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Production planning for controllable deteriorating items with price and stock dependent demand rate

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Abstract

In this paper, an Economic Production Quantity (EPQ) inventory model for deteriorating items with price-stock dependent demand rate under complete and partial backlog is developed, in which the deterioration rate is controlled by investment in preservation technology. To fulfil the demand and to reduce the shortage we have considered the production rate proportional to the selling price of the product. This study is to maximize the total profit for seasonal deteriorating items by simultaneously determining the optimal selling price, the optimal production and the optimal preservation technology cost when the producer invests in the preservation technology to reduce the deterioration rate. We first show that for any given number of the production cycle, optimal selling price and preservation technology cost exists and are unique. Next, we show that the total profit is a jointly concave function of selling price and preservation technology cost. We provide some conditions to determine an optimal solution that maximizes profits for the EPQ model. We then provide a simple algorithm to figure out the optimality of total profit for the proposed model. Mathematical theorems are developed to determine optimal inventory policy. Numerical results demonstrate the advantages of the preservation technology, and further show the effects of different system parameters on the optimal variables and the maximal total profit. Finally, some managerial implications are provided.

Keywords: Inventory, Pricing, Deterioration, Variable production, Backordering, Preservation technology investment
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1 Introduction

Deteriorating inventory had been studied in the past decades, and usually focused on constant or variable deterioration rate and ways to reduce the effect of deterioration. Deteriorating items are items that deteriorate with time, resulting in a decreasing utility, quality, marginal value and quantity from the original ones. Such items include medicines, fruits, vegetables, fashion goods, blood, electronic equipments, etc. The supervising of inventories is one of the most significant jobs that every manager must do effectively and efficiently in any system. Nowadays, all organizations are interested in a globally competitive market and so these organizations are taking severely the actions related to managing their inventories. Thus, presently, researchers have been putting their interest in optimizing inventory decisions. Today's research is interested in focusing on inventory holding models which have real-life applications. Most of the classical inventory models did not take into account the effects of preservation technology on the deterioration rate. In real life

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business inventory holding areas via firm house, cold firms; preservation technology investment becomes a powerful tool to improve inventory stock and profits in an industry. But during the last 35 years, the economic situation of most of the countries has changed to such an extent due to large-scale investment in preservation technology. The pioneer in this direction was Murr and Morris [34], who showed that a lower temperature will increase the storage life and decrease decay. The models for pricing, preservation technology investment policies involving price-dependent and stock-varying demand patterns have received the attention of several researchers. The fundamental consequence in the evolution of economic order quantity models with selling price-sensitive and stock-varying demand patterns is that Mishra [32] presented a coordination scheme for an inventory of deteriorating items with revenue sharing by the manufacturer on preservation technology investment of retailer under price sensitive and stock dependent demand. Chakrabarti and Chaudhuri [5] proposed an EOQ model with linear demand and found that both the reorder number and the system cost decrease considerably as a result of allowing shortages in all cycles. After many years, Hsu et al. [16] developed a deteriorating inventory policy when the retailer invests on the preservation technology to reduce the rate of product deterioration. Several other researchers have extended their approach to various interesting situations by considering the time-varying deterioration rate, shortage, different (probabilistic and function of preservation technology investment) deterioration rate, etc.

Some of the items are either damaged or decayed or affected by some factors and is not in perfect condition to satisfy the demand. Food items, drugs, volatile liquids, blood, pharmaceuticals, radioactive substances are examples of such items where deterioration can take place during the normal storage period of the commodity. The deterioration rate of items in the above-mentioned papers is considered as an exogenous variable. Which is not subject to reduce or control, such an assumption is not always tenable in view of the technologies. In practice, the rate of deterioration of items can be controlled and reduced by various ways such as specialized equipment acquisition and applying preservation techniques. The analysis of deteriorating inventory started with Ghare and Schrader [12], who demonstrated the classical inventory model considering the constant rate of decay and no shortage. We cannot avoid the loss due to deterioration. But we knew that this can be reduced by developing a policy regarding preservation technology. In addition, some research has been done in optimal decisions of preservation technology investment and pricing for fresh food, with shortage allowed during the sales period and imperfectly competitive market environment screening by Liu *et al.* [24]. As a result, researchers including Hsu *et al.* [16], Zhang *et al.* [55], Dye and Hsieh [10], Mishra [31] and Li *et al.* [23] presented models treating the deterioration rate as a function of preservation technology investment.

The consequences of the sensitivity analysis in numerous studies (Pando et al. [37]; Sana et al. [41]; Dye [11]; Khedlekar and Singh [20]; Lee and Dye [22]; Tashakkor et al. [48]; Khedlekar et al. [21]; Yang et al. [54]) also presented that a lower deterioration rate is reasoned beneficial from an economic point of view. In real inventory systems, firms can bring down the rate of deterioration of products by means of effectual funds investment in preservation technology. So, the rate of deterioration is a function of preservation technology investment. Wu et al. [53] first integrated the phenomenon "non-instantaneous deterioration" into the inventory model. In reference to this phenomenon, Dye [11] proposed in a rigorous way that a higher preservation technology investment leads to a higher optimal service rate. Khedlekar et al. [19] recently presented an economic production quantity model with disruption considering shortage and time proportional demand. Hsieh and Dye [15] integrated the production-inventory model with time-varying demand and controllable deterioration rate by allowing preservation technology cost as a decision variable where shortages were forbidden over a planning horizon. They also established an algorithm for finding the optimal production and preservation technology investment policy for minimizing the total cost over the finite planning horizon. Chandra Das et al. [4] presented a deteriorated inventory model for selling price dependent demand rate, partial backlogging for two different backlogging rates, two separate preservation rates. This paper results if the products are sold-out quickly then less preservation cost is used.

Goel et al. [14] proposed a supply chain inventory model with variable deterioration and stock-dependent demand, where the unsatisfied demand is partially backlogged which depends on waiting time. Abad [1] showed that when the good is highly perishable, the reseller may need to backlog demand in order to market the good at a reasonable price. Papachristos and Skouri [38] generalized the model of Wee and Wang [52] for a constant rate of partial backlogging by assuming variable backlogging rate, as proposed by Abad [1]. Sarkar [42] extended the literature in supply chain management with the algebraical procedure and the probabilistic deterioration function. Cárdenas-Barrón et al. [3] discussed a note on economic production quantity (EPQ) model for coordinated planning with partial backlogging. Sarkar et al. [44] discussed preservation techniques for deteriorating seasonal products with stock-dependent consumption rate and shortages.

Production rate is an important factor of production inventory systems (Salookolaeia et al. [40]) . It plays a very crucial role in many production-inventory systems. Therefore, the consequence of production rate cannot be discarded. Abad [2] considered the problem of determining the lot size for a perishable good under finite production,

Table 1: Comparative study of related research

Authors	Demand pattern	Production	Deterioration	Preservation technology	Shortages	Planning horizon
Skouri et al. [47]	Time dependent	Yes (Proportional to demand rate)	Time dependent	No	Yes	Finite
Dye and Hsieh [10]	Constant	No	Time dependent	Yes	Partial Backordering	Infinite
Hsieh and Dye [15]	Time dependent	Yes (constant)	Constant	Yes	No	Finite
Zhang et al. [55]	Price dependent	No	Constant	Yes	No	Infinite
Mishra et al. [33]	Price and stock dependent	No	Constant	Yes	Complete and Partial backordering	Finite
Zhang et al. [56]	Price dependent	Yes (constant)	Constant	Yes	No	Infinite
Lu et al. [26]	Price and stock dependent	No	Constant	No	No Partial Backordering	Finite
Liuxin et al. [25]	Stock, Time Price dependent	No	Constant	No	No	Infinite
Singh et al. [46]	Time dependent	Yes Demand dependent	non-instantaneous	No	Partial Backordering	Finite
Proposed paper	Pice and Stock dependent	Yes (Proportional to demand rate)	Constant	Yes	Complete and Partial Backordering	Finite

exponential decay, partial backordering and lost sale. He showed that the total cost per time period is pseudoconvex. To incorporate the production rate, Wee [51] formulated an economic production plan for deteriorating items with partial back-ordering. Singh et al. [46] presented a production inventory system for non-instantaneous deteriorating items with partial backordering, demand dependent production rate, and demand rate exponentially increasing the function of time. Sarkar and Sarkar [43] proposed a production system which produces both proper and improper items with probabilistic deterioration and exponential demand over a finite time horizon. The production rate is a dynamic variable (varying with time) in a production system. Teng and Chang [49] who presented an EPQ model for deteriorating items in which the demand rate is a function of the on-display stock level and the selling price. Mahata [28] proposed an integrated production-inventory with backorder model in a fuzzy sense.

Chen and Chen [6] considered a single product that is subjected to continuous decay, faces a price-dependent and time-varying demand, and time-varying deteriorating rate, production rate, and variable production cost, with the objective to maximized the profit stream over the multi-period planning horizon. Wee [51] proposed and analyzed the single commodity economic production lot size model for deterioration items with partial back-ordering and finite replenishment. Palanivel and Uthayakumar [36] developed an economic production lot-size model for deteriorating items with a price- and advertisement-dependent demand, a variable production rate, deterioration follows a continuous probability distribution function, selling price is a mark-up over the unit production cost, shortages are allowed and partial backlogged. Dye and Hsieh [10] proposed an inventory model involving deteriorating items with a generalized deterioration rate and productivity of invested capital. They assumed preservation technology cost as a function of the length of the replenishment cycle. Kabirian [17] extended the Economic Lot-Size Scheduling (ELS) with pricing for a single item on a single machine. They assumed variable production cost as a decreasing function of economic lot-size. Mahata et al. [27] developed an economic production quantity (EPQ) based model for perishable items under the retailer's partial trade credit policy and price-dependent demand. Shah et al. [45] presented an inventory model for deteriorating items under two-level trade credit financing with price-sensitive and time-dependent demand rate in order to maximize the total profit per unit time of a retailer.

Wee and Wang [52] developed a model for variable-production cycles with time-varying demands and completely backlogged shortages. Zhou et al. [58] extended Wee and Wang [52] model in two directions. First, starting each cycle by aggregation backlogs, followed by inventory building via production and terminating with depletion of positive inventory. Second, model the situation in which some backlogged shortages can turn into lost sales. Skouri et al. [47] further extended the model of Manna and Chaudhuri [30] by considering a general function of time for the variable part of the demand rate and also, the demand rate is stabilized after the production stopping time, with and without shortages. Chung [8] presented an economic production quantity (EPQ) inventory model for deteriorating items under two levels of trade-credit.

Dye [9] discussed pricing and ordering policy for deteriorating items with time-dependent backlogging rate. Maihami and Kamalabadi [29] addressed the problem of simultaneously determining a pricing and inventory replenishment strategy for non-instantaneous deteriorating items. They considered price and time-dependent demand and variable backlogging rate, dependent on the waiting time for the next replenishment. Ray and Chaudhuri [39] discussed a deterministic inventory model for a deteriorating item having an inventory-level-dependent demand rate at any instant. The time value of money and different inflation rates for various costs associated with the inventory system are taken into account. The effect of price- and stock-dependent demand on the perishable products is noted in the works of Onal et al. [35] and Liuxin et al. [25].

An appropriate production and pricing inventory model for a controllable deteriorating item is presented in this paper. In the sum of all works on pricing, production and inventory control, models that consider deteriorating items, the demand functions are simple and dependent on price, stock or time, separately. But, in the proposed model, the price and the stock should be considered jointly. Because this form of demand function reflects a real situation. The main purpose of this paper is to amend the paper of Mishra et al., [33] with a view to making the model more relevant applicable in practice. In this paper, we consider the problem of determining the preservation technology cost and selling price for a deteriorating good under demand dependent production rate, price sensitive and stock dependent demand, exponential decay and backordering. The main objective is to determine the optimal selling price, the optimal ordering frequency and the optimal preservation technology cost simultaneously such that, the total profit is maximized. Then, we prove that the optimal selling price and preservation technology policy not only exists but are unique, for any given finite time horizon.

In this paper, we study the preservation technology investment, production cycle and selling price of the product for a controllable deterioration rate considering complete and partial backlogging, and perform a sensitivity analysis to understand how they depend on cost parameters. We also provided some useful results on finding the optimal production, selling price and preservation technology strategies. In this model, the graphical analysis approach is applied to indicate the concavity of the profit function.

The rest of this paper is organized as follows: Section 2 provides notations and assumptions used throughout this paper. Section 3 characterizes an EPQ model for complete and partial backordering. Section 4 illustrates the solution procedure. In Section 5, we obtain some useful theoretical results and incorporated an algorithm to find the optimal solution. In Section 6, a couple of numerical examples and managerial implications are provided to illustrate the proposed model. Section 7 concludes this study.

2 Model notations and assumptions

The mathematical model is developed on the basis of the following notations and assumptions.

2.1 Notations

We used the following notations for developing our model:

- n the number of production cycle (an integer number and a decision variable) over $[0, T]$,
- p the selling price (a decision variable) per unit, where $p > c$,
- ξ the preservation technology cost (a decision variable) per unit time, $0 \leq \xi \leq \bar{\xi}$, where $\bar{\xi}$ is the maximum cost of investment in preservation technology,
- α the stock dependent consumption rate parameter,
- θ the controllable deterioration rate (function of ξ, when there is investing on preservation technology), $\theta(\xi) = \tilde{\theta}e^{-\psi\xi}$,
- $\tilde{\theta}$ the deterioration rate without preservation technology investment per unit per unit time,
- ψ the sensitive parameter of investment to the deterioration rate,
- R the stock level (produced quantity) at time t_1,
- t_1 the length of time period during which production, demand and deterioration are allowed or the time point at which the production stops,
- $t_1{}'$ the time at which the inventory level reaches zero,
- T/n the period of one cycle,
- h the unit holding cost per unit time,
- c the unit production cost of the producer (or manufacturer),
- T the finite time horizon,

Q the produced quantity per inventory cycle,
Q_T the total produced quantity in complete interval $[0, T]$,
s the unit shortage cost of the product,
d the unit deterioration cost of the product,
P the production rate, which is dependent on $D(p)$, $P = D(p)/\lambda$, where $0 < \lambda < 1$,
a the market potential, $a > 0$,
b the price sensitivity of demand, $b > 0$,
π the goodwill cost of lost sales per unit,
ϑ the backordering parameter,
$I(t)$ the level of inventory at time t,
$L(t)$ the amount of lost sales at any time t,
$l_s(\frac{T}{n})$ the maximum shortage level for complete backordering,
$l_b(t)$ the backorder level at any time t for partial backordering,
$l_b(\frac{T}{n})$ the maximum backorder level for partial backordering,
$D(p, t)$ the demand rate per unit time function of the selling price p and the instantaneous stock level $I(t)$,
$\Pi(p, n, \xi)$ the total profit per unit time,
$R(n, p, \xi)$ the total sales revenue,
$P(n)$ the total production cost,
$H(n, p, \xi)$ the total holding cost,
$S(n, p)$ the total shortage cost,
$D(n, p, \xi)$ the total deterioration cost,
$P(\xi)$ the total preservation technology cost.

2.2 Assumptions

1. The planning horizon of the inventory model here is finite and is taken as T.
2. The manufacturer can bring down the deterioration rate of the product by making effectual funds investment in storehouse equipment. So, we considered controllable deterioration rate as an exponential function of preservation investment cost. If ξ is the preservation technology cost per unit time for reducing the deterioration rate in order to preserve the products then, the controllable deterioration rate, $\theta(\xi)$, is a continuous, decreasing function of preservation capital investment, i.e., $\partial\theta(\xi)/\partial\xi < 0$, $\partial^2\theta(\xi)/\partial\xi^2 > 0$. Additionally, it is natural to set $\theta(\xi) = \tilde{\theta}e^{-\psi\xi}$ for this research study. Moreover $\theta(0) = \tilde{\theta}$, where $\tilde{\theta}$ is the deterioration coefficient under natural specifications.
3. We assumed the rate of production is a function of selling price p (or $D(p) = a - bp$) and inversely proportional to the parameter λ; $0 < \lambda < 1$, i.e.,
$$P = \frac{D(p)}{\lambda} = \frac{a - bp}{\lambda}.$$
4. Shortages are allowed, and occur when demand crosses the inventory level during the lead time. Unsatisfied demand is backlogged.
5. The demand rate $D(p, t)$ is known, continuous, and differentiable function of the selling price and inventory level of products, taking the form of
$$D(p, t) = \begin{cases} D(p) + \alpha I_i(t), & I_i(t) > 0, i = 1, 2,\ 0 < \alpha < 1; \\ D(p), & I(t) \leq 0. \end{cases}$$
Here $D(p) = a - bp$, where $a > 0$ and $b > 0$; is a non-negative, linearly decreasing function of the selling price. The linear demand function has been widely put upon in some literature e.g., (Wang and Li [50]; Giri and Bardhan [13]; Chen and Chen [7]; Zhang et al., [56]; Zhang et al., [55]; Karray [18]; Zhang et al., [57]).
6. We divided the total time period T into n equal sub-intervals (cycles) performing similar action.
7. During the period, $[t_1, t_1']$, there is no production. In this period, the product deteriorates under the preservation investment cost. Besides, there is no replacement or repair of deteriorated units during the inventory cycle.
8. The production rate, demand rate and deterioration rate are always same in every cycle.
9. Holding cost is constant and known.

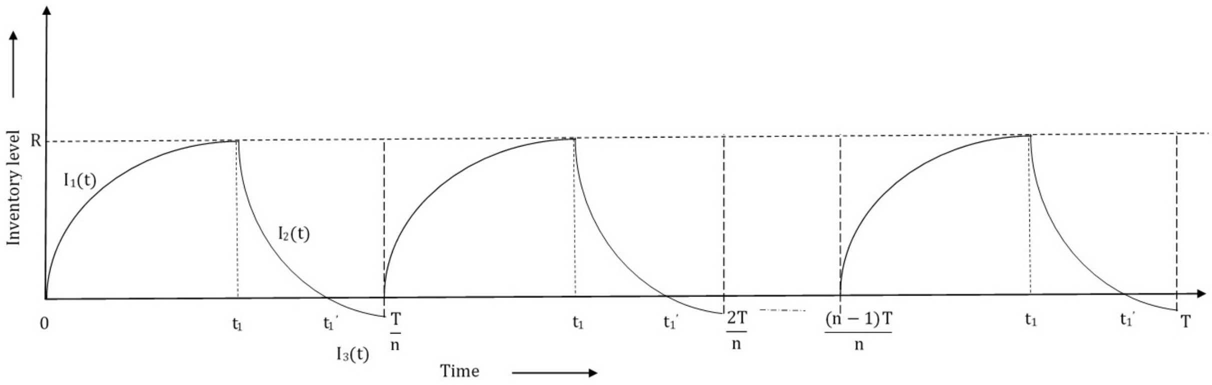

Figure 1: Production inventory model with the complete backlog

3 Mathematical Formulation

In the model formulation of a production inventory system for deteriorating item under investment on preservation techniques, two situations may be considered. First, the EPQ model with complete backordering and second, the EPQ model with partial backordering.

3.1 The EPQ inventory model with complete backordering

The demeanor of the inventory system can be depicted as in Fig. 1. The variance of the inventory takes place due to the combined effects of the production, demand, and deterioration, occurring simultaneously, during the interval $[0, t_1]$ of the first cycle. Because the inventory level in this interval is greater than zero, the corresponding demand rate is $D(p) + \alpha I(t)$. Implementing this, the variation of the inventory with respect to time t can be described by the following differential equation:

$$\frac{dI_1(t)}{dt} + \theta(\xi)I_1(t) = P - D(p) - \alpha I_1(t), \quad 0 \leq t \leq t_1, \tag{3.1}$$

with boundary condition $I_1(0) = 0$ and $I_1(t_1) = R$. On the other hand, the depletion of inventory takes place due to concerted effects of the demand and deterioration during the interval $[t_1, {t_1}^{'}]$. The variation of the inventory with respect to time t can be depicted by the following differential equation:

$$\frac{dI_2(t)}{dt} + \theta(\xi)I_2(t) = -D(p) - \alpha I_2(t), \quad t_1 \leq t \leq t_1^{'} \tag{3.2}$$

with boundary condition $I_2(t_1) = R$ and $I_2({t_1}^{'}) = 0$. Afterward, shortages are allowed to happen and entire demand in the period $[{t_1}^{'}, T/n]$ is completely backordered. Hence, the inventory level is governed by the following differential equation:

$$\frac{dI_3(t)}{dt} = -D(p), \quad {t_1}^{'} \leq t \leq T/n \tag{3.3}$$

with boundary condition $I_3({t_1}^{'}) = 0$.
Using boundary conditions, the solution of the Eqs. (3.1), (3.2) and (3.3) are given by:

$$I_1(t) = \frac{P - D(p)}{\theta(\xi) + \alpha}(1 - e^{-(\theta(\xi)+\alpha)t}), \quad 0 \leq t \leq t_1 \tag{3.4}$$

$$I_2(t) = \frac{D(p)}{\theta(\xi) + \alpha}(e^{-(\theta(\xi)+\alpha)({t_1}^{'}-t)} - 1), \quad t_1 \leq t \leq t_1^{'} \tag{3.5}$$

$$I_3(t) = D(p)(t_1{'} - t), \quad t_1{'} \le t \le T/n \tag{3.6}$$

Moreover, from the continuity of the inventory level at time t_1, i.e. considering Eqs. (3.4) and (3.5) and the condition $I_1(t_1) = I_2(t_1)$, we have the following result:

$$\frac{P - D(p)}{\theta(\xi) + \alpha}(1 - e^{-(\theta(\xi)+\alpha)t_1}) = \frac{D(p)}{\theta(\xi) + \alpha}(e^{-(\theta(\xi)+\alpha)(t_1{'} - t_1)} - 1), \tag{3.7}$$

which gives the expression of $t_1{'}$.
Total produced quantity (Q) per inventory cycle is

$$Q = Pt_1. \tag{3.8}$$

Total number of products that become deteriorated (D_θ) during $[0, T/n]$ is

$$D_\theta = Q - \int_0^{t_1{'}} D(p,t)\ dt. \tag{3.9}$$

The maximum amount of shortage per cycle is

$$l_s\left(\frac{T}{n}\right) = D(p)(T/n - t_1{'}). \tag{3.10}$$

The objective function includes the sales revenue $R(p,n,\xi)$, the production cost $P(n)$, the holding cost $H(p,n,\xi)$, the shortage cost $S(p,n)$, the deterioration cost $D(n,p,\xi)$, and the preservation technology cost $P(\xi)$. When the finite planning horizon is T. $\Pi(p,n,\xi)$ is the total profit function of the inventory system, then

Total Profit = Sales revenue-Production cost - Holding cost - Shortage cost - Deterioration cost - Preservation technology cost

$$\Pi(n,p,\xi) = R(n,p,\xi) - P(n) - H(n,p,\xi) - S(n,p) - D(n,p,\xi) - P(\xi),\ 0 \le t \le T \tag{3.11}$$

where

$$R(n,p,\xi) = pD(p)T + np\alpha \int_0^{t_1{'}} I(t)\ dt, \tag{3.12}$$

$$P(n) = nct_1P, \tag{3.13}$$

$$H(n,p,\xi) = nh\int_0^{t_1{'}} I(t)\ dt, \tag{3.14}$$

$$S(n,p) = nsD(p)\left(\frac{t_1{'}T}{n} - \frac{T^2}{2n^2} - \frac{t_1{'}^2}{2}\right), \tag{3.15}$$

$$D(n,p,\xi) = nd\left(Q - \int_0^{t_1{'}} D(p,t)\ dt\right) = ndt_1P - ndt_1{'}D(p) - nd\alpha\int_0^{t_1{'}} I(t)\ dt, \tag{3.16}$$

$$P(\xi) = \xi\ T. \tag{3.17}$$

Therefore, the total profit function is:

$$\Pi(n,p,\xi) = pD(p)T + np\alpha \int_0^{t_1{}'} I(t)\ dt - nct_1P - nh\int_0^{t_1{}'} I(t)\ dt - nsD(p)\left(\frac{t_1{}'T}{n} - \frac{T^2}{2n^2} - \frac{t_1{}'^2}{2}\right)$$
$$- ndt_1P + ndt_1{}'D(p) + nd\alpha\int_0^{t_1{}'} I(t)\ dt - \xi T$$

$$\begin{aligned}\Pi(n,p,\xi) &= pD(p)T - ncPt_1 - ndPt_1 + ndD(p)t_1{}' - \xi T - nsD(p)\left(\frac{t_1{}'T}{n} - \frac{T^2}{2n^2} - \frac{t_1{}'^2}{2}\right)\\ &+ n(p\alpha - h + d\alpha)\left[\frac{P}{(\theta(\xi)+\alpha)^2}\left\{(\theta(\xi)+\alpha)t_1 - 1 + e^{-(\theta(\xi)+\alpha)t_1}\right\}\right.\\ &\left.+\frac{D(p)}{(\theta(\xi)+\alpha)^2}\left\{(e^{(\theta(\xi)+\alpha)(t_1{}'-t_1)} - e^{-(\theta(\xi)+\alpha)t_1} - (\alpha+\theta(\xi))t_1{}')\right\}\right].\end{aligned} \tag{3.18}$$

If we consider the maximum constraint $\bar{\xi}$, then our problem can be formulated as:
Max: $\Pi(n,p,\xi)$
Subject to: $n \geq 1$, $p > 0$, $0 \leq \xi \leq \bar{\xi}$, and $D(p) > 0$.

It is worthwhile referring here that in order to optimize the above non-linear optimization problem first the objective function is simplified. Putting $t_1 = \frac{\rho T}{n}$, $0 < \rho < 1$, $P = \frac{D(p)}{\lambda}$, $0 < \lambda < 1$ and value of $t_1{}'$ in Eq. (3.18). For small value of x, the exponential function has an approximation of $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$, by the Taylor series expansion. Then, the total profit of the inventory system can be rewritten in a simplified form as follows:

$$\begin{aligned}\Pi(n,p,\xi) = \frac{T}{6}&\left[6D(p)p - 6\xi + \frac{3sTD(p)(\rho-\lambda)^2}{n\lambda^2} - \frac{6cD(p)\rho}{\lambda}\right.\\ &\left.+\frac{D(p)T\rho^2(h-d\alpha-p\alpha)(\lambda-1)\left\{3n\lambda - T\rho(2\lambda-1)(\alpha+\theta(\xi))\right\}}{n^2\lambda^3}\right]\end{aligned} \tag{3.19}$$

3.2 The EPQ inventory model with partial backordering

Following the earlier assumptions, the inventory level follows the pattern depicted in Fig. 2. In this inventory model the differential equation describing the inventory levels $I_1(t)$ over the production period $[0, t_1]$ and $I_2(t)$ during period $[t_1, t_1{}']$ of this system is similar to Section 3.1. The differential equation representing the backorder level during period $[t_1{}', \frac{T}{n}]$ is expressed as:

$$\frac{dl_b(t)}{dt} = D(p)e^{-\vartheta\left(\frac{T}{n}-t\right)};\quad t_1{}' \leq t \leq \frac{T}{n} \tag{3.20}$$

Now, the boundary condition is $l_b(t_1{}') = 0$. By using it, we obtain

$$l_b(t) = \frac{D(p)}{\vartheta}\left[e^{-\vartheta\left(\frac{T}{n}-t\right)} - e^{-\vartheta\left(\frac{T}{n}-t_1{}'\right)}\right];\quad t_1{}' \leq t \leq \frac{T}{n}. \tag{3.21}$$

The maximum backorder level per cycle is

$$l_b\left(\frac{T}{n}\right) = \frac{D(p)}{\vartheta}\left[1 - e^{-\vartheta\left(\frac{T}{n}-t_1{}'\right)}\right]. \tag{3.22}$$

The number of lost sales at time t is given by

$$L(t) = D(p)\left[(t - t_1{}') - \frac{1}{\vartheta}\left\{e^{-\vartheta\left(\frac{T}{n}-t\right)} - e^{-\vartheta\left(\frac{T}{n}-t_1{}'\right)}\right\}\right]. \tag{3.23}$$

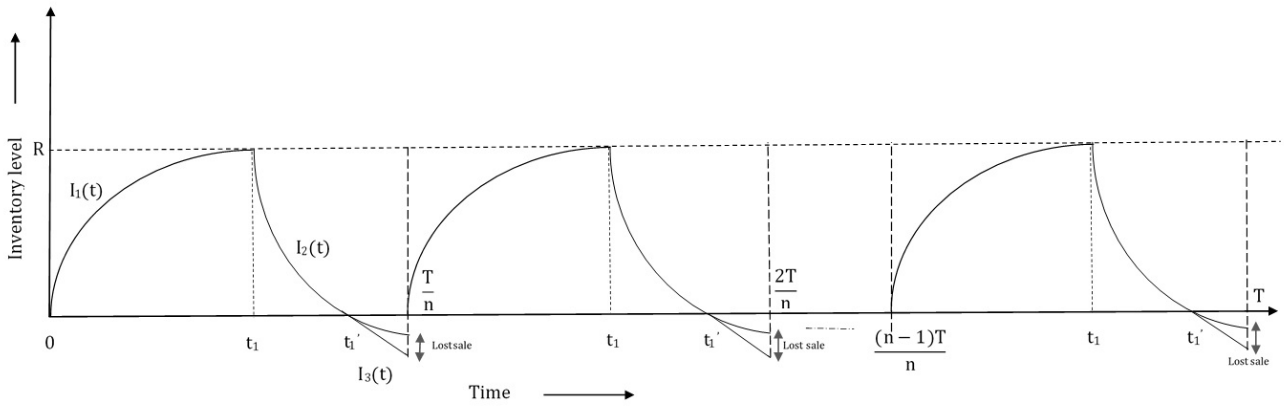

Figure 2: Production inventory model with the partial backlog

The quantity produce over the replenishment cycle is determined as

$$Q = Pt_1 + l_b\left(\frac{T}{n}\right) = Pt_1 + \frac{D(p)}{\vartheta}\left[1 - e^{-\vartheta\left(\frac{T}{n} - {t_1}'\right)}\right]. \tag{3.24}$$

The shortage cost is $S'(p,n) = ns\int_{{t_1}'}^{\frac{T}{n}} l_b(t)\ dt$

$$S'(n,p) = \frac{nsD(p)}{\vartheta}\left[\frac{1}{\vartheta}\left(1 - e^{-\vartheta\left(\frac{T}{n} - {t_1}'\right)}\right) - e^{-\vartheta\left(\frac{T}{n} - {t_1}'\right)}\left(\frac{T}{n} - {t_1}'\right)\right]. \tag{3.25}$$

The lost sale cost is $LS(n,p) = n\pi\int_{{t_1}'}^{\frac{T}{n}}\left[1 - e^{-\vartheta\left(\frac{T}{n} - t\right)}\right]D(p)\ dt$

$$LS(n,p) = n\pi D(p)\left[\left(\frac{T}{n} - {t_1}'\right) - \frac{1}{\vartheta}\left(1 - e^{-\vartheta\left(\frac{T}{n} - {t_1}'\right)}\right)\right] \tag{3.26}$$

The objective function includes the sale revenues $R(n,p,\xi)$, the production cost $P(n)$, the holding cost $H(n,p,\xi)$, the shortage cost $S'(n,p)$, the deterioration cost $D(n,p,\xi)$, the preservation technology cost $P(\xi)$, and the lost sale cost $LS(n,p)$. $\Pi(n,p,\xi)$ is the total profit function of the production inventory model, Q is the quantity produced with partial backordering over the cycle. Then, combined with the relevant cost components mentioned above, we can simplify the total profit of the inventory system as:
The Total Profit = sales revenue - production cost - holding cost - shortage cost - deterioration cost - preservation technology cost - lost sale cost

$$\begin{aligned}
\Pi(n,p,\xi) = {} & pTD(p) + np\alpha\int_0^{{t_1}'} I(t)\ dt - ncPt_1 - nh\int_0^{{t_1}'} I(t)\ dt - \frac{nsD(p)}{\vartheta}\left[\frac{1}{\vartheta}\left(1 - e^{-\vartheta\left(\frac{T}{n} - {t_1}'\right)}\right)\right. \\
& \left. - e^{-\vartheta\left(\frac{T}{n} - {t_1}'\right)}\left(\frac{T}{n} - {t_1}'\right)\right] - \left[ndPt_1 - ndD(p){t_1}' - nd\alpha\int_0^{{t_1}'} I(t)\ dt\right] - \xi T \\
& - n\pi D(p)\left[\left(\frac{T}{n} - {t_1}'\right) - \frac{1}{\vartheta}\left(1 - e^{-\vartheta\left(\frac{T}{n} - {t_1}'\right)}\right)\right]
\end{aligned}$$

$$\begin{aligned}\Pi(n,p,\xi) &= pTD(p) - ncPt_1 - \frac{nsD(p)}{\vartheta}\left[\frac{1}{\vartheta}\left(1-e^{-\vartheta\left(\frac{T}{n}-{t_1}'\right)}\right) - e^{-\vartheta\left(\frac{T}{n}-{t_1}'\right)}\left(\frac{T}{n}-{t_1}'\right)\right]\\
&- ndPt_1 + ndD(p){t_1}' - \xi T - n\pi D(p)\left[\left(\frac{T}{n}-{t_1}'\right) - \frac{1}{\vartheta}\left(1-e^{-\vartheta\left(\frac{T}{n}-{t_1}'\right)}\right)\right]\\
&+ n(p\alpha - h + d\alpha)\left[\frac{P}{(\theta(\xi)+\alpha)^2}\left\{(\theta(\xi)+\alpha)\,t_1 - 1 + e^{-(\theta(\xi)+\alpha)t_1}\right\}\right.\\
&\left.+\frac{D(p)}{(\theta(\xi)+\alpha)^2}\left\{(e^{(\theta(\xi)+\alpha)({t_1}'-t_1)} - e^{-(\theta(\xi)+\alpha)t_1} - (\alpha+\theta(\xi)){t_1}'\right\}\right].\end{aligned} \tag{3.27}$$

Hence the problem, can be formulated as follows:
max $\Pi(n,p,\xi)$
s.t. $n \geq 1$, $p > 0$, $0 \leq \xi \leq \bar{\xi}$, and $D(p) > 0$.

Let $t_1 = \frac{\rho T}{n}$. For a small x value, the Taylor series approximated the exponential function to $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$. Putting the value of ${t_1}'$, extracted from Eq. (3.7), in Eq. (5.16), it follows that

$$\begin{aligned}\Pi(n,p,\xi) &= pD(p)T - \frac{cD(p)\rho T}{\lambda} - \xi T - nD(p)\left[\frac{s}{\vartheta^2} + \pi\left(\frac{T}{n} - \frac{1}{\vartheta} - \frac{T\rho}{n\lambda}\right)\right]\\
&+ n(p\alpha - h + d\alpha)\left[\frac{\rho^2T^2D(p)}{6n^2}\left(\frac{1}{\lambda}-1\right)\left\{3 - (\theta(\xi)+\alpha)\frac{T\rho}{n}\right\} + \frac{D(p)}{6}\left(\frac{T\rho}{n\lambda} - \frac{T\rho}{n}\right)^2\right.\\
&\left.\left\{3 + \left(\frac{T\rho}{n\lambda} - \frac{T\rho}{n}\right)(\theta(\xi)+\alpha)\right\}\right] + \frac{nD(p)}{\vartheta}\left[s\left(\frac{1}{\vartheta} + \frac{T}{n} - \frac{T\rho}{n\lambda}\right) - \pi\right]\\
&\left[1 - \vartheta\left(\frac{T}{n} - \frac{T\rho}{n\lambda}\right) + \frac{\vartheta^2}{2}\left(\frac{T}{n} - \frac{T\rho}{n\lambda}\right)^2 - \frac{\vartheta^3}{6}\left(\frac{T}{n} - \frac{T\rho}{n\lambda}\right)^3\right].\end{aligned} \tag{3.28}$$

The purpose of this paper is to find the optimal production, selling price and preservation technology cost so that $\Pi(n,p,\xi)$ is maximum.

4 Solution procedure

Since the objective function $\Pi(n,p,\xi)$ has three variables n, p and ξ with the constraints $n \geq 1$, $p > 0$, and $0 \leq \xi \leq \bar{\xi}$, where n is a positive integer and p and ξ are real variables, the optimal p^* and ξ^* can be derived with a given n, then by computing the optimal profit Π for all possible n. n^* can be derived.
Since the ordering frequency, n, is a discrete variable, the necessary condition for the optimal profit $\Pi(n,p,\xi)$ at $n = n^*$, $p = p^*$, $\xi = \xi^*$ is

$$\Delta\Pi(n^*,p^*,\xi^*) < 0 < \Delta\Pi(n^*-1,p^*,\xi^*)$$

where

$$\Delta\Pi(n^*,p^*,\xi^*) = \Pi(n^*+1,p^*,\xi^*) - \Pi(n^*,p^*,\xi^*).$$

5 Computational results and optimal solution

5.1 The EPQ inventory model with complete backordering

Now, the problem is to determine n, p and ξ such that $\Pi(n,p,\xi)$ is maximum, put $\theta(\xi) = \tilde{\theta}e^{-\psi\xi} \approx \tilde{\theta}(1-\psi\xi)$ in Eq. (3.19) for $D(p) = a - bp > 0$, the equation reduces to

$$\begin{aligned}\Pi(n,p,\xi) &= \frac{T}{6}\left[6(a-bp)p - 6\xi + \frac{3sT(a-bp)(\rho-\lambda)^2}{n\lambda^2} - \frac{6c(a-bp)\rho}{\lambda}\right.\\
&\left.+\frac{(a-bp)T\rho^2(h-d\alpha-p\alpha)(\lambda-1)\left\{3n\lambda - T\rho(2\lambda-1)(\alpha+\tilde{\theta}-\tilde{\theta}\psi\xi)\right\}}{n^2\lambda^3}\right].\end{aligned} \tag{5.1}$$

To maximize the total profit of the inventory system, it is necessary to solve the following equations simultaneously.

$$\begin{aligned}\frac{\partial \Pi(n,p,\xi)}{\partial p} &= \frac{T}{6}\left[-6bp+6(a-bp)-\frac{3bsT(\rho-\lambda)^2}{n\lambda^2}+\frac{6bc\rho}{\lambda}\right.\\ &-\frac{(a-bp)T\alpha\rho^2(\lambda-1)\left\{3n\lambda-T\rho(2\lambda-1)(\alpha+\tilde{\theta}-\tilde{\theta}\psi\xi)\right\}}{n^2\lambda^3}\\ &\left.-\frac{bT(h-d\alpha-p\alpha)\rho^2(\lambda-1)\left\{3n\lambda-T\rho(2\lambda-1)(\alpha+\tilde{\theta}-\tilde{\theta}\psi\xi)\right\}}{n^2\lambda^3}\right]=0,\end{aligned} \tag{5.2}$$

$$\frac{\partial \Pi(n,p,\xi)}{\partial \xi} = \frac{T}{6}\left[-6+\frac{(a-bp)T^2\rho^3\tilde{\theta}\psi(h-d\alpha-p\alpha)(\lambda-1)(2\lambda-1)}{n^2\lambda^3}\right]=0, \tag{5.3}$$

and

$$\begin{aligned}\frac{\partial \Pi(n,p,\xi)}{\partial n} &= \frac{T}{6}\left[-\frac{3(a-bp)sT(\rho-\lambda)^2}{n^2\lambda^2}+\frac{3(a-bp)T(h-d\alpha-p\alpha)\rho^2(\lambda-1)}{n^2\lambda^2}\right.\\ &\left.-\frac{2(a-bp)T(h-d\alpha-p\alpha)\rho^2(\lambda-1)\left\{3n\lambda-T\rho(2\lambda-1)(\alpha+\tilde{\theta}-\tilde{\theta}\psi\xi)\right\}}{n^3\lambda^3}\right]=0.\end{aligned} \tag{5.4}$$

Proposition 5.1. For any given feasible n such that $1-3\lambda+2\lambda^2>0$ and $h-d\alpha-2p\alpha>0$,

1. The system of Eqs. (5.2) and (5.3) has a unique solution.
2. The solution in (a) satisfies the second-order conditions for a local maxima.

Proof . Eqs. (5.2) and (5.3) can be solved simultaneously to derive the values of p^* and ξ^*. The mathematical software MATHEMATICA and MATLAB are used in the analysis.

$$p^* = \frac{1}{2}\left[\frac{a}{b}-d+\frac{h}{\alpha}-\frac{\sqrt{k}}{bT^2\tilde{\theta}\alpha\rho^3\psi(1-3\lambda+2\lambda^2)}\right] \tag{5.5}$$

where

$$\begin{aligned}k &= T^2\tilde{\theta}\rho^3\psi(1-3\lambda+2\lambda^2)\left[T^2\tilde{\theta}(a\alpha+bh-db\alpha)^2\rho^3(1-3\lambda+2\lambda^2)\psi\right.\\ &\left.+4b\alpha\{6n^2\lambda^3-aT^2\tilde{\theta}(h-d\alpha)\rho^3(1-3\lambda+2\lambda^2)\psi\}\right].\end{aligned}$$

and

$$\xi^* = \frac{m}{\alpha S_q} \tag{5.6}$$

where

$$\begin{aligned}S_q &= T^2\tilde{\theta}\rho^3\psi(1-3\lambda+2\lambda^2)\left[a^2T^2\alpha^2\rho^3\tilde{\theta}\psi(1-3\lambda+2\lambda^2)+b^2T^2\tilde{\theta}\rho^3\psi(h-d\alpha)^2(1-3\lambda+2\lambda^2)\right.\\ &\left.+2b\alpha\left\{12n^2\lambda^3+aT^2(d\alpha-h)\rho^3\tilde{\theta}\psi(1-3\lambda+2\lambda^2)\right\}\right],\end{aligned}$$

$$
\begin{aligned}
m = {} & a^2T^2\tilde{\theta}\alpha^2\rho^3\psi(1-3\lambda+2\lambda^2)\left\{-3nT\alpha\rho^2\lambda(\lambda-1)+6n^2\lambda^3+T^2\alpha(1-3\lambda+2\lambda^2)(\tilde{\theta}+\alpha)\rho^3\right\}\\
&+b^2T^2\tilde{\theta}(h-d\alpha)^2\rho^3\psi(1-3\lambda+2\lambda^2)\left\{-3nT\alpha\rho^2\lambda(\lambda-1)+6n^2\lambda^3+T^2\alpha(1-3\lambda+2\lambda^2)(\tilde{\theta}+\alpha)\rho^3\right\}\\
&+b\Big\{-72n^3T\alpha^2\rho^2(\lambda-1)\lambda^4+144n^4\alpha\lambda^6+2aT^4\tilde{\theta}\alpha^2(\tilde{\theta}+\alpha)(d\alpha-h)\rho^6\psi(1-3\lambda+2\lambda^2)^2\\
&+3nT\alpha\lambda\left(2aT^2\tilde{\theta}\alpha(h-d\alpha)\rho^5(\lambda-1)^2(2\lambda-1)\psi-s(\rho-\lambda)^2\sqrt{S_q}\right)\\
&+6n^2\lambda^2\left(2T^2\alpha\rho^3\lambda(1-3\lambda+2\lambda^2)\{2\alpha(\tilde{\theta}+\alpha)+a\tilde{\theta}\psi(d\alpha-h)\}+(c\alpha\rho-h\lambda+d\alpha\lambda)\sqrt{S_q}\right)\Big\}.
\end{aligned}
$$

Therefore, the optimal solution (p^*,ξ^*) exists and is unique for any given n. This completes the proof of Part (a). Now, considering $\theta(\xi)\approx\tilde{\theta}(1-\psi\xi+\frac{\psi^2\xi^2}{2!})$ in Eq.(3.19). Taking the second-order partial derivatives of $\Pi(n,p,\xi)$ with respect to p and ξ yields

$$
\frac{\partial^2\Pi(p,\xi|n)}{\partial p^2}=-\frac{bT}{3}\left[6+\frac{T\alpha\rho^2(1-\lambda)\left\{3n\lambda+T\rho(1-2\lambda)(\alpha+\tilde{\theta}(1-\psi\xi+\frac{\psi^2\xi^2}{2}))\right\}}{n^2\lambda^3}\right], \tag{5.7}
$$

$$
\frac{\partial^2\Pi(p,\xi|n)}{\partial \xi^2}=\frac{-(a-bp)T^3\tilde{\theta}(h-d\alpha-p\alpha)\rho^3(1-\lambda)(1-2\lambda)\psi^2}{6n^2\lambda^3}, \tag{5.8}
$$

$$
\frac{\partial^2\Pi(p,\xi|n)}{\partial p\partial \xi}=\frac{T^3\tilde{\theta}\rho^3\psi(1-\lambda)(1-2\lambda)(\xi\psi-1)\left\{a\alpha+b(h-d\alpha-2p\alpha)\right\}}{6n^2\lambda^3}. \tag{5.9}
$$

It is easy to check that $\frac{\partial^2\Pi(p,\xi|n)}{\partial p^2}<0$, and $\frac{\partial^2\Pi(p,\xi|n)}{\partial \xi^2}<0$, by virtue of the fact $0<\lambda<1$, $h-d\alpha-p\alpha>0$, and thus the determinant of Hessian matrix becomes

$$
\begin{aligned}
det(H) &= \frac{\partial^2\Pi(p,\xi|n)}{\partial p^2}.\frac{\partial^2\Pi(p,\xi|n)}{\partial \xi^2}-\left[\frac{\partial^2\Pi(p,\xi|n)}{\partial p\partial \xi}\right]^2\\
&= \frac{T^4\tilde{\theta}\rho^3(1-\lambda)(1-2\lambda)\psi^2}{36n^4\lambda^6}\Big[b(a-bp)(h-d\alpha-p\alpha)\big\{12n^2\lambda^3\\
&\quad +6nT\alpha\rho^2\lambda(1-\lambda)+T^2\alpha\rho^3(1-3\lambda+2\lambda^2)\{2\alpha+\tilde{\theta}(2-2\xi\psi+\xi^2\psi^2)\}\big\}\\
&\quad -T^2\tilde{\theta}\rho^3(1-\lambda)(1-2\lambda)(\xi\psi-1)^2(a\alpha+bh-bd\alpha-2bp\alpha)^2\Big]\geq 0.
\end{aligned} \tag{5.10}
$$

This means the Hessian matrix is semi-negative definite. Consequently, we can get that the optimal point (p^*,ξ^*) gives the maximum value of $\Pi(n,p,\xi)$, for any given n. This completes the proof of Part (b). □

From the above synthesis, we have obtained that, for any given n, the point (p^*,ξ^*) maximizing the total profit not only exists but also is unique. Next, we will inquire the terms under which the optimal production cycle n^* not only exists but also is unique. For any given feasible n and p, taking the second partial derivative of (5.1) with respect to n yields

$$
\begin{aligned}
\frac{\partial^2\Pi(n|p,\xi)}{\partial n^2} &= -\frac{(a-bp)T^2}{n^4\lambda^3}\big[2n(h-d\alpha-p\alpha)\rho^2(\lambda-1)\lambda-ns(\rho-\lambda)^2\lambda\\
&\quad +(h-d\alpha-p\alpha)\rho^2(1-\lambda)\left\{3n\lambda-T\rho(2\lambda-1)(\alpha+\theta(\xi))\right\}\big].
\end{aligned} \tag{5.11}
$$

Proposition 5.2. There exists a unique n^* that maximizes total profit function $\Pi(n,p,\xi)$ for fixed values of p and ξ.

Proof . To acquire the optimal production that maximizes $\Pi(n|p,\xi)$, n^* should be selected to satisfy

$$\frac{\partial \Pi(n|p,\xi)}{\partial n} = \frac{T}{6}\Bigg[-\frac{3(a-bp)sT(\rho-\lambda)^2}{n^2\lambda^2} + \frac{3(a-bp)T(h-d\alpha-p\alpha)\rho^2(\lambda-1)}{n^2\lambda^2} - \frac{2(a-bp)T(h-d\alpha-p\alpha)\rho^2(\lambda-1)\left\{3n\lambda - T\rho(2\lambda-1)(\alpha+\tilde{\theta}-\tilde{\theta}\psi\xi)\right\}}{n^3\lambda^3}\Bigg] = 0 \tag{5.12}$$

provided it satisfies the sufficient condition

$$\left.\frac{\partial^2 \Pi(n|p,\xi)}{\partial n^2}\right|_{n=n^*} < 0,$$

for any feasible (p,ξ). Because $\Pi(n|p,\xi)$ is a very complicated function due to the high-power expression in n, it is difficult to solve the nonlinear equation (5.12) analytically. Although we are not accomplished to prove the uniqueness of the solution, we will consider a numerical approach to ensure the concavity of (5.1) and the unique solution to (5.12) in Section 6. □

Theorem 5.3. For any given feasible (n,p) such that $h-d\alpha-p^*\alpha>0$ and $0<\lambda<1/2$. If the reduced deterioration rate $\theta(\xi)$, is a strictly convex function of ξ (i.e. $\theta'(\xi)<0$ and $\theta''(\xi)>0$), then the total profit, $\Pi(\xi|n,p)$, is a strictly concave function with respect to ξ.

Proof . From $\theta(\xi)=\tilde{\theta}e^{-\psi\xi}$, we can know that θ is a function of ξ. Because, by assumptions, $\theta(\xi)$ is a strictly convex function of ξ, it is not difficult to check that $\theta''(\xi)>0$. Differentiating (3.19) with respect to ξ, it follows that

$$\frac{\partial^2 \Pi(\xi|n,p)}{\partial \xi^2} = \frac{-D(p)T^3\rho^3(h-d\alpha-p\alpha)(1-\lambda)(1-2\lambda)\theta''(\xi)}{6n^2\lambda^3}$$

using the results, $h-d\alpha-p\alpha>0$, and $0<\lambda<1/2$, we can conclude that $\partial^2\Pi(\xi|n,p)/\partial\xi^2<0$, for any feasible (n,p). This implies that $\Pi(\xi|n,p)$ is concave in ξ for given values of n and p and results in a maximum value. □

At present, we can realize that $\Pi(\xi|n,p)$ is strictly concave in ξ, and so there exists a unique value of ξ such that $\Pi(\xi|n,p)$ is maximum. Because ξ is bounded on $[0,\bar{\xi}]$, the above derivation also indicates that the optimal ξ^* should be selected to satisfy

$$\frac{\partial \Pi(\xi|n,p)}{\partial \xi} = 0;$$

otherwise

$$\xi^* = \begin{cases} 0, & \text{if } \left.\frac{\partial \Pi(\xi|n,p)}{\partial \xi}\right|_{\xi=0} < 0 \\ \bar{\xi}, & \text{if } \left.\frac{\partial \Pi(\xi|n,p)}{\partial \xi}\right|_{\xi=\bar{\xi}} > 0. \end{cases}$$

Proposition 5.4. For all (n,p) such that $h-d\alpha-p^*\alpha>0$ and $0<\lambda<1/2$, the total profit of the production-inventory $\Pi(n,p|\xi)$, is a strictly concave function of (n,p).

Proof . From Eq.(3.19), we obtain

$$\frac{\partial \Pi(n,p|\xi)}{\partial n} = \frac{T}{6}\Bigg[-\frac{3(a-bp)sT(\rho-\lambda)^2}{n^2\lambda^2} + \frac{3(a-bp)T(h-d\alpha-p\alpha)\rho^2(\lambda-1)}{n^2\lambda^2} - \frac{2(a-bp)T(h-d\alpha-p\alpha)\rho^2(\lambda-1)\left\{3n\lambda - T\rho(2\lambda-1)(\alpha+\theta(\xi))\right\}}{n^3\lambda^3}\Bigg] = 0 \tag{5.13}$$

$$\frac{\partial \Pi(n,p|\xi)}{\partial p} = \frac{T}{6}\Bigg[-6bp + 6(a-bp) - \frac{3bsT(\rho-\lambda)^2}{n\lambda^2} + \frac{6bc\rho}{\lambda} - \frac{(a-bp)T\alpha\rho^2(\lambda-1)\{3n\lambda - T\rho(2\lambda-1)(\alpha+\theta(\xi))\}}{n^2\lambda^3} - \frac{bT(h-d\alpha-p\alpha)\rho^2(\lambda-1)\left\{3n\lambda - T\rho(2\lambda-1)(\alpha+\theta(\xi))\right\}}{n^2\lambda^3}\Bigg] = 0 \tag{5.14}$$

Let (n^*, p^*) be the solution of (5.13) and (5.14), then the second-order condition for maximization becomes

$$\left.\frac{\partial^2\Pi(n,p|\xi)}{\partial n^2}\right|_{(n^*,p^*)} = -\frac{(a-bp^*)T^2}{n^{*4}\lambda^3}\left[(h-d\alpha-p^*\alpha)\rho^2(1-\lambda)\left\{n^*\lambda+T\rho(1-2\lambda)(\alpha+\theta(\xi))\right\}\right.$$
$$\left.-n^*s(\rho-\lambda)^2\lambda\right] < 0,$$

$$\left.\frac{\partial^2\Pi(n,p|\xi)}{\partial p^2}\right|_{(n^*,p^*)} = -\frac{bT}{6}\left[12+\frac{2T\alpha\rho^2(1-\lambda)\left\{3n^*\lambda+T\rho(1-2\lambda)(\alpha+\theta(\xi))\right\}}{n^{*2}\lambda^3}\right] < 0$$

and

$$\left.\frac{\partial^2\Pi(n,p|\xi)}{\partial p\partial n}\right|_{(n^*,p^*)} = \frac{T}{6}\left[\frac{3bsT(\rho-\lambda)^2}{n^{*2}\lambda^2}-\frac{3(a-bp^*)T\alpha\rho^2(\lambda-1)}{n^{*2}\lambda^2}-\frac{3bT(h-d\alpha-p^*\alpha)\rho^2(\lambda-1)}{n^{*2}\lambda^2}\right.$$
$$+\frac{2(a-bp^*)T\alpha\rho^2(\lambda-1)\left\{3n^*\lambda-T\rho(2\lambda-1)(\alpha-\theta(\xi))\right\}}{n^{*3}\lambda^3}$$
$$\left.+\frac{2bT(h-d\alpha-p^*\alpha)\rho^2(\lambda-1)\left\{3n^*\lambda-T\rho(2\lambda-1)(\alpha+\theta(\xi))\right\}}{n^{*3}\lambda^3}\right].$$

Thus, the determinant of the Hessian matrix at the stationary point (n^*, p^*) is

$$det(H) = \left.\frac{\partial^2\Pi(n,p|\xi)}{\partial p^2}\right|_{(n^*,p^*)}\cdot\left.\frac{\partial^2\Pi(n,p|\xi)}{\partial n^2}\right|_{(n^*,p^*)} - \left.\frac{\partial^2\Pi(n,p|\xi)}{\partial p\partial n}\right|_{(n^*,p^*)}$$
$$= \frac{T^3}{36n^{*6}\lambda^6}\left[12bn^{*2}(a-bp^*)\lambda^3\left\{n^*s(\rho-\lambda)^2\lambda+2n^*(h-d\alpha-p^*\alpha)\rho^2\lambda\right.\right.$$
$$\left.+(h-d\alpha-p^*\alpha)\rho^2(1-\lambda)\{3n^*\lambda+T\rho(1-2\lambda)(\alpha+\theta(\xi))\}\right\}$$
$$\left(6-\frac{T\alpha\rho^2(\lambda-1)\{3n^*\lambda-T\rho(2\lambda-1)(\alpha+\theta(\xi))\}}{n^{*2}\lambda^3}\right)$$
$$-T\left\{3n^*(a-bp^*)\alpha\rho^2(\lambda-1)\lambda-3bn^*s(\rho-\lambda)^2\lambda+3bn^*(h-d\alpha-p^*\alpha)\rho^2(\lambda-1)\lambda\right.$$
$$+2(a-bp^*)\alpha\rho^2(1-\lambda)\{3n^*\lambda+T\rho(1-2\lambda)(\alpha+\theta(\xi))\}$$
$$\left.\left.+2b(h-d\alpha-p^*\alpha)\rho^2(1-\lambda)\{3n^*\lambda-T\rho(2\lambda-1)(\alpha+\theta(\xi))\}\right\}^2\right] > 0.$$

Clearly, the Hessian matrix at point (n^*, p^*) is negative-definite and (n^*, p^*) represents maximum point of $\Pi(n, p|\xi)$. This completes the proof. The graph of $\Pi(n, p|\xi)$ with fixed parameters is depicted in Fig. 3 (using MATHEMATICA), which shows the concavity. □

Thus, combining Propositions 5.1 and 5.2, we exhibit the following iterative algorithm to find the solution (n^*, p^*, ξ^*).

Algorithm

Step 1. Set $i = 1$ and $\Pi_0 = 0$.

Step 2. Initialize $n_i = i$.

Step 3. Calculate the optimal selling price p_i from (5.5) for the given n_i.

Step 4. Calculate the optimal preservation technology cost ξ_i from (5.6), based on p_i, for the given n_i.

Step 5. Then calculate the total profit Π_i from (5.1) using (n_i, p_i, ξ_i). Also, check the optimality conditions.

Step 6. If the difference between Π_i and Π_{i-1} is positive, set $i = i + 1$ and go back to Step 2. Otherwise set $n^* = n_i$, $p^* = p_i$ and $\xi^* = \xi_i$.

Step 7. Then output (n^*, p^*, ξ^*) and stop.

Substituting the optimal strategies $n = n^*$, $p = p^*$ and $\xi = \xi^*$ obtained from Algorithm into (5.1) yields the optimal total profit of the inventory system $\Pi^*(n, p, \xi)$.

5.2 The EPQ inventory model with partial backordering

Using $\theta(\xi)=\tilde{\theta}e^{-\psi\xi}\approx\tilde{\theta}(1-\psi\xi)$ in (5.16) and taking the first-order partial derivative of $\Pi(n,p,\xi)$ with respect to n, p and ξ respectively, we have

$$\begin{aligned}\frac{\partial\Pi(n,p,\xi)}{\partial n}&=\frac{(a-bp)T^2}{6n^4\lambda^4}\left[3sT^2\vartheta^2(\lambda-\rho)^4+3n^2\lambda^2\left\{s(\lambda-\rho)^2+\pi\vartheta(\lambda-\rho)^2\right.\right.\\&\left.-(h-d\alpha-p\alpha)(\lambda-1)\rho^2\right\}-2nT\lambda\left\{2s\vartheta(\lambda-\rho)^3+\pi\vartheta^2(\lambda-\rho)^3\right.\\&\left.\left.-(h-d\alpha-p\alpha)(1-3\lambda+2\lambda^2)\rho^3(\alpha+\tilde{\theta}-\psi\tilde{\theta}\xi)\right\}\right]=0\end{aligned}\tag{5.15}$$

$$\begin{aligned}\frac{\partial\Pi(n,p,\xi)}{\partial p}&=\frac{T}{6n^3\lambda^4}\left[an\lambda\left\{6n^2\lambda^3-3nT\alpha(\lambda-1)\lambda\rho^2+T^2\alpha(1-3\lambda+2\lambda^2)\rho^3(\alpha+\tilde{\theta}-\psi\tilde{\theta}\xi)\right\}\right.\\&+b\left(sT^3\vartheta^2(\lambda-\rho)^4+6n^3\lambda^3(c\rho-2p\lambda)\right.\\&+3n^2T\lambda^2\left\{s(\lambda-\rho)^2+\pi\vartheta(\lambda-\rho)^2-(h-d\alpha-2p\alpha)(\lambda-1)\rho^2\right\}\\&+nT^2\lambda\left\{-2s\vartheta(\lambda-\rho)^3-\pi\vartheta^2(\lambda-\rho)^3\right.\\&\left.\left.\left.+(h-d\alpha-2p\alpha)(1-3\lambda+2\lambda^2)\rho^3(\alpha+\tilde{\theta}-\psi\tilde{\theta}\xi)\right\}\right)\right]=0\end{aligned}\tag{5.16}$$

$$\frac{\partial\Pi(n,p,\xi)}{\partial\xi}=-T+\frac{(a-bp)T^3(h-d\alpha-p\alpha)(1-3\lambda+2\lambda^2)\rho^3\tilde{\theta}\psi}{6n^2\lambda^3}=0\tag{5.17}$$

Our objective is to determine the optimal production cycle, the optimal selling price and the optimal preservation technology cost which maximizing the total profit of the EPQ model. To obtain our purpose, firstly, we prove that for any given n, the optimal solution of (p,ξ) not only exists but also is unique. Then for any given feasible value of (p,ξ), there exists a n which maximizes the total profit.

Proposition 5.5. For any given n with $\lambda<1/2$ and $h-d\alpha-p\alpha>0$, we have

(a) The system of equations $\frac{\partial\Pi(p,\xi|n)}{\partial p}=0$, and $\frac{\partial\Pi(p,\xi|n)}{\partial\xi}=0$ has a unique solution.
(b) The solution in (a) satisfies the second order conditions for maximum.

Proof . For any given n, the optimal solutions (p^*,ξ^*) should satisfy Eqs. (5.16) and (5.17) simultaneously, we get

$$p^*=\frac{1}{2}\left[\frac{a}{b}-d+\frac{h}{\alpha}-\frac{\sqrt{k}}{bT^2\tilde{\theta}\alpha\rho^3\psi(1-3\lambda+2\lambda^2)}\right]$$

where

$$\begin{aligned}k=&T^2\tilde{\theta}\rho^3\psi(1-3\lambda+2\lambda^2)\left[T^2\tilde{\theta}(a\alpha+bh-db\alpha)^2\rho^3(1-3\lambda+2\lambda^2)\psi\right.\\&\left.+4b\alpha\left\{6n^2\lambda^3-aT^2\tilde{\theta}(h-d\alpha)\rho^3(1-3\lambda+2\lambda^2)\psi\right\}\right]\end{aligned}$$

and

$$\xi^*=\frac{m}{n\lambda\alpha S_q}$$

where

$$m = a^2nT^2\tilde{\theta}\alpha^2\rho^3\psi\lambda(1-3\lambda+2\lambda^2)\left\{-3nT\alpha\rho^2\lambda(\lambda-1)+6n^2\lambda^3+T^2\alpha(1-3\lambda+2\lambda^2)(\tilde{\theta}+\alpha)\rho^3\right\}$$
$$+nb^2T^2\tilde{\theta}\lambda(h-d\alpha)^2\rho^3\psi(1-3\lambda+2\lambda^2)\left\{-3nT\alpha\rho^2\lambda(\lambda-1)+6n^2\lambda^3+T^2\alpha(1-3\lambda+2\lambda^2)(\tilde{\theta}+\alpha)\rho^3\right\}$$
$$+b\left\{-72n^4T\alpha^2\rho^2(\lambda-1)\lambda^5+144n^5\alpha\lambda^7+sT^3\vartheta^2\alpha(\lambda-\rho)^4\sqrt{S_q}\right.$$
$$+3n^2T\alpha\lambda^2\left(2aT^2\tilde{\theta}\alpha(h-d\alpha)\rho^5(\lambda-1)^2(2\lambda-1)\psi+(s+\pi\vartheta)(\rho-\lambda)^2\sqrt{S_q}\right)$$
$$+nT^2\alpha\lambda\left(2aT^2\alpha\rho^6\tilde{\theta}\psi(d\alpha-h)(1-3\lambda+2\lambda^2)(\tilde{\theta}+\alpha)+\vartheta(2s+\pi\vartheta)(\rho-\lambda)^3\sqrt{S_q}\right)$$
$$\left.+6n^3\lambda^3\left(2T^2\alpha\rho^3\lambda(1-3\lambda+2\lambda^2)\{2\alpha(\tilde{\theta}+\alpha)+a\tilde{\theta}\psi(d\alpha-h)\}+(c\alpha\rho-h\lambda+d\alpha\lambda)\sqrt{S_q}\right)\right\},$$

$$S_q = T^2\tilde{\theta}\rho^3\psi(1-3\lambda+2\lambda^2)\left\{a^2T^2\alpha^2\rho^3\tilde{\theta}\psi(1-3\lambda+2\lambda^2)+b^2T^2\tilde{\theta}\rho^3\psi(h-d\alpha)^2(1-3\lambda\right.$$
$$\left.+2\lambda^2)+2b\alpha\{12n^2\lambda^3+aT^2(d\alpha-h)\rho^3\tilde{\theta}\psi(1-3\lambda+2\lambda^2)\}\right\}.$$

Assuming $\theta(\xi)\approx\tilde{\theta}\left(1-\xi\psi+\frac{\xi^2\psi^2}{2}\right)$ in (5.16), so that $\frac{\partial^2\Pi(p,\xi|n)}{\partial\xi^2}\neq 0$. For any given n, taking the second order partial derivatives with respect to p and ξ

$$\frac{\partial^2\Pi(p,\xi|n)}{\partial\xi^2} = -\frac{(a-bp)T^3(h-d\alpha-p\alpha)(1-3\lambda+2\lambda^2)\rho^3\tilde{\theta}\psi^2}{6n^2\lambda^3}, \tag{5.18}$$

$$\frac{\partial^2\Pi(p,\xi|n)}{\partial p^2} = -\frac{bT}{6n^2\lambda^3}\left[12n^2\lambda^3+6nT\alpha(1-\lambda)\lambda\rho^2+T^2\alpha(\lambda-1)\rho^3\left\{2\alpha\lambda^2+(2\lambda-1)\tilde{\theta}(2+\xi\psi(\xi\psi-2))\right\}\right], \tag{5.19}$$

and

$$\frac{\partial^2\Pi(p,\xi|n)}{\partial p\partial\xi} = \frac{-T^3\rho^3\tilde{\theta}\psi(1-\xi\psi)(1-3\lambda+2\lambda^2)\{a\alpha+b(h-d\alpha-2p\alpha)\}}{6n^2\lambda^3}. \tag{5.20}$$

Since $h-d\alpha-p\alpha>0$, and $1-3\lambda+2\lambda^2>0$, it is easy to see that $\partial^2\Pi_P/\partial p^2<0$ and $\partial^2\Pi_P/\partial\xi^2<0$ and thus the determinant of Hessian matrix becomes

$$det(H) = \frac{\partial^2\Pi(p,\xi|n)}{\partial p^2}.\frac{\partial^2\Pi(p,\xi|n)}{\partial\xi^2} - \left[\frac{\partial^2\Pi(p,\xi|n)}{\partial p\partial\xi}\right]^2$$

For given n, the negative definite condition implying the concavity of Π with respect to (p,ξ), is hardly verified analytically, but the numerical simulation conducted in Section 6 has shown the concavity (see Fig. 4). □

Lemma 5.6. When the number of production cycle n and preservation technology cost ξ are fixed, the total profit of the inventory system $\Pi(p|n,\xi)$ is concave in selling price p.

The following proposition characterizes the optimal selling price for any given n and ξ.

Proposition 5.7. For any given n and ξ, the optimal selling price p^* can be derived from $\partial\Pi(p|n,\xi)/\partial p=0$, namely,

$$p^* = \frac{num}{den}, \text{if } \; p^* < a/b. \tag{5.21}$$

where

$$num = an\lambda\left\{6n^2\lambda^3-3nT\alpha(\lambda-1)\lambda\rho^2+T^2\alpha(\alpha+\theta(\xi))(1-3\lambda+2\lambda^2)\rho^3\right\}$$
$$+b\left(sT^3\vartheta^2(\lambda-\rho)^4+6cn^3\lambda^3\rho+3n^2T\lambda^2\left\{s(\lambda-\rho)^2+\pi\vartheta(\lambda-\rho)^2-(h-d\alpha)(\lambda-1)\rho^2\right\}\right.$$
$$\left.+nT^2\lambda\left\{-2sv(\lambda-\rho)^3-\pi\vartheta^2(\lambda-\rho)^3+(h-d\alpha)(\alpha+\theta(\xi))(1-3\lambda+2\lambda^2)\rho^3\right\}\right),$$

and

$$den = 2bn\lambda\left\{6n^2\lambda^3 - 3nT\alpha(\lambda-1)\lambda\rho^2 + T^2\alpha(\alpha+\theta(\xi))(1-3\lambda+2\lambda^2)\rho^3\right\}. \quad (5.22)$$

Otherwise the optimal selling price $p^* = a/b$.

Lemma 5.8. When selling price p and preservation technology cost ξ are fixed, total profit function of the inventory system $\Pi(n|p,\xi)$ is concave in number of production cycle n.

According to Lemma 5.8, the following proposition can be obtained immediately.

Proposition 5.9. For any given p and ξ, the production cycle n^* should satisfy

$$\begin{aligned}\frac{\partial\Pi(n|p,\xi)}{\partial n} &= \frac{(a-bp)T^2}{6n^4\lambda^4}\left[3sT^2\vartheta^2(\lambda-\rho)^4 + 3n^2\lambda^2\left\{s(\lambda-\rho)^2 + \pi\vartheta(\lambda-\rho)^2\right.\right.\\ &\left.-(h-d\alpha-p\alpha)(\lambda-1)\rho^2\right\} - 2nT\lambda\left\{2s\vartheta(\lambda-\rho)^3 + \pi\vartheta^2(\lambda-\rho)^3\right.\\ &\left.\left.-(h-d\alpha-p\alpha)(1-3\lambda+2\lambda^2)\rho^3(\alpha+\tilde{\theta}-\psi\tilde{\theta}\xi)\right\}\right] = 0,\end{aligned} \quad (5.23)$$

if $n^* \geq 1$. Otherwise, the production $n^* = 1$.

To verify the optimality of these solutions in (5.21) and (5.23), we calculate the Hessian matrix as

$$H = \begin{pmatrix} \frac{\partial^2\Pi(n,p|\xi)}{\partial p^2} & \frac{\partial^2\Pi(n,p|\xi)}{\partial p\partial n} \\ \frac{\partial^2\Pi(n,p|\xi)}{\partial n\partial p} & \frac{\partial^2\Pi(n,p|\xi)}{\partial n^2}\end{pmatrix}, \quad (5.24)$$

where expression of $\partial^2\Pi(n,p|\xi)/\partial p^2$ is shown in (5.19),

$$\begin{aligned}\frac{\partial^2\Pi(n,p|\xi)}{\partial n^2} &= -\frac{(a-bp)T^2}{n^5\lambda^4}\left[2sT^2\vartheta^2(\lambda-\rho)^4 + n^2\lambda^2\left\{s(\lambda-\rho)^2 + \pi\vartheta(\lambda-\rho)^2\right.\right.\\ &\left.-(h-d\alpha-p\alpha)(\lambda-1)\rho^2\right\} - nT\lambda\left\{2s\vartheta(\lambda-\rho)^3 + \pi\vartheta^2(\lambda-\rho)^3\right.\\ &\left.\left.-(h-d\alpha-2p\alpha)(1-3\lambda+2\lambda^2)\rho^3(\alpha+\tilde{\theta}-\psi\tilde{\theta}\xi)\right\}\right],\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2\Pi(n,p|\xi)}{\partial p\partial n} &= \frac{\partial^2\Pi(n,p|\xi)}{\partial n\partial p}\\ &= -\frac{T^2}{6n^4\lambda^4}\left[-an\alpha(\lambda-1)\lambda\rho^2\{3n\lambda - 2T(2\lambda-1)\rho(\alpha+\tilde{\theta}-\psi\tilde{\theta}\xi)\}\right.\\ &+b\left(3sT^2v^2(\lambda-\rho)^4 + 3n^2\lambda^2\left\{s(\lambda-\rho)^2 + \pi\vartheta(\lambda-\rho)^2 - (h-d\alpha-2p\alpha)(\lambda-1)\rho^2\right\}\right.\\ &\left.\left.-2nT\lambda\left\{2s\vartheta(\lambda-\rho)^3 + \pi\vartheta^2(\lambda-\rho)^3 - \rho^3(h-d\alpha-p\alpha)(1-3\lambda+2\lambda^2)(\alpha+\tilde{\theta}-\psi\tilde{\theta}\xi)\right\}\right)\right].\end{aligned}$$

Due to complexities of the expressions in Hessian matrix H, the concavity of $\Pi(n,p|\xi)$ with respect to (p,n) is difficult to verify by a mathematically analytic approach, but the numerical simulation conveyed in Section 6 has presented its concavity.

6 Computational results

In this section, we presented numerical examples to illustrate the above algorithm and related results:

Table 2: Optimal solution at fix $\xi^* = 1.30$

n	p^*	Total produced quantity Q_T	Total Profit Π^*
6	11.85	4465.24	32913.67
7	11.27	5669.65	38439.43
8	10.67	6922.02	42622.92
9	10.05	8209.79	45262.66
10	**9.42**	**9524.25**	**46218.51**
11	8.78	10859.19	45391.18
12	8.13	12210.14	42708.93
13	7.47	13573.81	38118.93
14	6.81	14947.74	31581.57

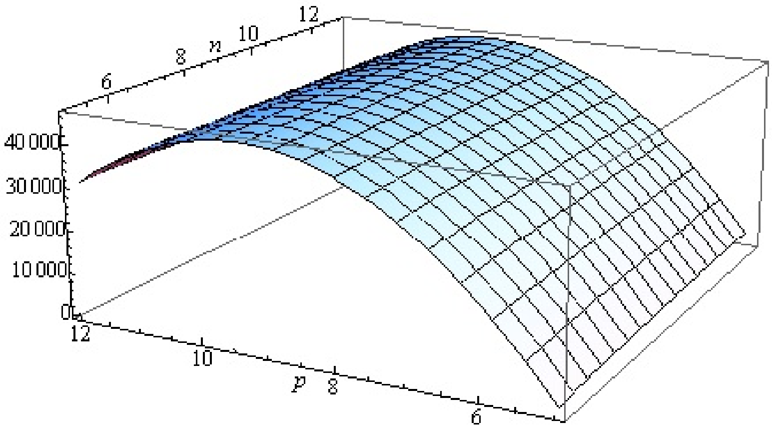

Figure 3: The total profit function, $\Pi(n, p, 1.30)$.

6.1 Numerical examples

Example 1. Complete Backordering: In order to illustrate the above solution procedure, we fix the following parameters for the benchmark case: $T = 220$ weeks, $a = 140$, $b = 10$, $c = \$5$ per unit, $d = \$0.1$ per unit, $h = \$0.02$ per unit per week, $s = \$0.1$ per unit per week, $\alpha = 0.001$, $\rho = 0.19$, $\tilde{\theta} = 0.08$, $\psi = 0.8$, $\lambda = 0.201$. With the given data, results can be obtained by using Mathematica and MS excel software. Then we obtain $n^* = 10$, $p^* = \$9.42$ per unit, $\xi^* = \$1.30$ per unit per week, $\Pi^*(n, p, \xi) = \$46218.5$. Also, $t_1 = 4.2$, $t_1{}' = 20.8$, $T/n = 23$, $R = 716.2$ units, Total demand in $[0, T] = 10165.8$ units.
Total profit is concave with respect to n and p; ξ and p: $\frac{\partial^2\Pi}{\partial n^2} = -21.9 < 0$, $\frac{\partial^2\Pi}{\partial p^2} = -4438.7 < 0$, $\frac{\partial^2\Pi}{\partial \xi^2} = -25659.7 < 0$, $\frac{\partial^2\Pi}{\partial p^2} \cdot \frac{\partial^2\Pi}{\partial n^2} - \left(\frac{\partial^2\Pi}{\partial p \partial n}\right)^2 = 96505.55 > 0$, $\frac{\partial^2\Pi}{\partial p^2} \cdot \frac{\partial^2\Pi}{\partial \xi^2} - \left(\frac{\partial^2\Pi}{\partial p \partial \xi}\right)^2 = 113940329.2 > 0$.

The numerical results of Table 2 indicate that $\Pi(n, p, \xi^*)$ is jointly concave in n and p, as shown in Fig. 3. Accordingly, we are certain that the local maximum obtained here is indeed the global maximum solution.

Example 2. Partial Backordering: In order to exemplify the proposed model, we deliberate an inventory situation with following data: $T = 220$ weeks, $a = 140$ units, $b = 10$, $c = \$5$ per unit, $d = \$0.1$ per unit, $h = \$0.02$ per unit per week, $s = \$0.1$ per unit per week, $\pi = \$7$ per unit, $\alpha = 0.001$, $\rho = 0.19$, $\tilde{\theta} = 0.08$, $\psi = 0.8$, $\lambda = 0.201$, $\vartheta = 0.6$. For distinct values of n, the total profit is computed. The computed results are show in Table 3. The results obtained for illustrative example show that the total profit is concave with respect to ordering frequency. With the given data, results can be obtained by using Mathematica and MS excel software. Then we obtain $n^* = 10$, $p^* = 9.42$, $\xi^* = 2.98$,

Table 3: Optimal solution at fix $\xi^* = 2.98$

n	p^*	Total produced quantity Q_T	Total Profit Π^*
6	11.85	4659.02	32642.40
7	11.27	5915.87	38126.70
8	10.67	7216.06	42235.80
9	10.05	8559.59	44810.80
10	**9.42**	**9924.79**	**45687.30**
11	8.78	11311.70	44781.90
12	8.13	12720.20	42009.80
13	7.47	14150.4	37284.80
14	6.81	15580.6	30634.90

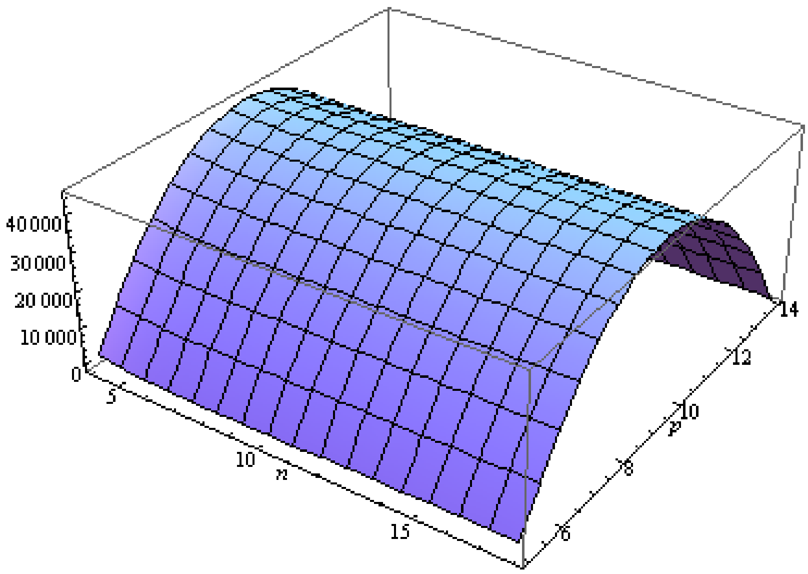

Figure 4: The total profit function, $\Pi(n, p, 2.98)$.

$\Pi^*(n, p, \xi) = \$45687.30$. Fig. 4 show that the total profit function of the inventory system is jointly concave in (n, p), for fixed ξ.

6.2 Sensitivity analysis

In this subsection, we go through sensitivity analysis on the equilibrium strategies with respect to system parameters a, b, c, h, d, s, α, ρ, $\tilde{\theta}$ and ψ. We vary one parameter once and keep other parameters fixed. Next, we have to analyze the sensitivity of the optimal solution by changing the values of the different parameters related to the model. Using the MS excel yields the results reported in Table 4. The results obtained for illustrative examples furnish certain insights about the problem studies. Some of them are as follows.

(i) Π^* increases with an increase in the values of parameters a, d and s while Π^* decreases with an increase in the value of b, c and ρ. Π^* shows concave nature with increase in the values of parameters h, α, $\tilde{\theta}$ and ψ. Π^* is highly sensitive to changes in a, b, c, α and ρ. It is less sensitive to changes in d, s, h, $\tilde{\theta}$ and ψ.

(ii) p^* increases with an increase in the values of parameters a, h, ρ, $\tilde{\theta}$ and ψ while p^* decreases with an increase in the value of b, d and α. p^* is highly sensitive to changes in a, b and ρ. It is less sensitive to changes in h, α, $\tilde{\theta}$ and ψ; and very less sensitive to changes in d; and no changes in c and s.

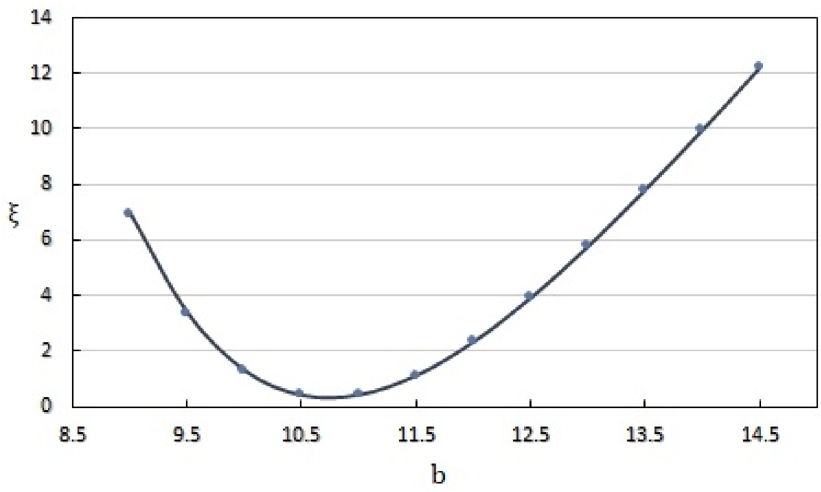

Figure 5: Graphical representation of ξ for distinct value of b.

(iii) ξ^* increases with an increase in the values of parameters c, d and α while ξ^* decreases with an increase in the values of parameters a, b, h, s, ρ, $\tilde{\theta}$ and ψ. ξ^* is highly sensitive to changes in c, d and α. It is less sensitive to changes in a, b, h, s, ρ, $\tilde{\theta}$ and ψ.

We can conclude the following from the sensitivity analysis:

(a) When the value of the market potential a rises and other parameters values are fixed, it can be resolved that the optimal total profit Π^*, the optimal selling price p^*, and the optimal total produced quantity Q_T increase. This means that when the demand parameter a rises, the demand rate will rise, making the organization increases the produced quantity per time and decreases capital investment on preservation ξ as to satisfy high demand rate, the stock will get out fast. Moreover, the organization will set a higher selling price to get more profit.

(b) When the price-sensitive parameter of demand rate b increases, the output parameters p, Q_T, R, and Π will decrease. From a managerial point of view, as b increases, it implies that the demand rate decreases. Hence, the optimal total profit decreases due to the lower selling price. If b is high enough, then the demand rate is very less, so the production is less. Also, ξ is convex in b (see Fig. 5).

(c) When the unit production cost c increases, the output parameters ξ and R will increase; however, Π will decreases. The value of p and Q_T will remain almost unchanged, which implies that the total produced quantity over complete time horizon is insensitive to unit production cost. From an economic point of view, if the producer provides a higher production cost per unit, the producer will produce larger quantity in order to take the profits of the reduction of deterioration rate by increasing the preservation technology cost.

(d) When the holding cost h per item increases, the output parameters ξ, Q_T and R will decrease; however, p will increases. It implies that when the holding cost increases, the retailer merchant should decrease the preservation technology cost to ward off too much inventory on hand. Graphical representation of total profit Π is represented in Fig. 6 with respect to h.

(e) When the unit deteriorating cost d increases, the output parameters ξ, Q_T, R and Π will increase; however, p will decreases.

(f) When the deterioration rate without preservation technology $\tilde{\theta}$ increases, Table 4 shows that Q_T and R decrease, but p increases. The variation in the total profit Π with respect to $\tilde{\theta}$ is shown in Fig. 7.

(g) When the investment cost coefficient ψ increases, the output parameters ξ, Q_T and R decrease, while p increases. Actually, facing a large investment cost coefficient, the retailer will charge a high selling price and bring down the preservation investment cost in order to extract more profits. Fig. 8 shows the concavity of Π in ψ.

Fig. 9 shows the changes in the total profit Π and the time t_1 during which production occurs at the rate $P = D(p)/\lambda$ for variable ρ. Figure shows the constraint ρ at 0.171, 0.1805, 0.19, 0.1995, and 0.209 with other variables unchanged. It is shown that as ρ increases, the total profit Π decreases, while the time t_1 increases.
Comparing the strategies obtained from complete and partial backlogging, we find that the preservation cost in the

Table 4: Sensitivity analysis on $\Pi(n, p, \xi)$ for Example 1.

Parameter	% changes	p^*	ξ^*	Q_T^*	R^*	Π^*
a	−10	−10.62%	+1429.23%	−8.72%	−3.21%	−28.84%
	−5	−5.20%	+697.70%	−4.50%	+1.25%	−14.89%
	+5	+5.10%	−654.62%	+4.80%	−98.96%	+15.89%
	+10	+10.08%	−1255.38%	+9.92%	−99.99%	+32.82%
b	−10	+6.82%	+434.62%	+7.03%	+13.42%	+23.27%
	−5	+3.61%	+159.23%	+3.31%	+8.31%	+10.96%
	+5	−3.40%	−68.46%	−2.96%	−8.56%	−9.79%
	+10	−6.58%	−66.92%	−5.61%	−10.91%	−18.56%
c	−10	−	−1158.46%	−	−99.99%	+10.30%
	−5	−	−579.23%	−	−97.83%	+5.15%
	+5	−	+579.23%	−	+6.01%	−5.15%
	+10	−	+1158.46%	−	+6.03%	−10.30%
h	−10	−7.66%	+3409.23%	+14.57%	+21.48%	−1.75%
	−5	−3.40%	+1593.08%	+6.95%	+13.41%	−0.31%
	+5	+3.08%	−1377.69%	−6.34%	−99.99%	−0.56%
	+10	+5.94%	−2553.08%	−12.12%		−1.78%
d	−10	+0.03%	−14.62%	−0.07%	−1.02%	−0.002%
	−5	+0.02%	−7.69%	−0.03%	−0.50%	−0.001%
	+5	−0.01%	+6.92%	+0.03%	+0.47%	+0.001%
	+10	−0.03%	+14.62%	+0.07%	+0.90%	+0.002%
s	−10	−	+0.85%	−	+0.05%	−0.007%
	−5	−	+0.46%	−	+0.02%	−0.004%
	+5	−	−0.39%	−	−0.02%	+0.004%
	+10	−	−0.77%	−	−0.05%	+0.007%
α	−10	+3.08%	−1400.00%	−6.22%	−99.99%	−0.55%
	−5	+1.49%	−706.15%	−3.14%	−99.44%	−0.18%
	+5	−1.59%	+712.31%	+3.18%	+9.39%	+0.02%
	+10	−3.18%	+1424.62%	+6.39%	+12.79%	+0.25%
ρ	−10	−11.68%	+5002.31%	+11.69%	+18.45%	+7.98%
	−5	−5.41%	+2079.23%	+5.64%	+12.02%	+4.87%
	+5	+4.78%	−1465.38%	−5.54%	−99.99%	−5.83%
	+10	+8.92%	−2479.23%	−10.16%		−12.10%
$\tilde{\theta}$	−10	−3.72%	+1816.92%	+7.56%	+14.04%	−0.58%
	−5	−1.80%	+844.62%	+3.62%	+9.87%	−0.13%
	+5	+1.59%	−736.92%	−3.35%	−99.61%	−0.11%
	+10	+3.18%	−1382.31%	−6.45%	−99.99%	−0.42%
ψ	−10	−3.72%	+1827.69%	+7.56%	+14.04%	−0.64%
	−5	−1.80%	+849.23%	+3.62%	+9.87%	−0.16%
	+5	+1.59%	−741.54%	−3.35%	−99.72%	−0.09%
	+10	+3.18%	−1390.77%	−6.45%	−99.99%	−0.36%

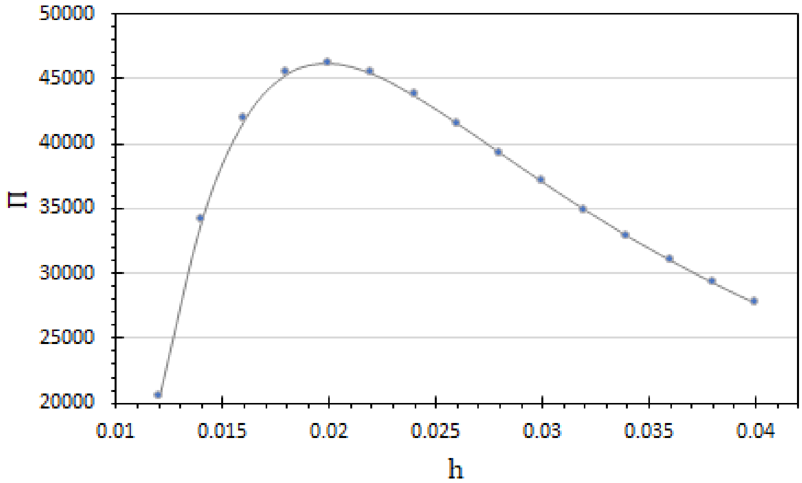

Figure 6: Graphical representation of Π for distinct value of h.

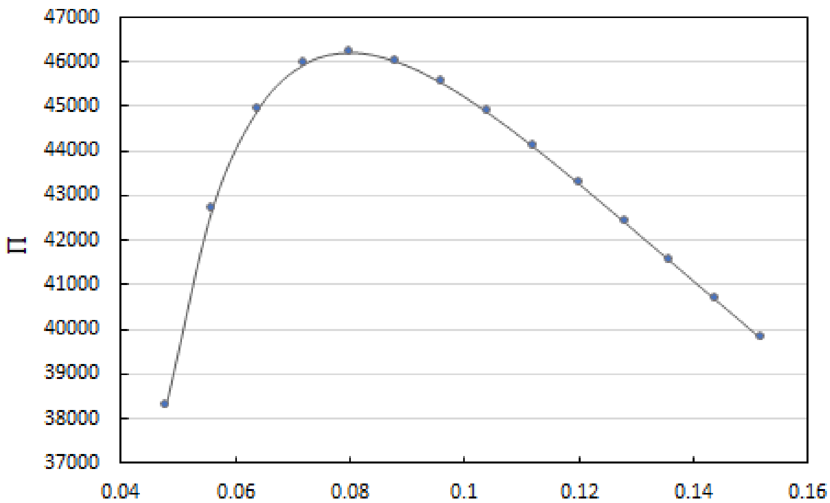

Figure 7: Graphical representation of Π for distinct value of $\tilde{\theta}$.

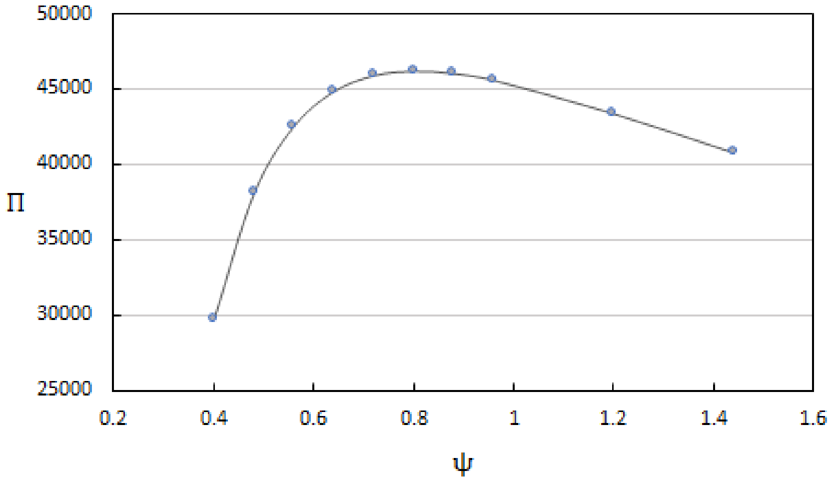

Figure 8: Graphical representation of Π for distinct value of ψ.

$T = 220,$	$a = 140,$	$b = 10,$	$c = 5,$	$d = 0.1,$
$h = 0.02,$	$s = 0.1,$	$\alpha = 0.001,$	$\lambda = 0.201,$	$\tilde{\theta} = 0.08,$
$\psi = 0.8,$	$\xi = 1.30$			
ρ	p	Q_T	Π	t_1
0.171	8.32	10637.84	49905.53	3.76
0.1805	8.91	10060.97	48467.69	3.97
0.19	9.42	9524.25	46218.51	4.18
0.1995	9.87	9023.88	43524.65	4.39
0.209	10.26	8556.60	40625.33	4.60

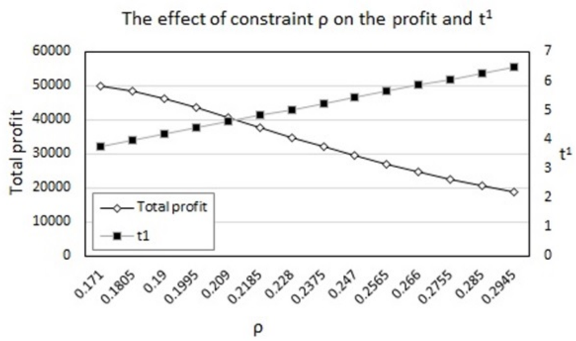

Figure 9: Sensitivity analysis for the constraint ρ

complete backlogging is less than that of the partial backlogging. Meanwhile, the total profit under complete backlogging is greater than the partial backlogging whereas the total quantity produced over the whole time horizon under the complete backlogging is less than that of the partial backlogging.

Managerial Insights

This study reveals a strong recommendation on how the investment of preservation techniques can improve inventory (or stock) within the framework of the production system, where generally shortage is expected. However, the management can decide the optimum sales price, based on the situation of huge or fewer products necessity. They can increase or decrease the production rate based on market demand. For that case, the chances of the deteriorating item will be reduced by the use of preservation techniques.

The selection of a number of production cycles should be taken care as major tasks are done by management. Thus, the management can get the proper quantity of stock of products with optimum profits.

7 Conclusions and future research

The purpose of this study is to present an economic production quantity inventory model with price-dependent and stock-varying demand pattern and controllable deterioration rate to extend the traditional EPQ model. In real markets, the rate of deterioration of the product can be reduced by making an effective capital investment in storehouse equipment or preservation technologies. This paper develops a finite time-horizon production-inventory model where investment is made in the preservation technology cost with complete and partial backlogged that comprises some realistic features. The effects of holding cost and the time value of money are taken into account, considering two separate backlogged.

In this research, we formulated the optimal pricing, optimal number of the production cycle and optimal preservation technology investment strategy for controlling the rate of deterioration and maximizing the total profit in conjunction with production policy. Sensitivity analysis indicates that when the market potential increases, the inventory total profit increases, and if the deterioration rate increases, more investment is needed. The application of the proposed model let to notable sales cost reduction, stock-level improvement by proper investment in preservation techniques, and an increase in profitability of the whole system. The consequences provide management managerial insights in the worth of investment of preservation technology facilities. The results clearly explained the importance of preservation technology investment. From the sensitivity analysis, the larger demand parameter a is beneficial for the organization, because of higher selling price and improved stock level. Thus, the preservation cost reduction is performed as the stock will get out fast.

The proposed model can be extended in several ways. One can extend the deterministic demand function to stochastic fluctuating demand patterns. This inventory model can further be extended by taking some characteristics such as quantity discounts, multiple products, probabilistic demand rate, partial credit trade policy. For instance, we may take the permissible delay in payments. Moreover, we can also see that any deterioration rate can be applied to the proposed model such as the three-parameter Weibull deterioration rate and Gamma deterioration rate. Therefore the usage of general deterioration rates builds the scope of the application wider.

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8 Appendix

Proof of Lemma 5.6. Taking the first-order and second-order partial derivatives of $\Pi(p|n,\xi)$ with respect to p yields

$$\begin{aligned}\frac{\partial\Pi(p|n,\xi)}{\partial p} &= \frac{T}{6n^3\lambda^4}\Big[an\lambda\Big\{6n^2\lambda^3 - 3nT\alpha(\lambda-1)\lambda\rho^2 + T^2\alpha(1-3\lambda+2\lambda^2)\rho^3(\alpha+\tilde{\theta}-\psi\tilde{\theta}\xi)\Big\}\\ &+b\left\{sT^3\vartheta^2(\lambda-\rho)^4 + 6n^3\lambda^3(c\rho-2p\lambda)\right.\\ &+3n^2T\lambda^2\{s(\lambda-\rho)^2+\pi\vartheta(\lambda-\rho)^2-(h-d\alpha-2p\alpha)(\lambda-1)\rho^2\}\\ &+nT^2\lambda\{-2s\vartheta(\lambda-\rho)^3-\pi\vartheta^2(\lambda-\rho)^3\\ &\left.+(h-d\alpha-2p\alpha)(1-3\lambda+2\lambda^2)\rho^3(\alpha+\tilde{\theta}-\psi\tilde{\theta}\xi)\}\Big\}\Big],\end{aligned} \tag{8.1}$$

$$\frac{\partial^2\Pi(p|n,\xi)}{\partial p^2} = -2bT - \frac{bT^2\alpha(1-\lambda)\rho^2}{3n^2\lambda^3}\{3n\lambda - T(\alpha+\theta(\xi))(2\lambda-1)\rho\} < 0. \quad \square$$

Proof of Proposition 5.7. Since $(\lambda-\rho)<0$ and $0<\lambda<1$, we have

$$\begin{aligned}p^* = &\left[an\lambda\left\{6n^2\lambda^3 - 3nT\alpha(\lambda-1)\lambda\rho^2 + T^2\alpha(\alpha+\theta(\xi))(1-3\lambda+2\lambda^2)\rho^3\right\}\right.\\ &+b\left\{sT^3\vartheta^2(\lambda-\rho)^4 + 6cn^3\lambda^3\rho + 3n^2T\lambda^2\{s(\lambda-\rho)^2+\pi\vartheta(\lambda-\rho)^2-(h-d\alpha)(\lambda-1)\rho^2\}\right.\\ &\left.\left.+nT^2\lambda\{-2sv(\lambda-\rho)^3-\pi\vartheta^2(\lambda-\rho)^3+(h-d\alpha)(\alpha+\theta(\xi))(1-3\lambda+2\lambda^2)\rho^3\}\right\}\right]/\\ &\left[2bn\lambda\left\{6n^2\lambda^3-3nT\alpha(\lambda-1)\lambda\rho^2+T^2\alpha(\alpha+\theta(\xi))(1-3\lambda+2\lambda^2)\rho^3\right\}\right]>0.\end{aligned} \tag{8.2}$$

Hence, if $p^* < a/b$, the optimal selling price is p^*. Otherwise, the optimal selling price $p^* = a/b$. $\square$

Proof of Lemma 5.8. Taking the second-order derivative of $\Pi(n|p,\xi)$ with respect to n, we can obtain

$$\begin{aligned}\frac{\partial^2\Pi(n|p,\xi)}{\partial n^2} = &-\frac{(a-bp)T^2}{n^5\lambda^4}\left[2sT^2\vartheta^2(\lambda-\rho)^4 + n^2\lambda^2\left\{s(\lambda-\rho)^2+\pi\vartheta(\lambda-\rho)^2\right.\right.\\ &\left.-(h-d\alpha-p\alpha)(\lambda-1)\rho^2\right\} - nT\lambda\left\{2s\vartheta(\lambda-\rho)^3+\pi\vartheta^2(\lambda-\rho)^3\right.\\ &\left.\left.-(h-d\alpha-p\alpha)(1-3\lambda+2\lambda^2)\rho^3(\alpha+\tilde{\theta}-\psi\tilde{\theta}\xi)\right\}\right],\end{aligned}$$

where $0 < \lambda < 1$ and $(h - d\alpha - p\alpha) > 0$, it suffices to verify that $\partial^2 \Pi(n|p,\xi)/\partial n^2 \leq 0$. As a consequence, $\Pi(n|p,\xi)$ is a concave function of n. □

Proof of Proposition 5.9. From Lemma 5.8, the result can be obtained easily. □