely continuous for all $y \in [0,1]$ almost everywhere on $[0,1], |(f_1)(y)| \leq 1$, or $|(f_2)(y)| \leq 1$;*
4. *All the conditions above provide that $C_{(a_o,a_1)}$ is absolutely continuous over $[0,1]$.*

The proof is clear, see [4, 8].

In fact, the second condition of the corollary above with respect to, for example, $f_1(y)$ can be shown as $3(y_2 + y_1) + 2(y_1^2 + y_1y_2 + y_2^2) \geq 0$, for all $y \in [0,1]$.

Moreover, it is trivial, but necessary to show that the copula $C_{(a_o,a_1)}$ can be obtained from its copula density. Let recall the formula in (3.7), with second condition (II). Then

$$C_{(a_o,a_1)}(x,y) = \int_0^x \int_0^y [1 + (1-2s)(a_o + 2(a_1 - a_o)t - 3a_1t^2)]dsdt$$

It is clear that solving the double integral of the function above yields our desired function in (3.6). Indeed, there are much more details of some properties related to copula, and survival copula like symmetry, asymmetry, continuity, and so on, see [4, 9, 11].

3.2. Calculations of measures of association

In this part, we show some calculations of the most popular measures of association that have been presented in literatures, see [9]. By implementing the constructing copulas there are four essential measures that are known by spearman's ρ, kendall's tau τ, Gini's correlation γ, and Blomqvist β. They are known as a measure of concordance and discordance, see [3, 5, 6, 7, 14].

In fact, those dependencies with any copula family represent a nonparametric approach or equivalently distribution free approach. In fact, these types of correlations have much details and valuable information that have been discussed in [1, 5, 7, 9, 10, 13, 14].

Theorem 3.6. *For all $x, y \in [0,1]$,$a_o \in [-1,1]$, and $a_o \in [-2,2]$, the yielding dependencies ρ_C, τ_C, γ_C, and β_C, respectively, in terms of the given copula in (3.6) are the following*

$$\rho_{C_{(a_o,a_1)}} = \frac{(2a_0 + a_1)}{6} \tag{3.15}$$

$$\tau_{C_{(a_o,a_1)}} = \frac{(2a_0 + a_1)}{9} \tag{3.16}$$

$$\beta_{C_{(a_o,a_1)}} = \frac{(2a_0 + a_1)}{8} \tag{3.17}$$

$$\gamma_{C_{(a_o,a_1)}} = \frac{(8a_0 + 4a_1)}{15} \tag{3.18}$$

The proof follows directly from the computations of each formula given in (2.4), (2.5), (2.6), and (2.7). Note that the obtained forms in (3.15), (3.16), and (3.17) within the intervals of the constants (a_o, a_1) *leads to the following results*

$$|\rho_{C_{(a_o,a_1)}}| \leq \frac{1}{3} \tag{3.19}$$

$$|\tau_{C_{(a_o,a_1)}}| \leq \frac{2}{9} \tag{3.20}$$

$$|\beta_{C_{(a_o,a_1)}}| \leq \frac{1}{4} \tag{3.21}$$

$$|\gamma_{C_{(a_o,a_1)}}| \leq \frac{4}{15} \tag{3.22}$$

4. Conclusion

The modification of the FGM copula via polynomial function led to illustrate a function that holds the conditions of copula. The constants of the constructed functions are linearly increasing. Lemma 3.1, Theorem 3.3, Theorem 3.4, and Corollary 3.5 provide almost the necessary and sufficient conditions of the 2-increasing property so that the function $C_{(a_o,a_1)}$ is being copula. The calculations of the desired dependencies yield new bound values that may differ from other bounds in other literature that have various extension over FGM-copula, see [8, 10]. There are future works related to the extension of FGM-copula with respect to various cases of n that may yield new different values of the measures of association.

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