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# Fuzzy partial differential equations

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**Abstract**

We will consider a type of elementary fuzzy partial differential equation that we wish to solve. The classical solution and the extension solution are discussed.

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## 1 Introduction

To define the elementary fuzzy partial differential equation, we are interested in. Let $I_1 = [0, M_1]$ and $I_2 = [0, M_2]$ for some $M_1, M_2 > 0$, $f(x, y, k)$ be a continuous function for $(x, y) \in I_1 \times I_2$ and $k = (k_1, k_2, \ldots, k_n)$ a vector of constants with $k_i$ in the interval $J_i$, $1 \leq i \leq n$. The operator $\varphi(D_x, D_y)$ will be a polynomial, with constant coefficients, in $D_x$ and $D_y$, where $D_x$, $D_y$ stands for the partial derivative with respect to $x, y$ respectively.

Also, let $u(x, y)$ be a continuous function, having continuous partial derivatives with respect to both $x$ and $y$, with $(x, y) \in I_1 \times I_2$. The crisp partial differential equation is

$$\varphi(D_x, D_y)\ u(x, y) = f(x, y, k) \tag{1.1}$$

subject to certain boundary conditions. These boundary conditions can come in a variety of forms such as $u(0, y) = c_1, u(x, 0) = c_2, u(M_1, y) = c_3, \ldots, u(0, y) = r_l(y; c_4), u(x, 0) = h_l(x; c_5), \ldots, u_x(x, 0) = h_2(x; c_6), u_y(0, y) = r_2(y; c_7, c_8)$, .... At this point, we will not give any explicit structure to the boundary conditions except to say they depend on constants $c_l, \ldots, c_m$ with the $c_i$ in intervals $L_i$, $1 \leq i \leq m$. Let $c = (c_l, \ldots, c_m)$ be the vector of these constants. We assume that problem (1.1) with associated boundary conditions has a solution

$$u(x, y) = g(x, y, k, c), \tag{1.2}$$

with $\varphi(\ D_x, D_y) g(x, y, k, c)$ continuous for $(x, y) \in I_1 I_2$, $k \in J = \prod J_i$ and $c \in L = \prod L_i$.

By "**elementary**" we mean that the solution g in (1.2) is not defined in terms of a series. That is, there are no Fourier series used to define g. Since we will need to fuzzify g we do not wish to fuzzify Fourier series. We need the solution g to be fairly simple. So, we also assume that Bessel functions and Legendre functions are not used in g. The constants $k_j$ and $c_i$ are not known exactly so there will be uncertainty in their values. We will model this uncertainty using fuzzy numbers. So, we will substitute triangular fuzzy numbers $k_i$ for $k_i, K_i$ in $J_i$, $1 \leq i \leq n$, and substitute triangular fuzzy numbers $C_i$ for $c_i$, $C_i$ in $L_i$, $1 \leq i \leq m$.

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If we fuzzify (1.1), then we obtain the elementary fuzzy partial differential equation we wish to consider, which is:

$$\phi(D_x, D_y)\ U(x,y) = F(x,y,K) \tag{1.3}$$

subject to certain boundary conditions. The boundary conditions can be of the form $U(0,y) = C_1, U(x,0) = C_2, U(M_1,y) = C_3, \ldots, U(0,y) = R_1(y;C_4), U(x,0) = H_1(x;C_5), \ldots, U_x(x,0) = H_2(x;C_6), U_y(0,y) = R_2(y;C_7,C_8), \ldots$. The $R_i$ and $H_i$ are the extension principle extensions of $r_i$ and $h_i$ respectively [2]. We wish to solve (1.3) to certain fuzzy boundary conditions. We first introduce the classical solution [9].

## 2 Classical Solution

Let $Y(x,y)$ be the classical solution, $[Y(x,y)]^\alpha = [y_1(x,y,\alpha),\ y_2(x,y,\alpha)]$, $(x,y) \in I_1 \times I_2$, $\alpha \in [0,1]$. We assume that $\varphi(\ D_x, D_y)y_i(x,y,\alpha)$ is continuous for $(x,y) \in I_1 \times I_2$ , $\alpha \in [0,1]$, $i = 1,2$. Substituting the $\alpha-$cuts of $Y(x,y)$ in (1.3) we have:

$$\varphi(\ D_x, D_y)[y_1(x,y,\alpha), y_2(x,y,\alpha)] = [F_1(x,y,\alpha), F_2(x,y,\alpha)], \tag{2.1}$$

assuming the fuzzy boundary conditions are $U(0,y) = C_1, U(M_1,y) = C_2$. That is

$$y_1(0,y,\alpha) = c_{11}(\alpha),$$
$$y_2(0,y,\alpha) = c_{12}(\alpha),$$
$$y_1(M,y,\alpha) = c_{21}(\alpha),$$
$$y_2(M,y,\alpha) = c_{22}(\alpha),$$

where $[C_1]^\alpha = [c_{11}(\alpha), c_{12}(\alpha)]$, $[C_2]^\alpha = [c_{21}(\alpha), c_{22}(\alpha)]$. Then we find $y_i(x,y,\alpha)$, $i = 1,2$. We sat that $Y(x,y)$ is a solution if $[y_1(x,y,\alpha),\ y_2(x,y,\alpha)]$ defines a triangular fuzzy shaped number [6]. That is for all $(x,y) \in I_1 \times I_2$,

$$\partial y_1(x,y,\alpha)/\partial\alpha > 0,\ \ \partial y_2(x,y,\alpha)/\partial\alpha < 0, \quad 0 < \alpha < 1, \quad y_1(x,y,1) = y_2(x,y,1)$$

**Example 2.1.** Consider the elementary partial differential equation:

$$u_{xy} = k_1 xy + k_2 e^x, \tag{2.2}$$

for $k_1 \in [0, M_3], k_2 \in [0, M_4]$, $M_3, M_4 > 0$. The initial conditions are

$$u(x,0) = c_1,$$
$$u_y(0,y) = c_2 y,$$

for $c_1 \in [0, M_5], c_2 \in [0, M_6]$, $M_5, M_6 > 0$. Now, assuming $c_1, c_2, k_1, k_2$ are fuzzy triangular numbers, we have:

$$[C_1]^\alpha = [c_{11}(\alpha), c_{12}(\alpha)],\ \ [C_2]^\alpha = [c_{21}(\alpha), c_{22}(\alpha)],$$
$$[K_1]^\alpha = [k_{11}(\alpha), k_{12}(\alpha)],\ \ [K_2]^\alpha = [k_{21}(\alpha), k_{22}(\alpha)], \text{then}$$
$$\partial^2 y_1(x,y,\alpha)/\partial y \partial x = k_{11}(\alpha)xy + k_{21}(\alpha)e^x$$
$$\partial^2 y_2(x,y,\alpha)/\partial y \partial x = k_{12}(\alpha)xy + k_{22}(\alpha)e^x$$
$$y_1(x,0,\alpha) = c_{11}(\alpha)$$
$$y_2(x,0,\alpha) = c_{12}(\alpha)$$
$$\partial y_1(0,y,\alpha)/\partial y = c_{21}(\alpha)y,$$
$$\partial y_2(0,y,\alpha)/\partial y = c_{22}(\alpha)y,$$

with solutions,

$$y_1(x,y,\alpha) = (k_{11}(\alpha)/4)x^2y^2 + k_{21}(\alpha)ye^x + c_{11}(\alpha) + c_{21}(\alpha)y^2/2 - k_{21}(\alpha)y,$$

and

$$y_2(x,y,\alpha) = (k_{12}(\alpha)/4)x^2y^2 + k_{22}(\alpha)ye^x + c_{12}(\alpha) + c_{22}(\alpha)y^2/2 - k_{22}(\alpha)y.$$

Since $\partial y_1 \partial\alpha > 0, \partial y_2/\partial\alpha < 0, 0 < \alpha < 1, y_1(x,y,1) = y_2(x,y,1)$, we have $Y(x,y)$ is a solution, that can be written as

$$Y(x,y) = (x^2y^2/4)K_1 + ye^x K_2 + C_1 + (y^2/2)C_2 - yK_2$$

for $(x,y) \in I_1 \times I_2, K_i \in J_i$, $C_i \in L_i$, $i = 1,2$. These are true for $M_i > 0$, $1 \le i \le 6$.

## 3 Extension Solution

Let $[Y(x,y)]^\alpha = [y_1(x,y,\alpha),\ y_2(x,y,\alpha)],\ \ [F(x,y,K)]^\alpha = [f_1(x,y,\alpha),\ f_2(x,y,\alpha)]$ for all $x,y$ and $\alpha$, where

$$\begin{aligned} y_1(x,y,\alpha) &= \min\{g(x,y,k,c),\ \ k\in[K]^\alpha,\ \ c\in[C]^\alpha\}\\ y_2(x,y,\alpha) &= \max\{g(x,y,k,c),\ \ k\in[K]^\alpha,\ \ c\in[C]^\alpha\}\\ f_1(x,y,\alpha) &= \min\{g(x,y,k,c),\ \ k\in[K]^\alpha\}\\ f_1(x,y,\alpha) &= \max\{g(x,y,k,c),\ \ k\in[K]^\alpha\}. \end{aligned}$$

Assume that the $y_i(x,y,\alpha)$ have continuous partial derivatives, define

$$\Gamma(x,y,\alpha) = [\varphi(D_x,D_y)]y_1(x,y,\alpha), \varphi(D_x,D_y)y_2(x,y,\alpha), \tag{3.1}$$

for all $(x,y)\in I_1\times I_2,\ \alpha\in[0,1]$. If for each fixed $(x,y)\in I_1\times I_2,,\ \Gamma(x,y,\alpha)$ defines the $\alpha-$cuts of a fuzzy number, then we will say that $Y(x,y)$ is differentiable and write

$$[\varphi(D_x,D_y)Y(x,y)]^\alpha = \Gamma(x,y,\alpha)$$

for all $(x,y)\in I_1\times I_2$ and all $\alpha$. Sufficient conditions for $\Gamma(x,y,\alpha)$ to define $\alpha-$cuts of a fuzzy number are:

(1) $\varphi(D_x,D_y)y_1(x,y,\alpha)$ is an increasing function of $\alpha$ for each $(x,y)\in I_1\times I_2$
(2) $\varphi(D_x,D_y)y_2(x,y,\alpha)$ is an decreasing function of $\alpha$ for each $(x,y)\in I_1\times I_2$
(3) $\varphi(D_x,D_y)y_1(x,y,1)\le\varphi(D_x,D_y)y_2(x,y,1)$ for all $(x,y)\in I_1\times I_2$.

For $Y(x,y)$ to be an extension solution [3] to the fuzzy partial differential equation we need the following:

(i) $Y(x,y)$ is differentiable,
(ii) Equation (1.3) holds for $U(x,y)=Y(x,y)$, that is

$$\varphi(D_x,D_y)y_1(x,y,\alpha) = f_1(x,y,\alpha), \tag{3.2}$$
$$\varphi(D_x,D_y)y_2(x,y,\alpha) = f_2(x,y,\alpha), \tag{3.3}$$

for all $(x,y)\in I_1\times I_2$ and all $\alpha\in[0,1]$.
(iii) $Y(x,y)$ satisfies the boundary conditions, when boundary conditions are specified.

These conditions define a triangular shaped fuzzy number since the endpoints of $\Gamma(x,y,\alpha)$ are continuous. If the extension solution satisfying the boundary conditions is $Y(x,y)$, then $Y(x,y)$ is also the classical solution. Now we will present a sufficient condition for the extension solution to exist. Since there are such a variety of possible boundary conditions we will omit them from the following Theorem:

**Theorem 3.1.** Assume $Y(x,y)$ is differentiable

(a) If for all $i,\ 1\le i\le n,\ g(x,y,k)$ *and* $f(x,y,k)$ are both increasing(or both decreasing) in $k_i$ for $(x,y)\in I_1\times I_2$ and $k\in j$, then $Y(x,y)$ is an extension solution.
(b) If there is an $i,\ 1\le i\le n$, such that for $k_i, g(x,y,k)$ is strictly increasing and increasing), for $(x,y)\in I_1\times I_2$ and $k\in j$, then $Y(x,y)$ is not an extension solution.

**Proof .** (a) Without loss of generality, assume that $n=2$ and $g(x,y,k)$ is increasing in $k_1, f(x,y,k)$ is increasing in $k_1, g(x,y,k)$ is decreasing in $k_2$ and $f(x,y,k)$ is also decreasing in $k_2$. The other cases are similar. We have:

$$y_1(x,y,\alpha) = g(x,y,k_{11}(\alpha),k_{22}(\alpha)), \tag{3.4}$$
$$y_2(x,y,\alpha) = g(x,y,k_{12}(\alpha),k_{21}(\alpha)), \tag{3.5}$$
$$f_1(x,y,\alpha) = f(x,y,k_{11}(\alpha),k_{22}(\alpha)), \tag{3.6}$$
$$f_2(x,y,\alpha) = f(x,y,k_{12}(\alpha),k_{21}(\alpha)), \tag{3.7}$$
(3.8)

for all $\alpha$ where $[K_1]^\alpha=[k_{11}(\alpha),k_{12}(\alpha)],\ [K_2]^\alpha=[k_{21}(\alpha),k_{22}(\alpha)]$. Now $g$ solves (1.1) means

$$\varphi(D_x,D_y)g(x,y,k_1,k_2) = f(x,y,k_1,k_2), \tag{3.9}$$

for all $(x, y) \in I_1 \times I_2$ and $k_1 \in J_1, k_2 \in J_2$. But $k_{1j}(\alpha) \in J_1,\ k_{2j}(\alpha) \in J_2$ for all $\alpha,\ j = 1, 2$. So,

$$\varphi(D_x, D_y)y_1(x, y, \alpha) = f_1(x, y, \alpha), \tag{3.10}$$
$$\varphi(D_x, D_y)y_2(x, y, \alpha) = f_2(x, y, \alpha), \tag{3.11}$$

for all $(x, y) \in I_1 \times I_2$ and $\alpha$. Thus (3.2) and (3.3) are satisfied and $Y(x, y)$ is an extension solution.

(b) Suppose also $n = 2$ and $g(x, y, k)$ is strictly increasing in $k_1, f(x, y, k)$ is strictly decreasing in $k_1$, both $g$ and $f$ are strictly decreasing in $k_2$. Equations (3.4) and (3.5) are still true but equations (3.6) and (3.7) become:

$$f_1(x, y, \alpha) = f(x, y, k_{12}(\alpha), k_{22}(\alpha)),$$
$$f_2(x, y, \alpha) = f(x, y, k_{11}(\alpha), k_{21}(\alpha)),$$

for all $\alpha$. Thus, (3.10) and (3.11) do not hold, that is $Y(x, y)$ is not extension solution. □

**Corollary 3.2.** Assume that $Y(x, y)$ is differentiable.

(a) $Y(x, y)$ is an extension solution if, $(\partial g/\partial k_i)(\partial f/\partial k_i) > 0$ for $i = 1, 2, \cdots, n$ for $(x, y) \in I_1 \times I_2$ and $k \in j$.
(b) If $(\partial g/\partial k_i)(\partial f/\partial k_i) < 0$ for some $i$, for $(x, y) \in I_1 \times I_2, k \in j$, then $Y(x, y)$ is in not an extension solution.

**Example 3.3.** Consider the partial differential equation:

$$u_{yx} - u_x = k, \tag{3.12}$$

where the constant $k \geq 0$. Initial conditions are

$$u(0, y) = c_1,$$
$$u_x(x, 0) = c_x^2,$$

for $c_1 \in [0, M_3],\ c_2 \in [0, M_4],\quad M_3 > 0, M_4 > 0$. A crisp solution is

$$g(x, y, k, c) = c_2x^3e^x/3 + kx(e^y - 1) + c_1.$$

Now, assuming $c_1, c_2, k$ are fuzzy triangular numbers, we have:

$$g_1(x, y, \alpha) = c_{21}(\alpha)x^3e^y/3 + k_1(\alpha)x(e^x - 1) + c_{11}(\alpha),$$
$$g_2(x, y, \alpha) = c_{22}(\alpha)x^3e^y/3 + k_2(\alpha)x(e^x - 1) + c_{12}(\alpha).$$

One also can easily check that for $y_i = g_i, i = 1, 2$, we have:

$$\varphi(D_x, D_y)y_1(x, y, \alpha) = k_1(\alpha),$$
$$\varphi(D_x, D_y)y_2(x, y, \alpha) = k_2(\alpha).$$

where $\varphi(D_x, D_y) = D_xD_y - D_x$. Also, we have

$$y_1(0, y, \alpha) = c_{11}(\alpha),$$
$$y_2(0, y, \alpha) = c_{12}(\alpha),$$
$$\partial y_1(0, y, \alpha)/\partial x = c_{21}x^2,$$
$$\partial y_2(0, y, \alpha)/\partial x = c_{22}x^2.$$

hold. One can check easily that, $(\partial g/\partial k)(\partial f/\partial k) > 0$. So,

$$Y(x, y) = C_2x^3e^y/3 + Kx(e^y - 1) + C_1$$

is an extension solution for all $x, y \in [0, \infty)$.

Now we introduce an example where the extension solution fails to exist but the classical solution exists in some region in the domain.

**Example 3.4.**

$$u_{yy} = k_1 x^2 \cos y + k_2, \tag{3.13}$$

with boundary conditions

$$u(x,0) = c_1,$$
$$u(x,\pi/2) = c_2,$$

where $x \in I_1 = [0, M_1],\ y \in I_2 = [0, \pi/2]$, with $M_1 > 0$. The values of the parameters $k_1, k_2, c_1$ and $c_2$ are in intervals $[0, M_i], 2 \le i \le 5$, respectively, for all $M_i > 0$. Therefore,

$$\varphi(D_x, D_y) = D_y^2 \text{ and } f(x,y,k) = k_1 x^2 \cos y + k_2.$$

A crisp solution is,

$$g(x,y,k,c) = k_1 x^2 (1 - \cos y - (2/\pi)y) + k_2 y/2(y - \pi/2) + c_1(1 - 2/\pi)y + c_2(2/\pi)y,$$

for $(x,y) \in I_1 \times I_2, k_i \in j, c_i \in L$. We have $Y(x,y)$ in not an extension solution since $(\partial g/\partial k_i)(\partial f/\partial k_i) < 0$, for $i = 1, 2$, where $\partial g/\partial k_1 < 0,\ \ \partial g/\partial k_2 < 0,\ \ \partial f/\partial k_1 > 0,\ \partial f/\partial k_2 > 0$. We proceed to look for the classical solution. We must solve

$$\partial^2 u_1(x,y,\alpha)/\partial y^2 = k_{11}(\alpha) x^2 \cos y + k_{21}(\alpha)$$
$$\partial^2 u_2(x,y,\alpha)/\partial y^2 = k_{12}(\alpha) x^2 \cos y + k_{22}(\alpha),$$

subject to

$$u_1(x,0,\alpha) = c_{11}(\alpha)$$
$$u_2(x,0,\alpha) = c_{12}(\alpha)$$
$$u_1(x,\pi/2,\alpha) = c_{21}(\alpha)$$
$$u_2(x,\pi/2,\alpha) = c_{22}(\alpha).$$

The solution is

$$u_i(x,y,k,c) = k_{1i}(\alpha) x^2 (1 - \cos y - (2/\pi)y) + k_{2i}(\alpha) y/2(y - \pi/2) + c_{1i}(\alpha)(1 - 2/\pi)y + c_{2i}(\alpha)(2/\pi)y,$$

for $i = 1, 2$. Since the $u_i$ are continuous and $u_1(x,y,1) = u_2(x,y,1)$, we only want to check if $\partial u_1/\partial\alpha > 0$ and $\partial u_2/\partial\alpha < 0$. So, we have a situation that there is a region $\tilde{R}$ contained in $I_1 \times I_2$ for which the classical solution exists depending on the fuzzy numbers $K_i$ and $C_i,\ i = 1, 2$.

To illustrate this, we pick simple fuzzy parameters that have base on the interval $[a-1, a+1]$ with vertex at $a$, then $k'_{i1}(\alpha) = 1,\ k'_{i2}(\alpha) = -1,\ c'_{i1}(\alpha) = 1,\ c'_{i1}(\alpha) = -1,\ i = 1, 2$. Then, for a classical solution to exist we require

$$x^2(1 - \cos y - (2/\pi)y) + y/2(y - \pi/2) + 1 > 0. \tag{3.14}$$

Since $(1 - \cos y - (2/\pi)y) \le 0$ and $y/2(y - \pi/2) \le 0$, for $0 \le y \le \pi/2$, we see as $x$ grows larger and larger, eventually (3.14) will be false. We find that

$$\begin{aligned} &\min\{(1 - \cos y - (2/\pi)y) :\ 0 \le y \le \pi/2\} = -0.2105 \textit{ and} \\ &\min\{y/2(y - \pi/2) :\ 0 \le y \le \pi/2\} = -0.3084. \textit{ Hence} \\ &x^2(1 - \cos y - (2/\pi)y) + y/2(y - \pi/2) + 1 > -0.21050x^2 + 0.6916. \end{aligned} \tag{3.15}$$

The region $\{(x,y) :\ \ 0 \le x \le 1.8126,\ \ 0 \le y \le \pi/2\}$, where the classical solution exists.

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