Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 1377–1397
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2020.20716.2195

Legendre spectral projection methods for linear second kind Volterra integral equations with weakly singular kernels

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(Communicated by Javad Vahidi)

Abstract

In this paper, Galerkin and iterated Galerkin methods are applied to approximate the linear second kind Volterra integral equations with weakly singular algebraic kernels using Legendre polynomial basis functions. We discuss the convergence results in both L^2 and infinity norms in two cases: when the exact solution is sufficiently smooth and non-smooth. We also apply Legendre multi-Galerkin and iterated Legendre multi-Galerkin methods and derive the superconvergence rates. Numerical results are given to verify the theoretical results.

Keywords: Volterra integral equations, Galerkin method, Multi-Galerkin method, Weakly singular kernels, Legendre polynomials.
2020 MSC: 45D05, 45A05, 65R20

1 Introduction

In this article, we consider the linear Volterra integral equation

$$\varphi(\sigma) = \int_{-1}^{\sigma} k(\sigma, s)\varphi(s)\, ds + f(\sigma), \quad \sigma \in [-1, 1], \tag{1.1}$$

with weakly singular kernel of the form

$$k(\sigma, s) = (\sigma - s)^{-\gamma}\tilde{m}(\sigma, s), \quad 0 < \gamma < 1, \tag{1.2}$$

where $\tilde{m}(.,.)$ and f both are known and sufficiently smooth functions and φ is the unknown function to be approximated.

Volterra integral equation of the type (1.1) arises in many applications in science and technology such as electrostatics, potential theory, reformulation of boundary value problems, physical applications etc. In general, it is not simple to find the explicit solutions for (1.1). So some numerical approximation methods are needed to apply for solving these integral equations. Various approximation methods such as collocation, Galerkin, Nyström, petrov-Galerkin

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Received: June 2020 *Accepted:* August 2020

methods, etc are available in literature to determine the numerical solution of linear second kind weakly singular Volterra integral equation of the type (1.1). Many authors (see [9], [10], [11], [12], [20], [22], [31]) have studied the approximation of linear weakly singular Volterra integral equations and their convergence analysis. In [9], H. Brunner discussed the non polynomial spline collocation method to solve (1.1). In ([10], [11]), H. Brunner obtained the numerical solution of the integral equation (1.1) by applying collocation and iterated collocation methods on graded mesh using piecewise polynomials basis functions of degree $\leq r-1$ and derived the convergence rates as $\mathcal{O}(h^r)$ and $\mathcal{O}(h^{r+1-\gamma})$, respectively, where h denotes the norm of the partition. In ([2], [3], [4], [5], [6]), Assari et al. discussed meshless Galerkin and meshless collocation methods to solve the weakly singular integral equations. A hybrid collocation method was proposed in [14] by Yanzhao Cao et al. and they obtained the convergence rate as $\mathcal{O}(n^{-r})$, where n denotes the number of partitions. However, for better accuracy in projection methods based on piecewise polynomials, we need to increase the number of partition points and hence a huge system of linear equations will have to be solved, which is computationally very much expensive. For minimization of this computational complexity, subspace of global polynomials can be used as the approximating subspace $\mathbb{X}_n$. In particular, Legendre polynomials can be used as the basis functions for the subspace $\mathbb{X}_n$. The Legendre polynomials posses nice property of orthogonality and we can generate them recursively with ease. In ([18], [19]), Das et al. accomplished Legendre spectral Galerkin and Legendre spectral collocation methods to solve Fredholm integral equations. In [28], Panigrahi et al. discussed Legendre Galerkin method for solving weakly singular Fredholm integral equations. In [32], Zhang Xiao-yong discussed the Jacobi spectral Galerkin method for linear Volterra integral equation with weakly singular kernel and derived the orders of convergence as $\mathcal{O}(n^{\frac{3}{4}-r})$ for infinity norm and $\mathcal{O}(n^{\frac{1}{2}-r})$ for weighted L^2-norm. In [26], M. Mandal et al. discussed the Legendre spectral Galerkin and iterated Legendre spectral Galerkin methods for solving linear second kind Volterra integral equation with smooth kernel and obtained the orders of convergence as $\mathcal{O}(n^{-r})$ in L^2 norm and $\mathcal{O}(n^{-r+\frac{1}{2}})$ in infinity norm for Legendre spectral Galerkin method and $\mathcal{O}(n^{-2r})$ in both L^2 and infinity norms for iterated Legendre spectral Galerkin method, where n denotes the highest degree of Legendre polynomials employed in approximation. The main motivation to consider this paper is to obtain the better superconvergence rates to linear Volterra integral equation with weakly singular kernel using Legendre polynomial basis functions.

In this paper, firstly Galerkin and iterated Galerkin methods using Legendre polynomial basis functions are applied to obtain the better convergence rates for the approximate solution of linear weakly singular Volterra integral equation (1.1). Here in Legendre Galerkin method, the convergence of approximate solutions are discussed in two cases, firstly, for sufficiently smooth exact solution and secondly, for non-smooth exact solution. Let φ be the exact solution of the Volterra integral equation (1.1). Then we show that, in Legendre Galerkin method, the approximate solution converges to φ with the orders $\mathcal{O}(n^{\frac{1}{2}-r})$, $0<\gamma<1$ in infinity norm and $\mathcal{O}(n^{-r})$ for $0<\gamma<\frac{1}{2}$ and $\mathcal{O}(n^{-\frac{1}{2}-r+\gamma})$ for $\frac{1}{2}\leq\gamma<1$ in L^2-norm, respectively, where r denotes the smoothness of the exact solution and n denotes the highest degree of the Legendre polynomials used in the approximation. Also for non-smooth exact solution, the Legendre Galerkin approximation approaches to φ with $\mathcal{O}(n^{-(1-\gamma)}\log n)$, $0<\gamma<1$, in infinity norm and $\mathcal{O}(n^{-(1-\gamma)})$, $0<\gamma<1$ in L^2-norm. Also we show that for both infinity and L^2-norms, the iterated Legendre Galerkin solution converges with orders $\mathcal{O}(n^{-\frac{1}{2}-r+\gamma})$, $0<\gamma<1$, and $\mathcal{O}(n^{-2(1-\gamma)}\log n)$, $0<\gamma<1$, respectively, for sufficiently smooth exact solution and for non-smooth exact solution.

In ([16], [23], [24], [25], [27], [29]), the multi-projection (multi-collocation and multi-Galerkin) methods are described to solve the second kind Fredholm integral equations. Here we also discuss the Legendre multi-Galerkin and iterated Legendre multi-Galerkin methods for solving the weakly singular Volterra integral equation (1.1) to improve the order of convergence and obtain superconvergence results. We show that for both infinity and L^2-norms, the approximate solution in iterated Legendre multi-Galerkin method converges with the orders $\mathcal{O}(n^{\frac{1}{2}-r-2(1-\gamma)}\log n)$, $0<\gamma<1$, for sufficiently smooth exact solution, and $\mathcal{O}(n^{-3(1-\gamma)}(\log n)^2)$, $0<\gamma<1$, for non-smooth exact solution. Hence we can conclude that the iterated Legendre multi-Galerkin approximate solution provides better convergence results than iterated Legendre Galerkin and Legendre Galerkin approximate solutions.

We present this paper as follows. In Section 2, Legendre Galerkin method and its iterated version to the equation (1.1) are applied and the convergence rates are derived in Section 3. In Section 4, we apply the Legendre multi-Galerkin and iterated Legendre multi-Galerkin methods for obtaining the superconvergence results. Numerical results are provided in Section 5 to justify our theoretical results. Throughout the paper, it is assumed that C denotes a generic constant.

2 Legendre Galerkin method: linear weakly singular Volterra integral equation

Let $\mathbb{X} = \mathcal{C}[-1,1]$. We consider the linear second kind Volterra integral equation

$$\varphi(\sigma) = \int_{-1}^{\sigma} k(\sigma,s)\varphi(s)\,ds + f(\sigma), \quad \sigma \in [-1,1], \tag{2.1}$$

with weakly singular kernels

$$k(\sigma,s) = (\sigma - s)^{-\gamma}\tilde{m}(\sigma,s), \quad s \in [-1,1],\ 0 < \gamma < 1, \tag{2.2}$$

where the functions $\tilde{m}(\sigma,s)$ and f are known sufficiently smooth functions and φ is the unknown function to be approximated.

Now for obtaining superconvergence results, we convert the domain of integral from $[-1,\sigma]$ to $[-1,1]$. To do this, we consider a transformation $s(.,.) : ([-1,1]\times[-1,1]) \to [-1,1]$ by

$$s = \frac{1+\sigma}{2}\zeta + \frac{\sigma-1}{2}.$$

Then the equation (2.1) becomes

$$\varphi(\sigma) = \int_{-1}^{1} H(\sigma, s(\sigma,\zeta))\varphi(s(\sigma,\zeta))\,d\zeta + f(\sigma), \quad \sigma \in [-1,1], \tag{2.3}$$

where

$$H(\sigma, s(\sigma,\zeta)) = (1-\zeta)^{-\gamma} m(\sigma, s(\sigma,\zeta)), \quad 0 < \gamma < 1, \tag{2.4}$$

and

$$m(\sigma, s(\sigma,\zeta)) = \left(\frac{1+\sigma}{2}\right)^{1-\gamma} \tilde{m}(\sigma, s(\sigma,\zeta)), \quad 0 < \gamma < 1. \tag{2.5}$$

Define

$$\mathcal{K}\varphi(\sigma) = \int_{-1}^{1} H(\sigma, s(\sigma,\zeta))\varphi(s(\sigma,\zeta))\,d\zeta. \tag{2.6}$$

Then (2.3) in operator form can be written as

$$\varphi - \mathcal{K}\varphi = f. \tag{2.7}$$

Next to discuss the Legendre Galerkin method and its iterated version to solve the equation (2.7), we consider the sequence of approximating subspaces $\mathbb{X}_n = \operatorname{span}\{\psi_0, \psi_1, \psi_2, \dots, \psi_n\}$, the Legendre polynomial subspaces of $\mathbb{X}$ of degree $\le n$. Here ψ_j's are defined by

$$\psi_j(p) = \sqrt{\frac{2j+1}{2}} L_j(p), \quad j = 0, 1, \dots, n,$$

where L_j's denote the Legendre polynomials of degree j and these polynomials can be generated by the three terms recurrence relation given by

$$L_0(p) = 1, L_1(p) = p, \quad p \in [-1,1],$$

and for $j = 1, 2, \dots, n-1$,

$$(j+1)L_{j+1}(p) = (2j+1)pL_j(p) - jL_{j-1}(p), \quad p \in [-1,1].$$

We define the orthogonal projection operator $\mathcal{P}_n : \mathbb{X} \to \mathbb{X}_n$ as

$$\langle \mathcal{P}_n\varphi, w\rangle = \langle \varphi, w\rangle, \quad \varphi \in \mathbb{X},\ w \in \mathbb{X}_n, \tag{2.8}$$

where $\langle \varphi, w\rangle = \int_{-1}^{1} \varphi(\sigma)w(\sigma)\,d\sigma$.

Lemma 2.1. ([8], [13]) Let $\mathcal{P}_n : \mathbb{X} \to \mathbb{X}_n$ be the orthogonal projection operator defined by (2.8). Then there hold

i)
$$||\mathcal{P}_n\varphi||_{L^2} \leq p_1||\varphi||_\infty, \tag{2.9}$$
where p_1 (independent of n) is a constant.

ii) For any $\varphi \in \mathbb{X}$, $\exists\, C > 0$ (constant independent of n) such that
$$||\mathcal{P}_n\varphi - \varphi||_{L^2} \leq C \inf_{v\in\mathbb{X}_n} ||\varphi - v||_\infty \to 0 \quad \text{as } n \to \infty.$$

Lemma 2.2. ([8], [13]) Let $\mathcal{P}_n : \mathbb{X} \to \mathbb{X}_n$ be the projection operator defined by (2.8). Then for any $\varphi \in \mathcal{C}^r([-1, 1])$, there hold
$$||\varphi - \mathcal{P}_n\varphi||_{L^2} \leq cn^{-r}||\varphi^{(r)}||_\infty, \tag{2.10}$$
$$||\varphi - \mathcal{P}_n\varphi||_\infty \leq cn^{\frac{3}{4}-r}||\varphi^{(r)}||_\infty, \tag{2.11}$$
where c (independent of n) is a constant. Also we get
$$||\varphi - \mathcal{P}_n\varphi||_\infty \leq cn^{\frac{1}{2}-r}V(\varphi^{(r)}), \tag{2.12}$$
where $V(\varphi^{(r)})$ indicates the total variation of $\varphi^{(r)}$.

Note that for an orthogonal projection operator we have
$$||\mathcal{P}_n||_\infty \leq p_2 \log n, \tag{2.13}$$
where p_2 (independent of n) is a constant (see Ben-yu Guo et al. [8]).

The Legendre Galerkin method to solve (2.7) is finding an approximation $\varphi_n \in \mathbb{X}_n$ such that
$$\varphi_n - \mathcal{P}_n\mathcal{K}\varphi_n = \mathcal{P}_n f. \tag{2.14}$$
Iterated Legendre Galerkin approximation is defined as
$$\tilde{\varphi}_n = \mathcal{K}\varphi_n + f. \tag{2.15}$$
Using $\mathcal{P}_n\tilde{\varphi}_n = \varphi_n$, we can write (2.15) as
$$\tilde{\varphi}_n = \mathcal{K}\mathcal{P}_n\tilde{\varphi}_n + f. \tag{2.16}$$
In the next section we will discuss convergence analysis of the Legendre Galerkin and iterated Legendre Galerkin methods.

3 Convergence analysis

Here we discuss the uniqueness and existence of the Legendre Galerkin and iterated Legendre Galerkin approximations and their convergence results. At first we quote the following theorem from Brunner [10] to the equation (2.1), which helps us to prove our next theorem.

Theorem 3.1. ([11]) Let φ be the exact solution of the Volterra integral equation (2.1) and it is non-smooth, then for any $f \in \mathcal{C}^r[-1, 1]$, it has the uniformly convergent series expansion
$$\varphi(\sigma) = f(\sigma) + \sum_{k=1}^{\infty} \phi_k\left(\frac{1}{2}(1+\sigma)\right)\left(\frac{1}{2}(1+\sigma)\right)^{k(1-\gamma)}, \quad 0 < \gamma < 1,$$
where $\phi_k(\cdot)$ are smooth functions.

Note that from Theorem 3.1, the non-smooth exact solution φ behaves at $\sigma = -1$ as $\varphi(\sigma) \sim (1 + \sigma)^{1-\gamma}$ or $\varphi'(\sigma) \sim (1 + \sigma)^{-\gamma}$, $0 < \gamma < 1$.

Theorem 3.2. Let φ be the non-smooth exact solution of (2.1) which behaves $\varphi'(\sigma) \sim (1+\sigma)^{-\gamma}$, $0<\gamma<1$, then there holds

$$\omega_{1,\infty}\left(\varphi, \frac{1}{n}\right) \leq \sup_{0\leq \delta \leq \frac{1}{n}} ||\Delta_\delta \varphi(\sigma)||_\infty \leq Cn^{-(1-\gamma)},$$

where $\omega_{1,\infty}$ denotes the first order modulus of smoothness w.r.t. infinity norm.

Proof . From Theorem 3.1, the exact solution of the transformed equation becomes

$$\varphi(s(\sigma,\zeta)) = f(s(\sigma,\zeta)) + \sum_{k=1}^{\infty} \phi_k\left(\frac{1}{2}(1+s(\sigma,\zeta))\right)\left(\frac{1}{2}(1+s(\sigma,\zeta))\right)^{k(1-\gamma)}.$$

Let $\tilde{\phi}_k(1+s(\sigma,\zeta)) = \left(\frac{1}{2}\right)^{k(1-\gamma)} \phi_k\left(\frac{1}{2}(1+s(\sigma,\zeta))\right)$, $M = \max_{\sigma\in[-1,1]} |f'(s(\sigma,\zeta))|$. Consider for any $\zeta \in [-1,1]$ and $\delta>0$ such that $\zeta+\delta \in [-1,1]$,

$$\begin{aligned}
&|\varphi(s(\sigma,\zeta+\delta)) - \varphi(s(\sigma,\zeta))| \\
&\leq |f(s(\sigma,\zeta+\delta)) - f(s(\sigma,\zeta))| + \sum_{k=1}^{\infty}\Big|\tilde{\phi}_k(1+s(\sigma,\zeta+\delta))(1+s(\sigma,\zeta+\delta))^{k(1-\gamma)} \\
&\quad -\tilde{\phi}_k(1+s(\sigma,\zeta))(1+s(\sigma,\zeta))^{k(1-\gamma)}\Big| \\
&\leq |f'(s(\sigma,\zeta+\xi_1\delta))(s(\sigma,\zeta+\delta)-s(\sigma,\zeta))| \\
&\quad + \sum_{k=1}^{\infty}\Big\{\Big|\tilde{\phi}_k(1+s(\sigma,\zeta+\delta))(1+s(\sigma,\zeta+\delta))^{k(1-\gamma)} \\
&\quad -\tilde{\phi}_k(1+s(\sigma,\zeta+\delta))(1+s(\sigma,\zeta))^{k(1-\gamma)}\Big| \\
&\quad + \Big|\tilde{\phi}_k(1+s(\sigma,\zeta+\delta))(1+s(\sigma,\zeta))^{k(1-\gamma)} \\
&\quad -\tilde{\phi}_k(1+s(\sigma,\zeta))(1+s(\sigma,\zeta))^{k(1-\gamma)}\Big|\Big\} \\
&\leq M\left|\frac{\sigma+1}{2}\delta\right| + \sum_{k=1}^{\infty}\Big\{\Big|\tilde{\phi}_k(1+s(\sigma,\zeta+\delta))\Big|\Big|(1+s(\sigma,\zeta+\delta))^{k(1-\gamma)} - (1+s(\sigma,\zeta))^{k(1-\gamma)}\Big| \\
&\quad + \Big|\tilde{\phi}_k(1+s(\sigma,\zeta+\delta)) - \tilde{\phi}_k(1+s(\sigma,\zeta))\Big|\Big|(1+s(\sigma,\zeta))^{k(1-\gamma)}\Big|\Big\} \\
&\leq M\delta + \sum_{k=1}^{\infty}\Big\{\Big|\tilde{\phi}_k(1+s(\sigma,\zeta+\delta))\Big|\Big|\left(1+\frac{1+\sigma}{2}(\zeta+\delta)+\frac{\sigma-1}{2}\right)^{k(1-\gamma)} \\
&\quad -\left(1+\frac{1+\sigma}{2}\zeta+\frac{\sigma-1}{2}\right)^{k(1-\gamma)}\Big| + \Big|\tilde{\phi}_k^{(1)}(1+s(\sigma,\zeta+\xi_2\delta))(s(\sigma,\zeta+\delta)-s(\sigma,\zeta))\Big| \\
&\quad \times\Big|(1+s(\sigma,\zeta))^{k(1-\gamma)}\Big|\Big\} \\
&\leq M\delta + \sum_{k=1}^{\infty}\Big\{\Big|\phi_k\left(\frac{1}{2}(1+s(\sigma,\zeta+\delta))\right)\left(\frac{1}{2}\right)^{k(1-\gamma)}\Big| \\
&\quad \times\Big|\left(\frac{1+\sigma}{2}(\zeta+\delta)+\frac{1+\sigma}{2}\right)^{k(1-\gamma)} - \left(\frac{1+\sigma}{2}\zeta+\frac{1+\sigma}{2}\right)^{k(1-\gamma)}\Big| \\
&\quad + \Big|\phi_k^{(1)}\left(\frac{1}{2}(1+s(\sigma,\zeta+\xi_2\delta))\right)\Big|\Big|\frac{\sigma+1}{2}\delta\Big|\Big|\left(\frac{1+s(\sigma,\zeta)}{2}\right)^{k(1-\gamma)}\Big|\Big\} \\
&\leq M\delta + \sum_{k=1}^{\infty}\Big\{\Big|\phi_k\left(\frac{1}{2}(1+s(\sigma,\zeta+\delta))\right)\left(\frac{1+\sigma}{2}\right)^{k(1-\gamma)}\Big| \\
&\quad \times\Big|\left(\frac{1+\zeta+\delta}{2}\right)^{k(1-\gamma)} - \left(\frac{1+\zeta}{2}\right)^{k(1-\gamma)}\Big|
\end{aligned}$$

$$+\left|\phi_k^{(1)}\left(\frac{1}{2}(1+s(\sigma,\zeta+\xi_2\delta))\right)\delta\left(\frac{1+s(\sigma,\zeta)}{2}\right)^{k(1-\gamma)}\right|\Bigg\}, \tag{3.1}$$

where $0<\xi_1,\xi_2<1$. Next we prove that

$$\left|\left(\frac{1+\zeta+\delta}{2}\right)^{k(1-\gamma)}-\left(\frac{1+\zeta}{2}\right)^{k(1-\gamma)}\right|=\mathcal{O}(\delta^{1-\gamma}). \tag{3.2}$$

we prove (3.2) using the principle of mathematical induction. For $k=1$, from the mathematical inequality $(x_1+x_2)^d<x_1^d+x_2^d$, where $x_1,x_2>0$, $0<d<1$, we obtain

$$\left|\left(\frac{1+\zeta+\delta}{2}\right)^{(1-\gamma)}-\left(\frac{1+\zeta}{2}\right)^{(1-\gamma)}\right|<\delta^{1-\gamma}.$$

Therefore, the statement is true for $k=1$. Now, assume that the statement is true for $k=q-1$, then

$$\left|\left(\frac{1+\zeta+\delta}{2}\right)^{(q-1)(1-\gamma)}-\left(\frac{1+\zeta}{2}\right)^{(q-1)(1-\gamma)}\right|\le C\delta^{1-\gamma}.$$

For $k=q$, consider

$$\begin{aligned}
&\left|\left(\frac{1+\zeta+\delta}{2}\right)^{q(1-\gamma)}-\left(\frac{1+\zeta}{2}\right)^{q(1-\gamma)}\right|\\
&=\Bigg|\left(\frac{1+\zeta+\delta}{2}\right)^{q(1-\gamma)}-\left(\frac{1+\zeta+\delta}{2}\right)^{(1-\gamma)}\left(\frac{1+\zeta}{2}\right)^{(q-1)(1-\gamma)}\\
&\quad+\left(\frac{1+\zeta+\delta}{2}\right)^{(1-\gamma)}\left(\frac{1+\zeta}{2}\right)^{(q-1)(1-\gamma)}-\left(\frac{1+\zeta}{2}\right)^{q(1-\gamma)}\Bigg|\\
&\le\left|\left(\frac{1+\zeta+\delta}{2}\right)^{(1-\gamma)}\right|\left|\left(\frac{1+\zeta+\delta}{2}\right)^{(q-1)(1-\gamma)}-\left(\frac{1+\zeta}{2}\right)^{(q-1)(1-\gamma)}\right|\\
&\quad+\left|\left(\frac{1+\zeta}{2}\right)^{(q-1)(1-\gamma)}\right|\left|\left(\frac{1+\zeta+\delta}{2}\right)^{(1-\gamma)}-\left(\frac{1+\zeta}{2}\right)^{(1-\gamma)}\right|\\
&\le\left|\left(\frac{1+\zeta+\delta}{2}\right)^{(1-\gamma)}\right|C\delta^{1-\gamma}+\left|\left(\frac{1+\zeta}{2}\right)^{(q-1)(1-\gamma)}\right|C\delta^{1-\gamma}\\
&\le\left|(1)^{(1-\gamma)}\right|C\delta^{1-\gamma}+\left|(1)^{(q-1)(1-\gamma)}\right|C\delta^{1-\gamma}\\
&\le\tilde{C}\delta^{1-\gamma},
\end{aligned}$$

where $\tilde{C}=2C$. Hence by the principle of mathematical induction, (3.2) is proved.
Combining (3.1) and (3.2), we have

$$\begin{aligned}
&|\varphi(s(\sigma,\zeta+\delta))-\varphi(s(\sigma,\zeta))|\\
&\le M\delta+C\delta^{1-\gamma}\sum_{k=1}^{\infty}\left|\phi_k\left(\frac{1}{2}(1+s(\sigma,\zeta+\delta))\right)\left(\frac{1+\sigma}{2}\right)^{k(1-\gamma)}\right|\\
&\quad+\delta^{1-\gamma}\sum_{k=1}^{\infty}\left|\phi_k^{(1)}\left(\frac{1}{2}(1+s(\sigma,\zeta+\xi_2\delta))\right)\delta^{\gamma}\left(\frac{1+s(\sigma,\zeta)}{2}\right)^{k(1-\gamma)}\right|.
\end{aligned} \tag{3.3}$$

Since $\sum_{k=1}^{\infty}\left|\phi_k\left(\frac{1}{2}(1+s(\sigma,\zeta+\delta))\right)\left(\frac{1+\sigma}{2}\right)^{k(1-\gamma)}\right|$, and $\sum_{k=1}^{\infty}\left|\phi_k^{(1)}\left(\frac{1}{2}(1+s(\sigma,\zeta+\xi_2\delta))\right)\delta^{\gamma}\left(\frac{1+s(\sigma,\zeta)}{2}\right)^{k(1-\gamma)}\right|$ are uniformly convergent series and are uniformly convergent to ℓ_1 and ℓ_2 (say), respectively, from (3.3), we have

$$|\varphi(s(\sigma,\zeta+\delta))-\varphi(s(\sigma,\zeta))|\le M\delta+C\delta^{1-\gamma}\ell_1+\ell_2\delta^{1-\gamma}=\mathcal{O}(\delta^{1-\gamma}).$$

Using this, we obtain

$$||\Delta_\delta \varphi(s(\sigma,\zeta))||_\infty = \max_{-1\le\sigma,\zeta\le 1} |\varphi(s(\sigma,\zeta+\delta)) - \varphi(s(\sigma,\zeta))| = \mathcal{O}(\delta^{1-\gamma}),$$

and hence we get

$$\omega_{1,\infty}\left(\varphi, \frac{1}{n}\right) \le \sup_{0\le\delta\le\frac{1}{n}} ||\Delta_\delta \varphi(s(\sigma,\zeta))||_\infty \le Cn^{-(1-\gamma)}.$$

Hence the proof. □

Lemma 3.3. Let φ be the nonsmooth exact solution of (2.7), which behaves like $\varphi'(\sigma) \sim (1+\sigma)^{-\gamma}$, $0<\gamma<1$, then there hold

$$||(I-\mathcal{P}_n)\varphi||_\infty = \mathcal{O}(n^{-(1-\gamma)}\log n), \tag{3.4}$$

and

$$||(I-\mathcal{P}_n)\varphi||_{L^2} = \mathcal{O}(n^{-(1-\gamma)}). \tag{3.5}$$

Proof . Using estimate (2.13), we obtain

$$\begin{aligned}||(I-\mathcal{P}_n)\varphi||_\infty &= ||(I-\mathcal{P}_n)(\varphi-\chi_n)||_\infty\\ &\le (1+||\mathcal{P}_n||_\infty)||\varphi-\chi_n||_\infty\\ &\le (1+p_2\log n)||\varphi-\chi_n||_\infty.\end{aligned}$$

Since this holds for any $\chi_n \in \mathbb{X}_n$, using Jackson's theorem (see. [17], p. 144, 147, Theorems III and V), and Theorem 3.2, we have

$$\begin{aligned}||(I-\mathcal{P}_n)\varphi||_\infty &\le (1+p_2\log n)\inf_{\chi_n\in\mathbb{X}_n} ||\varphi-\chi_n||_\infty\\ &= (1+p_2\log n)\omega_{1,\infty}\left(\varphi,\frac{1}{n}\right)\\ &\le (1+p_2\log n)Cn^{-(1-\gamma)}\\ &= \mathcal{O}(n^{-(1-\gamma)}\log n).\end{aligned}$$

On similar lines, using (2.9), we can prove (3.5). □ For the rest part of the paper, we assume that the kernel $m(\sigma, s(\sigma,\zeta))$ is r-times continuously differentiable with respect to ζ.

Lemma 3.4. Let $H(\sigma, s(\sigma,\zeta))$, $0<\gamma<1$ be the kernel of the form (2.4) of the integral operator defined by (2.6), then for each $\sigma\in[-1,1]$, there exists a polynomial $v_\sigma \in L^1([-1,1])$ of degree $\le n$ such that

$$||H(\sigma,s(\sigma,\cdot)) - v_\sigma||_{L^1} = \mathcal{O}(n^{-(1-\gamma)}), \quad 0<\gamma<1.$$

Proof . Note that $H(\sigma,s(\sigma,\zeta))\in L^1([-1,1])$. Then from (Schumaker [30], p. 92), for $r=1, p=1$, we get

$$||H(\sigma,s(\sigma,\cdot)) - v_\sigma||_{L^1} \le C\omega_{1,1}\left(H(\sigma,s(\sigma,\zeta)),\frac{1}{n}\right),$$

where $\omega_{1,1}$ denotes the first order modulus of smoothness w.r.t. L^1 norm. Let $I_\delta=[-1,1-\delta]$, then using the definition of modulus of smoothness (cf. Schumaker [30], p. 55), we have

$$\begin{aligned}||H(\sigma,s(\sigma,\cdot)) - v_\sigma||_{L^1} &\le C\omega_{1,1}\left(H(\sigma,s(\sigma,\zeta)),\frac{1}{n}\right)\\ &= C\sup_{0\le\delta\le\frac{1}{n}} ||\Delta_\delta H(\sigma,s(\sigma,\zeta))||_{L^1(I_\delta)}\\ &= C\sup_{0\le\delta\le\frac{1}{n}} \int_{-1}^{1-\delta} |H(\sigma,s(\sigma,\zeta+\delta)) - H(\sigma,s(\sigma,\zeta))|\,d\zeta.\end{aligned} \tag{3.6}$$

Using $H(\sigma, s(\sigma,\zeta)) = (1-\zeta)^{-\gamma} m(\sigma, s(\sigma,\zeta))$, $0<\gamma<1$, we have

$$\begin{aligned}
&\int_{-1}^{1-\delta} |H(\sigma, s(\sigma,\zeta+\delta)) - H(\sigma, s(\sigma,\zeta))|\, d\zeta \\
&= \int_{-1}^{1-\delta} \left| \frac{m(\sigma, s(\sigma,\zeta+\delta))}{(1-(\zeta+\delta))^{\gamma}} - \frac{m(\sigma, s(\sigma,\zeta))}{(1-\zeta)^{\gamma}} \right| d\zeta \\
&\leq \int_{-1}^{1-\delta} \left| \frac{m(\sigma, s(\sigma,\zeta+\delta))}{(1-(\zeta+\delta))^{\gamma}} - \frac{m(\sigma, s(\sigma,\zeta+\delta))}{(1-\zeta)^{\gamma}} \right| d\zeta \\
&\quad + \int_{-1}^{1-\delta} \left| \frac{m(\sigma, s(\sigma,\zeta+\delta))}{(1-\zeta)^{\gamma}} - \frac{m(\sigma, s(\sigma,\zeta))}{(1-\zeta)^{\gamma}} \right| d\zeta \\
&\leq \sup_{\substack{-1\leq\sigma\leq 1\\ -1\leq\zeta\leq\zeta+\delta\leq 1}} |m(\sigma, s(\sigma,\zeta+\delta))| \int_{-1}^{1-\delta} \left| \frac{1}{(1-(\zeta+\delta))^{\gamma}} - \frac{1}{(1-\zeta)^{\gamma}} \right| d\zeta \\
&\quad + \sup_{\substack{-1\leq\sigma\leq 1\\ -1\leq\zeta\leq\zeta+\delta\leq 1}} |m(\sigma, s(\sigma,\zeta+\delta)) - m(\sigma, s(\sigma,\zeta))| \int_{-1}^{1-\delta} \left| \frac{1}{(1-\zeta)^{\gamma}} \right| d\zeta. \qquad (3.7)
\end{aligned}$$

Now, using mean value theorem, we have

$$\begin{aligned}
&\sup_{\substack{-1\leq\sigma\leq 1\\ -1\leq\zeta\leq\zeta+\delta\leq 1}} |m(\sigma, s(\sigma,\zeta+\delta)) - m(\sigma, s(\sigma,\zeta))| \\
&= \sup_{\substack{-1\leq\sigma\leq 1\\ -1\leq\zeta\leq\zeta+\delta\leq 1}} |m^{(0,1)}(\sigma,\xi)(s(\sigma,\zeta+\delta) - s(\sigma,\zeta))| \\
&= \sup_{\substack{-1\leq\sigma\leq 1\\ -1\leq\zeta\leq\zeta+\delta\leq 1}} |m^{(0,1)}(\sigma,\xi)| \left| \frac{\sigma+1}{2}(\zeta+\delta) - \frac{\sigma+1}{2}(\zeta) \right| \\
&\leq \sup_{\substack{-1\leq\sigma\leq 1\\ -1\leq\zeta\leq\zeta+\delta\leq 1}} |m^{(0,1)}(\sigma,\xi)||\delta| \\
&\leq C_1|\delta|, \qquad (3.8)
\end{aligned}$$

where $s(\sigma,\zeta) < \xi < s(\sigma,\zeta+\delta)$ and $C_1 = \sup_{-1\leq\sigma,\zeta\leq 1} |m^{(0,1)}(\sigma, s(\sigma,\zeta))|$.

Also note that

$$\int_{-1}^{1-\delta} \left| \frac{1}{(1-\zeta)^{\gamma}} \right| d\zeta = \int_{-1}^{1-\delta} \frac{1}{(1-\zeta)^{\gamma}}\, d\zeta = \frac{2^{-\gamma+1} - \delta^{-\gamma+1}}{1-\gamma} < \infty, \qquad (3.9)$$

and

$$\int_{-1}^{1-\delta} \left| \frac{1}{(1-(\zeta+\delta))^{\gamma}} - \frac{1}{(1-\zeta)^{\gamma}} \right| d\zeta \leq \frac{\delta^{1-\gamma}}{1-\gamma}. \qquad (3.10)$$

Then combining the estimates (3.7–3.10) with the estimate (3.6), we obtain

$$||H(\sigma, s(\sigma,\cdot)) - v_\sigma||_{L^1} = \mathcal{O}(n^{-(1-\gamma)}).$$

Hence the proof. □

Lemma 3.5. Let $H(\sigma, s(\sigma,\zeta))$ be the kernel of the form (2.4) of the integral operator defined by (2.6), then for each $\sigma \in [-1,1]$, $\exists$ a polynomial v_σ of degree $\leq n$ such that

$$||H(\sigma, s(\sigma,\cdot)) - v_\sigma||_{L^2} = \mathcal{O}(n^{-(\frac{1}{2}-\gamma)}), \quad 0<\gamma<\frac{1}{2}.$$

Proof . Note that $H(\sigma, s(\sigma,\zeta)) \in L^2([-1,1])$ for $0<\gamma<\frac{1}{2}$. Then, from (Schumaker [30], p. 92), for $r=1, p=2$, we get

$$||H(\sigma, s(\sigma,\cdot)) - v_\sigma||_{L^2} \leq C\omega_{1,2}\left(H(\sigma, s(\sigma,\zeta)), \frac{1}{n}\right),$$

where $\omega_{1,2}$ denotes the first order modulus of smoothness w.r.t. L^2 norm. Let $I_\delta = [-1, 1-\delta]$, then using the definition of modulus of smoothness (see Schumaker [30], p. 55), we get

$$\begin{aligned}
||H(\sigma, s(\sigma, \cdot)) - v_\sigma||_{L^2} &\leq C\omega_{1,2}\left(H(\sigma, s(\sigma,\zeta)), \frac{1}{n}\right) \\
&= C \sup_{0\leq\delta\leq\frac{1}{n}} ||\Delta_\delta H(\sigma, s(\sigma,\zeta))||_{L^2(I_\delta)} \\
&= C \sup_{0\leq\delta\leq\frac{1}{n}} \left[\int_{-1}^{1-\delta} |H(\sigma, s(\sigma,\zeta+\delta)) - H(\sigma, s(\sigma,\zeta))|^2\, d\zeta\right]^{\frac{1}{2}}. \qquad (3.11)
\end{aligned}$$

Now using $H(\sigma, s(\sigma,\zeta)) = (1-\zeta)^{-\gamma} m(\sigma, s(\sigma,\zeta))$, we have

$$\begin{aligned}
&\int_{-1}^{1-\delta} |H(\sigma, s(\sigma,\zeta+\delta)) - H(\sigma, s(\sigma,\zeta))|^2\, d\zeta \\
&= \int_{-1}^{1-\delta} \left|\frac{m(\sigma, s(\sigma,\zeta+\delta))}{(1-(\zeta+\delta))^\gamma} - \frac{m(\sigma, s(\sigma,\zeta))}{(1-\zeta)^\gamma}\right|^2 d\zeta \\
&\leq \int_{-1}^{1-\delta} \left[\left|\frac{m(\sigma, s(\sigma,\zeta+\delta))}{(1-(\zeta+\delta))^\gamma} - \frac{m(\sigma, s(\sigma,\zeta+\delta))}{(1-\zeta)^\gamma}\right|\right. \\
&\qquad \left. + \left|\frac{m(\sigma, s(\sigma,\zeta+\delta))}{(1-\zeta)^\gamma} - \frac{m(\sigma, s(\sigma,\zeta))}{(1-\zeta)^\gamma}\right|\right]^2 d\zeta \\
&\leq \int_{-1}^{1-\delta} \left|\frac{m(\sigma, s(\sigma,\zeta+\delta))}{(1-(\zeta+\delta))^\gamma} - \frac{m(\sigma, s(\sigma,\zeta+\delta))}{(1-\zeta)^\gamma}\right|^2 d\zeta \\
&\qquad + \int_{-1}^{1-\delta} \left|\frac{m(\sigma, s(\sigma,\zeta+\delta))}{(1-\zeta)^\gamma} - \frac{m(\sigma, s(\sigma,\zeta))}{(1-\zeta)^\gamma}\right|^2 d\zeta \\
&\qquad + 2\int_{-1}^{1-\delta} \left[\left|\frac{m(\sigma, s(\sigma,\zeta+\delta))}{(1-(\zeta+\delta))^\gamma} - \frac{m(\sigma, s(\sigma,\zeta+\delta))}{(1-\zeta)^\gamma}\right|\right. \\
&\qquad \times \left.\left|\frac{m(\sigma, s(\sigma,\zeta+\delta))}{(1-\zeta)^\gamma} - \frac{m(\sigma, s(\sigma,\zeta))}{(1-\zeta)^\gamma}\right|\right] d\zeta \\
&= I_1 + I_2 + I_3. \qquad (3.12)
\end{aligned}$$

Now consider

$$\begin{aligned}
I_1 &= \int_{-1}^{1-\delta} \left|\frac{m(\sigma, s(\sigma,\zeta+\delta))}{(1-(\zeta+\delta))^\gamma} - \frac{m(\sigma, s(\sigma,\zeta+\delta))}{(1-\zeta)^\gamma}\right|^2 d\zeta \\
&\leq \sup_{-1\leq\sigma,\zeta+\delta\leq 1} |m(\sigma, s(\sigma,\zeta+\delta))|^2 \int_{-1}^{1-\delta} \left|\frac{1}{(1-(\zeta+\delta))^\gamma} - \frac{1}{(1-\zeta)^\gamma}\right|^2 d\zeta \\
&\leq C_1 \int_{-1}^{1-\delta} \left|\frac{1}{(1-(\zeta+\delta))^\gamma} - \frac{1}{(1-\zeta)^\gamma}\right|^2 d\zeta, \qquad (3.13)
\end{aligned}$$

where $C_1 = \sup_{-1\leq\sigma,\zeta+\delta\leq 1} |m(\sigma, s(\sigma,\zeta+\delta))|^2$.

Now

$$\begin{aligned}
&\int_{-1}^{1-\delta}\left|\frac{1}{(1-(\zeta+\delta))^{\gamma}}-\frac{1}{(1-\zeta)^{\gamma}}\right|^{2}d\zeta\\
&=\int_{-1}^{1-\delta}\left[\frac{1}{(1-(\zeta+\delta))^{\gamma}}-\frac{1}{(1-\zeta)^{\gamma}}\right]^{2}d\zeta\\
&=\int_{-1}^{1-\delta}\left[\frac{1}{(1-(\zeta+\delta))^{2\gamma}}+\frac{1}{(1-\zeta)^{2\gamma}}-\frac{2}{(1-(\zeta+\delta))^{\gamma}(1-\zeta)^{\gamma}}\right]d\zeta\\
&\leq\int_{-1}^{1-\delta}\left[\frac{1}{(1-(\zeta+\delta))^{2\gamma}}+\frac{1}{(1-\zeta)^{2\gamma}}-\frac{2}{(1-\zeta)^{2\gamma}}\right]d\zeta\\
&=\frac{1}{1-2\gamma}\left[(\delta)^{1-2\gamma}+(2-\delta)^{1-2\gamma}-(2)^{1-2\gamma}\right]. \qquad (3.14)
\end{aligned}$$

Hence combining the estimates (3.14 with the estimate (3.13), we obtain

$$\begin{aligned}
I_1&\leq C_1\frac{1}{1-2\gamma}\left[(\delta)^{-2\gamma+1}-\{2^{-2\gamma+1}-(2-\delta)^{-2\gamma+1}\}\right]\\
&\leq\frac{C_1}{(1-2\gamma)}\delta^{1-2\gamma}. \qquad (3.15)
\end{aligned}$$

Now, for I_2, we have

$$\begin{aligned}
I_2&=\int_{-1}^{1-\delta}\left|\frac{m(\sigma,s(\sigma,\zeta+\delta))}{(1-\zeta)^{\gamma}}-\frac{m(\sigma,s(\sigma,\zeta))}{(1-\zeta)^{\gamma}}\right|^{2}d\zeta\\
&\leq\sup_{\substack{-1\leq\sigma\leq1\\-1\leq\zeta\leq\zeta+\delta\leq1}}|m(\sigma,s(\sigma,\zeta+\delta))-m(\sigma,s(\sigma,\zeta))|^{2}\int_{-1}^{1-\delta}\left|\frac{1}{(1-\zeta)^{\gamma}}\right|^{2}d\zeta. \qquad (3.16)
\end{aligned}$$

Consider

$$\begin{aligned}
&\sup_{\substack{-1\leq\sigma\leq1\\-1\leq\zeta\leq\zeta+\delta\leq1}}|m(\sigma,s(\sigma,\zeta+\delta))-m(\sigma,s(\sigma,\zeta))|^{2}\\
&=\sup_{\substack{-1\leq\sigma\leq1\\-1\leq\zeta\leq\zeta+\delta\leq1}}\left|m^{(0,1)}(\sigma,\xi)(s(\sigma,\zeta+\delta)-s(\sigma,\zeta))\right|^{2}\\
&=\sup_{\substack{-1\leq\sigma\leq1\\-1\leq\zeta\leq\zeta+\delta\leq1}}\left|m^{(0,1)}(\sigma,\xi)\right|^{2}\left|\frac{\sigma+1}{2}\right|^{2}|\delta|^{2}\\
&\leq\sup_{\substack{-1\leq\sigma\leq1\\-1\leq\zeta\leq\zeta+\delta\leq1}}\left|m^{(0,1)}(\sigma,\xi)\right|^{2}|\delta|^{2}\\
&\leq C_2|\delta|^{2}, \qquad (3.17)
\end{aligned}$$

where $s(\sigma,\zeta)<\xi<s(\sigma,\zeta+\delta)$ and $C_2=\sup_{-1\leq\sigma,\zeta\leq1}|m^{(0,1)}(\sigma,s(\sigma,\zeta))|^{2}$.

Also we have

$$\int_{-1}^{1-\delta}\left|\frac{1}{(1-\zeta)^{\gamma}}\right|^{2}d\zeta=\int_{-1}^{1-\delta}\frac{1}{(1-\zeta)^{2\gamma}}d\zeta=\frac{2^{-2\gamma+1}-\delta^{-2\gamma+1}}{1-2\gamma}<\infty,\ 0<\gamma<\frac{1}{2}. \qquad (3.18)$$

Then using (3.17) and (3.18) in the estimate (3.16), we obtain

$$I_2\leq C_2\frac{2^{-2\gamma+1}-\delta^{-2\gamma+1}}{1-2\gamma}\delta^{2}. \qquad (3.19)$$

Now for I_3, we have

$$\begin{aligned} I_3 &= 2\int_{-1}^{1-\delta}\left[\left|\frac{m(\sigma,s(\sigma,\zeta+\delta))}{(1-(\zeta+\delta))^{\gamma}}-\frac{m(\sigma,s(\sigma,\zeta+\delta))}{(1-\zeta)^{\gamma}}\right|\right.\\ &\qquad\times\left.\left|\frac{m(\sigma,s(\sigma,\zeta+\delta))}{(1-\zeta)^{\gamma}}-\frac{m(\sigma,s(\sigma,\zeta))}{(1-\zeta)^{\gamma}}\right|\right]d\zeta\\ &\le 2\sup_{\substack{-1\le\sigma\le 1\\ -1\le\zeta\le\zeta+\delta\le 1}}|m(\sigma,s(\sigma,\zeta+\delta))(m(\sigma,s(\sigma,\zeta+\delta))-m(\sigma,s(\sigma,\zeta)))|\\ &\qquad\times\int_{-1}^{1-\delta}\left|\frac{1}{(1-\zeta)^{\gamma}}\right|\left|\frac{1}{(1-(\zeta+\delta))^{\gamma}}-\frac{1}{(1-\zeta)^{\gamma}}\right|d\zeta. \end{aligned} \tag{3.20}$$

Now, for the first term of I_3, we have

$$\begin{aligned} &\sup_{\substack{-1\le\sigma\le 1\\ -1\le\zeta\le\zeta+\delta\le 1}}|m(\sigma,s(\sigma,\zeta+\delta))\,(m(\sigma,s(\sigma,\zeta+\delta))-m(\sigma,s(\sigma,\zeta)))|\\ &=\sup_{\substack{-1\le\sigma\le 1\\ -1\le\zeta\le\zeta+\delta\le 1}}\left|m(\sigma,s(\sigma,\zeta+\delta))m^{(0,1)}(\sigma,\xi)(s(\sigma,\zeta+\delta)-s(\sigma,\zeta))\right|\\ &=\sup_{\substack{-1\le\sigma\le 1\\ -1\le\zeta\le\zeta+\delta\le 1}}\left|m(\sigma,s(\sigma,\zeta+\delta))m^{(0,1)}(\sigma,\xi)\right|\left|\frac{\sigma+1}{2}\right||\delta|\\ &\le\sup_{\substack{-1\le\sigma\le 1\\ -1\le\zeta\le\zeta+\delta\le 1}}\left|m(\sigma,s(\sigma,\zeta+\delta))m^{(0,1)}(\sigma,\xi)\right||\delta|\\ &\le C_3|\delta|, \end{aligned} \tag{3.21}$$

where $s(\sigma,\zeta)<\xi<s(\sigma,\zeta+\delta)$ and $C_3=\sup\limits_{\substack{-1\le\sigma\le 1\\ -1\le\zeta\le\zeta+\delta\le 1}}|m(\sigma,s(\sigma,\zeta+\delta))m^{(0,1)}(\sigma,s(\sigma,\zeta))|^2$.

Also, note that

$$\begin{aligned} &\int_{-1}^{1-\delta}\left|\frac{1}{(1-\zeta)^{\gamma}}\right|\left|\frac{1}{(1-(\zeta+\delta))^{\gamma}}-\frac{1}{(1-\zeta)^{\gamma}}\right|d\zeta\\ &=\int_{-1}^{1-\delta}\left[\frac{1}{(1-\zeta)^{\gamma}\,(1-(\zeta+\delta))^{\gamma}}-\frac{1}{(1-\zeta)^{2\gamma}}\right]d\zeta\\ &\le\int_{-1}^{1-\delta}\left[\frac{1}{(1-(\zeta+\delta))^{2\gamma}}-\frac{1}{(1-\zeta)^{2\gamma}}\right]d\zeta\\ &=\frac{1}{1-2\gamma}\left[(\delta)^{-2\gamma+1}-\{2^{-2\gamma+1}-(2-\delta)^{-2\gamma+1}\}\right]\\ &\le\frac{\delta^{1-2\gamma}}{1-2\gamma}. \end{aligned} \tag{3.22}$$

Hence using (3.21) and (3.22) in the estimate (3.20), we obtain

$$I_3\le 2C_3\delta\frac{\delta^{1-2\gamma}}{1-2\gamma}. \tag{3.23}$$

Then combining the estimates (3.12), (3.15), (3.19) and (3.23) with the estimate (3.11), we obtain

$$||H(\sigma,s(\sigma,\cdot))-v_{\sigma}||_{L^2}=\mathcal{O}(n^{-\frac{1}{2}(1-2\gamma)})=\mathcal{O}(n^{-(\frac{1}{2}-\gamma)}).$$

Hence the proof. □

Lemma 3.6. Let $\mathcal{P}_n:\mathbb{X}\to\mathbb{X}_n$ be the orthogonal projection operator defined by (2.8) and φ be the exact solution of (2.7). Then there hold

(i) when $\varphi \in \mathcal{C}^r[-1,1]$,

$$||\mathcal{K}(I-\mathcal{P}_n)\varphi||_\infty = \mathcal{O}(n^{-\frac{1}{2}-r+\gamma}), \tag{3.24}$$
$$||\mathcal{K}(I-\mathcal{P}_n)\varphi||_{L^2} = \mathcal{O}(n^{-\frac{1}{2}-r+\gamma}), \tag{3.25}$$

(ii) when φ is non-smooth, i.e., $\varphi'(\sigma) \sim (1+\sigma)^{-\gamma}$, $0<\gamma<1$,

$$||\mathcal{K}(I-\mathcal{P}_n)\varphi||_\infty = \mathcal{O}(n^{-2(1-\gamma)}\log n), \tag{3.26}$$
$$||\mathcal{K}(I-\mathcal{P}_n)\varphi||_{L^2} = \mathcal{O}(n^{-2(1-\gamma)}\log n). \tag{3.27}$$

Moreover,

$$||\mathcal{K}(I-\mathcal{P}_n)||_\infty = \mathcal{O}(n^{-(1-\gamma)}\log n), \quad 0<\gamma<1, \tag{3.28}$$
$$||\mathcal{K}(I-\mathcal{P}_n)||_{L^2} = \mathcal{O}(n^{-(\frac{1}{2}-\gamma)}), \quad 0<\gamma<\frac{1}{2}. \tag{3.29}$$

Proof . Consider for any $\sigma \in [-1,1]$,

$$\begin{aligned} |(\mathcal{K}\mathcal{P}_n-\mathcal{K})\varphi(\sigma)| &= \left| \int_{-1}^{1} H(\sigma,s(\sigma,\zeta))(\mathcal{P}_n-I)\varphi(s(\sigma,\zeta))\,d\zeta \right| \\ &= |\langle H(\sigma,s(\sigma,\zeta))-\varphi_\sigma, (\mathcal{P}_n-I)\varphi\rangle| \qquad (3.30)\\ &\le ||H(\sigma,s(\sigma,\zeta))-\varphi_\sigma||_{L^1}||(\mathcal{P}_n-I)\varphi||_\infty. \qquad (3.31)\end{aligned}$$

Now when $\varphi \in \mathcal{C}^r[-1,1]$, using Lemma 3.4 and estimate (2.12) in (3.31), we have

$$\begin{aligned} ||(\mathcal{K}\mathcal{P}_n-\mathcal{K})\varphi||_\infty &\le ||H(\sigma,s(\sigma,\zeta))-\varphi_\sigma||_{L^1}||(\mathcal{P}_n-I)\varphi||_\infty \\ &\le Cn^{-(1-\gamma)}n^{\frac{1}{2}-r}V(\varphi^{(r)}) \\ &= \mathcal{O}(n^{-\frac{1}{2}-r+\gamma}), \qquad (3.32)\end{aligned}$$

which proves (3.24).
And for L^2 norm, from (3.32), it follows that

$$||(\mathcal{K}\mathcal{P}_n-\mathcal{K})\varphi||_{L^2} \le \sqrt{2}||(\mathcal{K}\mathcal{P}_n-\mathcal{K})\varphi||_\infty = \mathcal{O}(n^{-\frac{1}{2}-r+\gamma}),$$

which proves (3.25).
Also when φ is non-smooth, using Lemma 3.4 and estimate (3.4) in (3.31), we have

$$\begin{aligned} ||(\mathcal{K}\mathcal{P}_n-\mathcal{K})\varphi||_\infty &\le ||H(\sigma,s(\sigma,\zeta))-\varphi_\sigma||_{L^1}||(\mathcal{P}_n-I)\varphi||_\infty \\ &\le Cn^{-(1-\gamma)}\mathcal{O}(n^{-(1-\gamma)}\log n) \\ &= \mathcal{O}(n^{-2(1-\gamma)}\log n),\end{aligned}$$

which proves (3.26), and this implies that

$$||(\mathcal{K}\mathcal{P}_n-\mathcal{K})\varphi||_{L^2} \le \sqrt{2}||(\mathcal{K}\mathcal{P}_n-\mathcal{K})\varphi||_\infty = \mathcal{O}(n^{-2(1-\gamma)}\log n),$$

which proves (3.27).
Moreover, from estimate (3.31), using Lemma 3.4 and estimate (2.13), we have

$$\begin{aligned} ||(\mathcal{K}\mathcal{P}_n-\mathcal{K})\varphi||_\infty &\le ||H(\sigma,s(\sigma,\zeta))-\varphi_\sigma||_{L^1}||(\mathcal{P}_n-I)\varphi||_\infty \\ &\le Cn^{-(1-\gamma)}(1+p_2\log n)||\varphi||_\infty, \qquad (3.33)\end{aligned}$$

and this follows that

$$||(\mathcal{K}\mathcal{P}_n-\mathcal{K})||_\infty \le Cn^{-(1-\gamma)}(1+p_2\log n), \tag{3.34}$$

which proves (3.28).
Again for any $\sigma \in [-1,1]$, using Lemma 3.5, from (3.30), we have

$$\begin{aligned} |(\mathcal{K}\mathcal{P}_n - \mathcal{K})\varphi(\sigma)| &= |\langle H(\sigma, s(\sigma,\zeta)) - \varphi_\sigma, (\mathcal{P}_n - I)\varphi\rangle| \\ &\leq ||H(\sigma, s(\sigma,\zeta)) - \varphi_\sigma||_{L^2} ||(\mathcal{P}_n - I)\varphi||_{L^2} \quad (3.35) \\ &\leq Cn^{-(\frac{1}{2}-\gamma)} ||\varphi||_{L^2}. \quad (3.36) \end{aligned}$$

From this it follows that

$$||\mathcal{K}(\mathcal{P}_n - I)\varphi||_\infty \leq Cn^{-(\frac{1}{2}-\gamma)} ||\varphi||_{L^2}.$$

Hence we obtain

$$||\mathcal{K}(\mathcal{P}_n - I)\varphi||_{L^2} \leq \sqrt{2} ||\mathcal{K}(I - \mathcal{P}_n)\varphi||_\infty \leq \sqrt{2} Cn^{-(\frac{1}{2}-\gamma)} ||\varphi||_{L^2}.$$

This implies

$$||\mathcal{K}(\mathcal{P}_n - I)||_{L^2} \leq \sqrt{2} Cn^{-(\frac{1}{2}-\gamma)}, \quad (3.37)$$

which proves (3.29).
This completes the proof. □

Theorem 3.7. Let $\mathcal{P}_n : \mathbb{X} \to \mathbb{X}_n$ be the orthogonal projection operator defined by (2.8) and $\mathcal{K}$ be the integral operator defined by (2.6). Also assume 1 is not an eigen value of $\mathcal{K}$. Then for $0 < \gamma < 1$, $||(I - \mathcal{K}\mathcal{P}_n)^{-1}||_\infty$ exists and uniformly bounded i.e., $\exists$ a constant $\mathcal{L} > 0$ such that $||(I - \mathcal{K}\mathcal{P}_n)^{-1}||_\infty \leq \mathcal{L} < \infty$ for n sufficiently large.

Also for $0 < \gamma < \frac{1}{2}$, for n large enough, $||(I - \mathcal{K}\mathcal{P}_n)^{-1}||_{L^2}$ exists and it is uniformly bounded i.e., $\exists\ \mathcal{L}_1 > 0$ such that $||(I - \mathcal{K}\mathcal{P}_n)^{-1}||_{L^2} \leq \mathcal{L}_1 < \infty$.

Proof . From Lemma 3.6, we have

$$||\mathcal{K}\mathcal{P}_n - \mathcal{K}||_\infty \leq Cn^{-(1-\gamma)}(1 + p_2 \log n) \to 0 \ \text{ as } n \to \infty \quad \text{for } 0 < \gamma < 1, \quad (3.38)$$

and

$$||\mathcal{K}\mathcal{P}_n - \mathcal{K}||_{L^2} \leq \sqrt{2} Cn^{-(\frac{1}{2}-\gamma)} \to 0 \ \text{ as } n \to \infty, \quad \text{for } 0 < \gamma < \frac{1}{2}. \quad (3.39)$$

Since 1 is not an eigenvalue of $\mathcal{K}$, $||\mathcal{K}\mathcal{P}_n - \mathcal{K}||_\infty \to 0$ for $0 < \gamma < 1$ and $||\mathcal{K}\mathcal{P}_n - \mathcal{K}||_{L^2} \to 0$ for $0 < \gamma < \frac{1}{2}$, from Ahues et al. [1], this implies that for n large enough, $(I - \mathcal{K}\mathcal{P}_n)^{-1}$ exists and uniformly bounded in infinity norm for $0 < \gamma < 1$ and in L^2-norm for $0 < \gamma < \frac{1}{2}$, respectively, i.e., $\exists$ constants $\mathcal{L}, \mathcal{L}_1 > 0$ such that $||(I - \mathcal{K}\mathcal{P}_n)^{-1}||_\infty \leq \mathcal{L} < \infty$, $0 < \gamma < 1$, and $||(I - \mathcal{K}\mathcal{P}_n)^{-1}||_{L^2} \leq \mathcal{L}_1 < \infty$, $0 < \gamma < \frac{1}{2}$. This completes the proof. □

Now we will discuss the convergence of the iterated Legendre Galerkin solution to the exact solution in both cases: for sufficiently smooth exact solution and for non-smooth exact solution i.e., $\varphi'(\sigma) \sim (1+\sigma)^{-\gamma}$, $0 < \gamma < 1$.

Theorem 3.8. Let $\varphi \in \mathcal{C}^r([-1,1])$ be the exact solution of (2.7) and $\tilde{\varphi}_n$ be the iterated Legendre Galerkin approximate solution of φ. Then for the kernel $H(\sigma, s(\sigma,\zeta))$ of the integral operator (2.6) and $f \in \mathcal{C}^r([-1,1])$, there hold

$$||\varphi - \tilde{\varphi}_n||_\infty = \mathcal{O}(n^{-\frac{1}{2}-r+\gamma}), \quad 0 < \gamma < 1,$$

and

$$||\varphi - \tilde{\varphi}_n||_{L^2} = \mathcal{O}(n^{-\frac{1}{2}-r+\gamma}), \quad 0 < \gamma < 1.$$

Proof . Using (2.7) and (2.16), we obtain

$$\begin{aligned} \varphi - \tilde{\varphi}_n &= (I - \mathcal{K})^{-1} f - (I - \mathcal{K}\mathcal{P}_n)^{-1} f \\ &= (I - \mathcal{K}\mathcal{P}_n)^{-1} \left[(I - \mathcal{K}\mathcal{P}_n) - (I - \mathcal{K})\right] (I - \mathcal{K})^{-1} f \\ &= (I - \mathcal{K}\mathcal{P}_n)^{-1} (\mathcal{K} - \mathcal{K}\mathcal{P}_n)\varphi. \quad (3.40) \end{aligned}$$

Then using the estimate (3.24) and Theorem 3.7, we obtain

$$\begin{aligned} ||\varphi - \tilde{\varphi}_n||_\infty &\leq ||(I - \mathcal{K}\mathcal{P}_n)^{-1}||_\infty ||(\mathcal{K} - \mathcal{K}\mathcal{P}_n)\varphi||_\infty \\ &\leq \mathcal{L} ||\mathcal{K}(I - \mathcal{P}_n)\varphi||_\infty \quad (3.41) \\ &= \mathcal{O}(n^{-\frac{1}{2}-r+\gamma}). \quad (3.42) \end{aligned}$$

For L^2 error bounds, for $0 < \gamma < 1$, from (3.42), we get

$$||\varphi - \tilde{\varphi}_n||_{L^2} \leq \sqrt{2}||\varphi - \tilde{\varphi}_n||_\infty \leq \sqrt{2}\mathcal{L}||\mathcal{K}(I - \mathcal{P}_n)\varphi||_\infty = \mathcal{O}(n^{-\frac{1}{2}-r+\gamma}).$$

Hence the proof. □

Theorem 3.9. Let φ be the nonsmooth exact solution of (2.7) which behaves $\varphi'(\sigma) \sim (1+\sigma)^{-\gamma}$, $0 < \gamma < 1$, and $\tilde{\varphi}_n$ be the iterated Legendre Galerkin approximation of the exact solution φ. Then for the kernel $H(\sigma, s(\sigma, \zeta))$ of the integral operator (2.6) and $f \in \mathcal{C}^r([-1, 1])$, there hold

$$||\varphi - \tilde{\varphi}_n||_\infty = \mathcal{O}(n^{-2(1-\gamma)} \log n), \quad 0 < \gamma < 1,$$

and

$$||\varphi - \tilde{\varphi}_n||_{L^2} = \mathcal{O}(n^{-2(1-\gamma)} \log n), \quad 0 < \gamma < 1.$$

Proof . Using estimate (3.26), from (3.41), we obtain

$$||\varphi - \tilde{\varphi}_n||_\infty \leq \mathcal{L}||\mathcal{K}(I - \mathcal{P}_n)\varphi||_\infty = \mathcal{O}(n^{-2(1-\gamma)} \log n). \tag{3.43}$$

For L^2-norm, from above, it implies that for $0 < \gamma < 1$,

$$||\varphi - \tilde{\varphi}_n||_{L^2} \leq \sqrt{2}||\varphi - \tilde{\varphi}_n||_\infty \leq \sqrt{2}\mathcal{L}||\mathcal{K}(I - \mathcal{P}_n)\varphi||_\infty = \mathcal{O}(n^{-2(1-\gamma)} \log n).$$

Hence the proof. □

Next we derive the convergence rates for Legendre Galerkin approximate solution in both infinity and L^2-norms.

Theorem 3.10. Let $\varphi \in \mathcal{C}^r([-1, 1])$ be the exact solution of the equation (2.7) and φ_n be the Legendre Galerkin approximate solution of the exact solution φ. Then for the kernel $H(\sigma, s(\sigma, \zeta))$ of the integral operator (2.6) and $f \in \mathcal{C}^r([-1, 1])$, there hold

$$||\varphi - \varphi_n||_\infty = \mathcal{O}(n^{\frac{1}{2}-r}), \quad 0 < \gamma < 1,$$

and

$$||\varphi - \varphi_n||_{L^2} = \begin{cases} \mathcal{O}(n^{-r}), & \text{for } 0 < \gamma \leq \frac{1}{2}, \\ \mathcal{O}(n^{-\frac{1}{2}-r+\gamma}), & \text{for } \frac{1}{2} \leq \gamma < 1. \end{cases}$$

Proof . Using $\varphi_n = \mathcal{P}_n\tilde{\varphi}_n$, we have

$$\varphi - \varphi_n = \varphi - \mathcal{P}_n\tilde{\varphi}_n = \varphi - \mathcal{P}_n\varphi + \mathcal{P}_n\varphi - \mathcal{P}_n\tilde{\varphi}_n. \tag{3.44}$$

Then using estimates (2.12), (2.13), (3.42) and noting that $n^{-(1-\gamma)} \log n \to 0$ as $n \to \infty$, we obtain

$$\begin{aligned} ||\varphi - \varphi_n||_\infty &\leq ||(I - \mathcal{P}_n)\varphi||_\infty + ||\mathcal{P}_n(\varphi - \tilde{\varphi}_n)||_\infty \\ &\leq Cn^{\frac{1}{2}-r}V(\varphi^{(r)}) + (p_2 \log n)||\varphi - \tilde{\varphi}_n||_\infty \\ &\leq Cn^{\frac{1}{2}-r}V(\varphi^{(r)}) + (p_2 \log n)n^{-\frac{1}{2}-r+\gamma}V(\varphi^{(r)}) \\ &= \mathcal{O}(n^{\frac{1}{2}-r}). \end{aligned} \tag{3.45}$$

Again, using Theorem 3.8 and estimates (2.9), (2.10), for $0 < \gamma < 1$, we have

$$\begin{aligned} ||\varphi - \varphi_n||_{L^2} &\leq ||(I - \mathcal{P}_n)\varphi||_{L^2} + ||\mathcal{P}_n(\varphi - \tilde{\varphi}_n)||_{L^2} \\ &\leq Cn^{-r}||\varphi^{(r)}||_\infty + p_1||\varphi - \tilde{\varphi}_n||_\infty \\ &\leq Cn^{-r}||\varphi^{(r)}||_\infty + p_1 n^{-\frac{1}{2}-r+\gamma}V(\varphi^{(r)}) \\ &= \begin{cases} \mathcal{O}(n^{-r}), & \text{for } 0 < \gamma \leq \frac{1}{2}, \\ \mathcal{O}(n^{-\frac{1}{2}-r+\gamma}), & \text{for } \frac{1}{2} \leq \gamma < 1. \end{cases} \end{aligned}$$

Hence the proof. □

Theorem 3.11. Let φ be the nonsmooth exact solution of (2.7) which behaves $\varphi'(\sigma) \sim (1+\sigma)^{-\gamma}$, $0<\gamma<1$, and φ_n be the Legendre Galerkin approximation of the exact solution φ. Then for the kernel $H(\sigma, s(\sigma,\zeta))$ of the integral operator defined in (2.6) and $f \in \mathcal{C}^r([-1,1])$, there hold

$$||\varphi - \varphi_n||_\infty = \mathcal{O}(n^{-(1-\gamma)}\log n), \quad 0<\gamma<1,$$

and

$$||\varphi - \varphi_n||_{L^2} = \mathcal{O}(n^{-(1-\gamma)}), \quad 0<\gamma<1.$$

Proof . When φ is non-smooth, then using estimates (2.13), (3.4), (3.43) and noting that $n^{-(1-\gamma)}\log n \to 0$ as $n \to \infty$, from (3.44), we obtain

$$\begin{aligned} ||\varphi - \varphi_n||_\infty &\leq ||(I-\mathcal{P}_n)\varphi||_\infty + ||\mathcal{P}_n(\varphi - \tilde{\varphi}_n)||_\infty \\ &\leq \mathcal{O}(n^{-(1-\gamma)}\log n) + p_2(\log n)||\varphi - \tilde{\varphi}_n||_\infty \\ &\leq \mathcal{O}(n^{-(1-\gamma)}\log n) + p_2(\log n)\mathcal{O}(n^{-2(1-\gamma)}\log n) \\ &= \mathcal{O}(n^{-(1-\gamma)}\log n). \end{aligned} \tag{3.46}$$

Also, for L^2-norms, using Theorem 3.9 and estimates (2.9), (3.5), we have

$$\begin{aligned} ||\varphi - \varphi_n||_{L^2} &\leq ||(I-\mathcal{P}_n)\varphi||_{L^2} + ||\mathcal{P}_n(\varphi - \tilde{\varphi}_n)||_{L^2} \\ &\leq \mathcal{O}(n^{-(1-\gamma)}) + p_1||\varphi - \tilde{\varphi}_n||_\infty \\ &\leq \mathcal{O}(n^{-(1-\gamma)}) + p_1\mathcal{O}(n^{-2(1-\gamma)}\log n) \\ &= \mathcal{O}(n^{-(1-\gamma)}). \end{aligned} \tag{3.47}$$

Hence the proof. □

Remark 3.12. From Theorems 3.8, 3.9, 3.10 and 3.11, we note that iterated Legendre Galerkin method gives better convergence rates than Legendre Galerkin method in both cases, when the exact solution is smooth and nonsmooth.

In the next section, we will improve these convergence rates further for iterated Legendre multi-Galerkin method.

4 Superconvergence results by Legendre multi-Galerkin method

In this section, we define the Legendre multi-Galerkin and iterated Legendre multi-Galerkin methods for solving (2.7) and obtain the superconvergence results over Galerkin and iterated Galerkin approximate solutions. Using the projection operator $\mathcal{P}_n$, we define the multi-projection operator $\mathcal{K}_n^M$ ([16], [26]) by

$$\mathcal{K}_n^M = \mathcal{P}_n\mathcal{K}\mathcal{P}_n + \mathcal{P}_n\mathcal{K}(I-\mathcal{P}_n) + (I-\mathcal{P}_n)\mathcal{K}\mathcal{P}_n. \tag{4.1}$$

Then the Legendre multi-Galerkin method for solving (2.7), is finding an approximation $\varphi_n^M \in \mathbb{X}$ such that

$$\varphi_n^M - \mathcal{K}_n^M\varphi_n^M = f. \tag{4.2}$$

To get more accurate approximation, we define the iterated Legendre multi-Galerkin approximate solution as

$$\tilde{\varphi}_n^M = \mathcal{K}\varphi_n^M + f. \tag{4.3}$$

Now, to solve the equation (4.2), applying $\mathcal{P}_n$ and $(I-\mathcal{P}_n)$ to the equation (4.2), we obtain

$$\mathcal{P}_n\varphi_n^M = \mathcal{P}_n\mathcal{K}\mathcal{P}_n\varphi_n^M + \mathcal{P}_n\mathcal{K}(I-\mathcal{P}_n)\varphi_n^M + \mathcal{P}_n f. \tag{4.4}$$

$$(I-\mathcal{P}_n)\varphi_n^M = (I-\mathcal{P}_n)\mathcal{K}\mathcal{P}_n\varphi_n^M + (I-\mathcal{P}_n)f. \tag{4.5}$$

$$\Rightarrow \varphi_n^M = \mathcal{P}_n\varphi_n^M + (I-\mathcal{P}_n)\mathcal{K}\mathcal{P}_n\varphi_n^M + (I-\mathcal{P}_n)f. \tag{4.6}$$

Using (4.6) in (4.4), we obtain

$$\mathcal{P}_n\varphi_n^M - (\mathcal{P}_n\mathcal{K} + \mathcal{P}_n\mathcal{K}(I-\mathcal{P}_n)\mathcal{K})\mathcal{P}_n\varphi_n^M = \mathcal{P}_n\mathcal{K}(I-\mathcal{P}_n)f + \mathcal{P}_n f. \tag{4.7}$$

Now, letting $\nu_n^M = \mathcal{P}_n\varphi_n^M$, we can find ν_n^M from the equation

$$(I - \mathcal{S}_n\mathcal{K})\nu_n^M = \mathcal{S}_n f, \tag{4.8}$$

where $\mathcal{S}_n = (\mathcal{P}_n + \mathcal{P}_n\mathcal{K}(I - \mathcal{P}_n))$. Using this ν_n^M, we can find φ_n^M from the equation (4.6) as

$$\varphi_n^M = \nu_n^M + (I - \mathcal{P}_n)(\mathcal{K}\nu_n^M + f). \tag{4.9}$$

Next we prove the following theorem which proves the existence and uniqueness of φ_n^M. Note that

$$\mathcal{K} - \mathcal{K}_n^M = (I - \mathcal{P}_n)\mathcal{K}(I - \mathcal{P}_n). \tag{4.10}$$

Theorem 4.1. Let $\varphi \in \mathbb{X}$ and $\mathcal{K}_n^M$ be the multi-projection operator defined by (4.1). Also let 1 is not an eigen value of $\mathcal{K}$. Then for n large enough, $||(I - \mathcal{K}_n^M)^{-1}||_\infty$ exists and uniformly bounded on $\mathbb{X}$ i.e., $\exists$ a constant $\mathcal{L}_2 > 0$ such that $||(I - \mathcal{K}_n^M)^{-1}||_\infty \leq \mathcal{L}_2 < \infty$ for all $0 < \gamma < 1$.

And also for $0 < \gamma < \frac{1}{2}$ and n large enough, $||(I - \mathcal{K}_n^M)^{-1}||_{L^2}$ exists and it is uniformly bounded i.e., $\exists\ \mathcal{L}_3 > 0$ such that $||(I - \mathcal{K}_n^M)^{-1}||_{L^2} \leq \mathcal{L}_3 < \infty$.

Proof . Using estimate (2.13) and Lemma 3.6, we have

$$\begin{aligned}||\mathcal{K} - \mathcal{K}_n^M||_\infty &= ||(I - \mathcal{P}_n)\mathcal{K}(I - \mathcal{P}_n)||_\infty \\ &\leq (1 + ||\mathcal{P}_n||_\infty)||\mathcal{K}(I - \mathcal{P}_n)||_\infty \\ &\leq (1 + p_2 \log n)||\mathcal{K}(I - \mathcal{P}_n)||_\infty \\ &\leq (1 + p_2 \log n)^2 n^{-(1-\gamma)} \to 0 \quad \text{as } n \to \infty, \quad \text{for } 0 < \gamma < 1.\end{aligned} \tag{4.11}$$

Also, using the estimate (3.29) of Lemma 3.6, we obtain

$$\begin{aligned}||\mathcal{K} - \mathcal{K}_n^M||_{L^2} &= ||(I - \mathcal{P}_n)\mathcal{K}(I - \mathcal{P}_n)||_{L^2} \\ &\leq (1 + ||\mathcal{P}_n||_{L^2})||\mathcal{K}(I - \mathcal{P}_n)||_{L^2} \\ &\leq C||\mathcal{K}(I - \mathcal{P}_n)||_{L^2} \\ &\leq C n^{-(\frac{1}{2}-\gamma)} \to 0 \quad \text{as } n \to \infty, \quad \text{for } 0 < \gamma < \frac{1}{2}.\end{aligned} \tag{4.12}$$

Hence using Ahues et al. [1], from (4.11) and (4.12), it implies that $(I - \mathcal{K}_n^M)^{-1}$ exists and uniformly bounded in both infinity and L^2 norms, i.e., $\exists$ constants $\mathcal{L}_2, \mathcal{L}_3 > 0$ such that $||(I - \mathcal{K}_n^M)^{-1}||_\infty \leq \mathcal{L}_2 < \infty$, $0 < \gamma < 1$, and $||(I - \mathcal{K}_n^M)^{-1}||_{L^2} \leq \mathcal{L}_3 < \infty$, $0 < \gamma < \frac{1}{2}$, for n sufficiently large. Hence the proof. □

In the next two theorems, we derive the superconvergence rates of iterated Legendre multi-Galerkin method, for sufficiently smooth exact solution and for non-smooth exact solution, respectively.

Theorem 4.2. Let $\varphi \in \mathcal{C}^r([-1, 1])$ be the exact solution of the equation (2.7). Also let φ_n^M and $\tilde{\varphi}_n^M$ be the Legendre multi-Galerkin and iterated Legendre multi-Galerkin approximations defined in (4.2) and (4.3), respectively. Then for the kernel $H(\sigma, s(\sigma, \zeta))$ of the integral operator defined in (2.6) and $f \in \mathcal{C}^r([-1, 1])$, there hold

$$\begin{aligned}||\varphi - \varphi_n^M||_\infty &= \mathcal{O}(n^{-\frac{1}{2}-r+\gamma} \log n), \quad 0 < \gamma < 1. \\ ||\varphi - \varphi_n^M||_{L^2} &= \begin{cases}\mathcal{O}(n^{-\frac{1}{2}-r+\gamma}), & \text{for } 0 < \gamma < \frac{1}{2}, \\ \mathcal{O}(n^{-\frac{1}{2}-r+\gamma} \log n), & \text{for } \frac{1}{2} \leq \gamma < 1.\end{cases} \\ ||\varphi - \tilde{\varphi}_n^M||_\infty &= \mathcal{O}(n^{\frac{1}{2}-r-2(1-\gamma)} \log n), \quad 0 < \gamma < 1. \\ ||\varphi - \tilde{\varphi}_n^M||_{L^2} &= \mathcal{O}(n^{\frac{1}{2}-r-2(1-\gamma)} \log n), \quad 0 < \gamma < 1.\end{aligned}$$

Proof . Note that

$$\begin{aligned}\varphi - \varphi_n^M &= (I - \mathcal{K})^{-1} f - (I - \mathcal{K}_n^M)^{-1} f \\ &= (I - \mathcal{K}_n^M)^{-1}(\mathcal{K} - \mathcal{K}_n^M)(I - \mathcal{K})^{-1} f \\ &= (I - \mathcal{K}_n^M)^{-1}(\mathcal{K} - \mathcal{K}_n^M)\varphi.\end{aligned} \tag{4.13}$$

Then using Theorem 4.1 and estimates (2.13), (3.24) and (4.13), for $0 < \gamma < 1$, we have

$$\begin{aligned}||\varphi - \varphi_n^M||_\infty &= ||(I - \mathcal{K}_n^M)^{-1}(\mathcal{K} - \mathcal{K}_n^M)\varphi||_\infty \\ &\le ||(I - \mathcal{K}_n^M)^{-1}||_\infty ||(\mathcal{K} - \mathcal{K}_n^M)\varphi||_\infty \\ &\le \mathcal{L}_2 ||(I - \mathcal{P}_n)\mathcal{K}(I - \mathcal{P}_n)\varphi||_\infty \\ &\le \mathcal{L}_2 (1 + ||\mathcal{P}_n||_\infty)||\mathcal{K}(I - \mathcal{P}_n)\varphi||_\infty \qquad (4.14) \\ &\le \mathcal{L}_2(1 + p_2 \log n)\mathcal{O}(n^{-\frac{1}{2}-r+\gamma}) \\ &= \mathcal{O}(n^{-\frac{1}{2}-r+\gamma}\log n). \qquad (4.15)\end{aligned}$$

Next for the L^2-norm, for $0 < \gamma < \frac{1}{2}$, using Theorem 4.1 and estimates (3.25), (4.10), (4.13), we obtain

$$\begin{aligned}||\varphi - \varphi_n^M||_{L^2} &= ||(I - \mathcal{K}_n^M)^{-1}(\mathcal{K} - \mathcal{K}_n^M)\varphi||_{L^2} \\ &\le ||(I - \mathcal{K}_n^M)^{-1}||_{L^2} ||(\mathcal{K} - \mathcal{K}_n^M)\varphi||_{L^2} \\ &\le \mathcal{L}_3 ||(I - \mathcal{P}_n)\mathcal{K}(I - \mathcal{P}_n)\varphi||_{L^2} \\ &\le \mathcal{L}_3 (1 + ||\mathcal{P}_n||_{L^2})||\mathcal{K}(I - \mathcal{P}_n)\varphi||_{L^2} \\ &\le \mathcal{L}_3 C ||\mathcal{K}(I - \mathcal{P}_n)\varphi||_{L^2} \qquad (4.16) \\ &= \mathcal{O}(n^{-\frac{1}{2}-r+\gamma}),\end{aligned}$$

and for $\frac{1}{2} \le \gamma < 1$, from (4.15), it follows that

$$||\varphi - \varphi_n^M||_{L^2} \le \sqrt{2}||\varphi - \varphi_n^M||_\infty = \mathcal{O}(n^{-\frac{1}{2}-r+\gamma}\log n). \qquad (4.17)$$

Now consider

$$\begin{aligned}\varphi - \tilde{\varphi}_n^M &= \mathcal{K}(\varphi - \varphi_n^M) \\ &= \mathcal{K}(I - \mathcal{K})^{-1}(\mathcal{K} - \mathcal{K}_n^M)(I - \mathcal{K}_n^M)^{-1} f \\ &= (I - \mathcal{K})^{-1}\mathcal{K}(I - \mathcal{P}_n)\mathcal{K}(I - \mathcal{P}_n)(\varphi + \varphi_n^M - \varphi).\end{aligned}$$

This implies

$$\begin{aligned}||\varphi - \tilde{\varphi}_n^M||_\infty &\le ||(I - \mathcal{K})^{-1}||_\infty \left[||\mathcal{K}(I - \mathcal{P}_n)\mathcal{K}(I - \mathcal{P}_n)\varphi||_\infty \right. \\ &\quad \left. + ||\mathcal{K}(I - \mathcal{P}_n)\mathcal{K}(I - \mathcal{P}_n)(\varphi - \varphi_n^M)||_\infty\right] \\ &\le ||(I - \mathcal{K})^{-1}||_\infty \left[||\mathcal{K}(I - \mathcal{P}_n)||_\infty ||\mathcal{K}(I - \mathcal{P}_n)\varphi||_\infty \right. \\ &\quad \left. + ||\mathcal{K}(I - \mathcal{P}_n)||_\infty^2 ||\varphi - \varphi_n^M||_\infty\right]. \qquad (4.18)\end{aligned}$$

Then, using the estimates (3.24), (3.28) and (4.15) and noting that $n^{-(1-\gamma)}(\log n)^2 \to 0$ as $n \to \infty$, from (4.18), we obtain

$$||\varphi - \tilde{\varphi}_n^M||_\infty = \mathcal{O}(n^{\frac{1}{2}-r-2(1-\gamma)}\log n), \quad 0 < \gamma < 1, \qquad (4.19)$$

and this gives

$$||\varphi - \tilde{\varphi}_n^M||_{L^2} \le \sqrt{2}||\varphi - \tilde{\varphi}_n^M||_\infty = \mathcal{O}(n^{\frac{1}{2}-r-2(1-\gamma)}\log n), \quad 0 < \gamma < 1.$$

Hence the proof. □

Theorem 4.3. Let φ be the non-smooth exact solution of (2.7) which behaves $\varphi'(\sigma) \sim (1+\sigma)^{-\gamma}$, $0 < \gamma < 1$. Also let φ_n^M and $\tilde{\varphi}_n^M$ be the Legendre multi-Galerkin and iterated Legendre multi-Galerkin approximations defined in (4.2) and (4.3), respectively. Then for the kernel $H(\sigma, s(\sigma, \zeta))$ of the integral operator (2.6) and $f \in \mathcal{C}^r([-1, 1])$, there hold

$$||\varphi - \varphi_n^M||_\infty = \mathcal{O}(n^{-2(1-\gamma)}(\log n)^2), \quad 0 < \gamma < 1.$$

$$||\varphi - \varphi_n^M||_{L^2} = \begin{cases}\mathcal{O}(n^{-2(1-\gamma)}\log n), & \text{for } 0 < \gamma < \frac{1}{2}, \\ \mathcal{O}(n^{-2(1-\gamma)}(\log n)^2), & \text{for } \frac{1}{2} \le \gamma < 1.\end{cases}$$

$$||\varphi - \tilde{\varphi}_n^M||_\infty = \mathcal{O}(n^{-3(1-\gamma)}(\log n)^2), \quad 0 < \gamma < 1.$$

$$||\varphi - \tilde{\varphi}_n^M||_{L^2} = \mathcal{O}(n^{-3(1-\gamma)}(\log n)^2), \quad 0 < \gamma < 1.$$

Proof . When φ is non-smooth, for $0 < \gamma < 1$, using estimate (3.26) in (4.14), we get

$$\begin{aligned} ||\varphi - \varphi_n^M||_\infty &\leq \mathcal{L}_2(1 + p_2 \log n)||\mathcal{K}(I - \mathcal{P}_n)\varphi||_\infty \\ &\leq \mathcal{L}_2(1 + p_2 \log n)\mathcal{O}(n^{-2(1-\gamma)} \log n) \\ &= \mathcal{O}(n^{-2(1-\gamma)}(\log n)^2). \end{aligned} \tag{4.20}$$

For L^2-norms, for $0 < \gamma < \frac{1}{2}$, using (3.26), from (4.16), we obtain

$$\begin{aligned} ||\varphi - \varphi_n^M||_{L^2} &\leq \mathcal{L}_3 C||\mathcal{K}(I - \mathcal{P}_n)\varphi||_{L^2} \\ &\leq \sqrt{2}\mathcal{L}_3 C||\mathcal{K}(I - \mathcal{P}_n)\varphi||_\infty \\ &= \mathcal{O}(n^{-2(1-\gamma)} \log n), \end{aligned}$$

and for $\frac{1}{2} \leq \gamma < 1$, from (4.20), it follows that

$$||\varphi - \varphi_n^M||_{L^2} \leq \sqrt{2}||\varphi - \varphi_n^M||_\infty = \mathcal{O}(n^{-2(1-\gamma)}(\log n)^2).$$

Again, noting that $n^{-(1-\gamma)}(\log n)^2 \to 0$ as $n \to \infty$ and using estimates (3.26), (3.28) and (4.20), from (4.18) we obtain

$$||\varphi - \tilde{\varphi}_n^M||_\infty = \mathcal{O}(n^{-3(1-\gamma)}(\log n)^2), \quad 0 < \gamma < 1, \tag{4.21}$$

and this gives

$$||\varphi - \tilde{\varphi}_n^M||_{L^2} \leq \sqrt{2}||\varphi - \tilde{\varphi}_n^M||_\infty = \mathcal{O}(n^{-3(1-\gamma)}(\log n)^2), \quad 0 < \gamma < 1.$$

Hence the proof. □

Remark 4.4. From Theorems 3.8, 3.9, 3.10, 3.11, 4.2 and 4.3, we observe that in both cases: for smooth exact solution and for non-smooth exact solution, the iterated Legendre multi-Galerkin approximate solution provides better convergence results over the iterated Legendre Galerkin, Legendre Galerkin and Legendre multi-Galerkin approximate solutions.

5 Numerical results

Here we provide the numerical results. To do this, Legendre polynomials are taken as the basis functions of the approximating subspaces $\mathbb{X}_n$. We provide the errors of approximations and iterated approximations of Galerkin and multi-Galerkin methods in both infinity and L^2 norms. In the following tables, φ_n, $\tilde{\varphi}_n$, φ_n^M and $\tilde{\varphi}_n^M$ are the approximate solutions defined in (2.14), (2.15), (4.2) and (4.3), respectively. In Tables [1-4], n denotes the degree of the Legendre polynomials applied in the computations. The numerical algorithms were run on a PC with Intel Pentium 3.20GHz CPU, 4GB RAM and Matlab has been used to compile the programs.

In Tables 1 and 3, the corresponding errors of Examples 5.1 and 5.2 in Legendre Galerkin method and its iterated version are presented. We provide the corresponding errors in Legendre multi-Galerkin method and its iterated version for Examples 5.1 and 5.2 in Tables 2 and 4.

Example 5.1. Consider the weakly singular Volterra integral equation

$$u(\sigma) = \int_0^\sigma (\sigma - s)^{-\frac{1}{2}} u(s)\, ds + f(\sigma), \quad \sigma \in [0, 1],$$

with $f(\sigma) = \sigma^7 \left(1 - \frac{4096}{6435}\sigma^{\frac{1}{2}}\right)$, where the exact solution is given by $u(\sigma) = \sigma^7$, which is smooth. The transformed integral equation is

$$\varphi(\sigma) = \int_{-1}^1 (1 - \zeta)^{-\frac{1}{2}}\, m(\sigma, s(\sigma, \zeta))\varphi(s(\sigma, \zeta))\, d\zeta + \tilde{f}(\sigma), \quad \sigma \in [-1, 1]$$

where $m(\sigma, s(\sigma, \zeta)) = \frac{1}{2}(1 + \sigma)^{\frac{1}{2}}$, $\tilde{f}(\sigma) = \left(\frac{1+\sigma}{2}\right)^7 \left(1 - \frac{4096}{6435}\left(\frac{1+\sigma}{2}\right)^{\frac{1}{2}}\right)$ and the exact solution $\varphi(\sigma) = \left(\frac{1+\sigma}{2}\right)^7$.

Table 1: Legendre Galerkin and iterated Legendre Galerkin methods

n	$\|\|\varphi-\varphi_n\|\|_\infty$	$\|\|\varphi-\varphi_n\|\|_{L^2}$	$\|\|\varphi-\tilde{\varphi}_n\|\|_\infty$	$\|\|\varphi-\tilde{\varphi}_n\|\|_{L^2}$
2	$2.4211803324\times10^{-1}$	$1.8291676443\times10^{-1}$	$2.2668086050\times10^{-1}$	$1.5398825403\times10^{-1}$
3	$9.2535914737\times10^{-2}$	$4.1964190814\times10^{-2}$	$2.8095233616\times10^{-2}$	$2.0917323614\times10^{-2}$
4	$2.6253135271\times10^{-2}$	$1.0202727566\times10^{-2}$	$5.1776158732\times10^{-3}$	$4.4496979323\times10^{-3}$
5	$4.5441484597\times10^{-3}$	$1.5495313233\times10^{-3}$	$7.3387832521\times10^{-4}$	$6.1550295395\times10^{-4}$
6	$3.6182033615\times10^{-4}$	$1.1004775341\times10^{-4}$	$5.2225729447\times10^{-5}$	$4.1038460062\times10^{-5}$
7	$2.7496618147\times10^{-5}$	9.354349217×10^{-6}	4.650071140×10^{-6}	3.425749351×10^{-6}

Table 2: Legendre multi-Galerkin and iterated Legendre multi-Galerkin methods

n	$\|\|\varphi-\varphi_n^M\|\|_\infty$	$\|\|\varphi-\varphi_n^M\|\|_{L^2}$	$\|\|\varphi-\tilde{\varphi}_n^M\|\|_\infty$	$\|\|\varphi-\tilde{\varphi}_n^M\|\|_{L^2}$
2	$8.5800622107\times10^{-2}$	$5.7042887293\times10^{-2}$	$6.9726907676\times10^{-2}$	$4.3735381968\times10^{-2}$
3	$4.1260378814\times10^{-2}$	$1.4823397894\times10^{-2}$	$1.0097379445\times10^{-2}$	$7.2335471273\times10^{-3}$
4	$8.7051150408\times10^{-3}$	$3.4099314941\times10^{-3}$	$1.6861675874\times10^{-3}$	$1.5080652255\times10^{-3}$
5	$1.3278226956\times10^{-3}$	$4.8857679946\times10^{-4}$	$2.2232530968\times10^{-4}$	$1.9946804015\times10^{-4}$
6	$1.1655056314\times10^{-4}$	$3.3888826155\times10^{-5}$	$2.7598815822\times10^{-5}$	$1.5231965387\times10^{-5}$
7	$1.6959468777\times10^{-5}$	3.152457297×10^{-6}	2.750101353×10^{-6}	1.354333363×10^{-6}

Example 5.2. Consider the Volterra integral equation with weakly singular kernel

$$u(\sigma) = -\int_0^\sigma (\sigma-s)^{-\frac{1}{2}} u(s)\,ds + f(\sigma), \quad \sigma\in[0,1],$$

with $f(\sigma)=\frac{1}{2}\pi\sigma+\sigma^{\frac{1}{2}}$, where the exact solution is $u(\sigma)=\sigma^{\frac{1}{2}}$, which is non-smooth. The transformed integral equation is

$$\varphi(\sigma) = \int_{-1}^1 (1-\zeta)^{-\frac{1}{2}}\, m(\sigma,s(\sigma,\zeta))\varphi(s(\sigma,\zeta))\,d\zeta + \tilde{f}(\sigma), \quad \sigma\in[-1,1]$$

where $m(\sigma,s(\sigma,\zeta)) = -\frac{1}{2}\,(1+\sigma)^{\frac{1}{2}}$, $\tilde{f}(\sigma)=\frac{1}{4}\pi(1+\sigma)+\left(\frac{1+\sigma}{2}\right)^{\frac{1}{2}}$ and the exact solution $\varphi(\sigma)=\left(\frac{1+\sigma}{2}\right)^{\frac{1}{2}}$.

Table 3: Legendre Galerkin and iterated Legendre Galerkin methods

n	$\|\|\varphi-\varphi_n\|\|_\infty$	$\|\|\varphi-\varphi_n\|\|_{L^2}$	$\|\|\varphi-\tilde{\varphi}_n\|\|_\infty$	$\|\|\varphi-\tilde{\varphi}_n\|\|_{L^2}$
2	$1.6675509339\times10^{-1}$	$2.8887941729\times10^{-2}$	$2.1011076663\times10^{-2}$	$1.3177158573\times10^{-2}$
3	$1.2440569434\times10^{-1}$	$1.6000141254\times10^{-2}$	$1.1778815836\times10^{-2}$	$6.1381796972\times10^{-3}$
4	$9.9400136932\times10^{-2}$	$1.0161203970\times10^{-2}$	$7.5445544294\times10^{-3}$	$3.3719151780\times10^{-3}$
5	$8.2826675237\times10^{-2}$	$7.0251350834\times10^{-3}$	$5.2478946929\times10^{-3}$	$2.0583048522\times10^{-3}$
6	$7.1014520370\times10^{-2}$	$5.1468248869\times10^{-3}$	$3.8620610762\times10^{-3}$	$1.3518453295\times10^{-3}$
7	$6.2161765519\times10^{-2}$	$3.9331351751\times10^{-3}$	$2.9613350432\times10^{-3}$	$9.3747914771\times10^{-4}$

From Tables 1 and 3, we observe that the iterated Legendre Galerkin solution improves over the the Legendre Galerkin solution. From Tables [1-4], we also observe that the iterated Legendre multi-Galerkin approximate solution improves over the approximate solutions of iterated Legendre Galerkin, Legendre Galerkin and Legendre multi-Galerkin methods.

Table 4: Legendre multi-Galerkin and iterated Legendre multi-Galerkin methods

n	$\|\|\varphi - \varphi_n^M\|\|_\infty$	$\|\|\varphi - \varphi_n^M\|\|_{L^2}$	$\|\|\varphi - \tilde{\varphi}_n^M\|\|_\infty$	$\|\|\varphi - \tilde{\varphi}_n^M\|\|_{L^2}$
2	$2.4482819759 \times 10^{-2}$	$1.1261901456 \times 10^{-2}$	$8.2734114009 \times 10^{-3}$	$5.8068048122 \times 10^{-3}$
3	$1.2516791756 \times 10^{-2}$	$5.3063883050 \times 10^{-3}$	$3.2680082969 \times 10^{-3}$	$2.4400024013 \times 10^{-3}$
4	$7.3966069774 \times 10^{-3}$	$2.9430103345 \times 10^{-3}$	$1.5515648235 \times 10^{-3}$	$1.2271976971 \times 10^{-3}$
5	$4.7754730089 \times 10^{-3}$	$1.8103110143 \times 10^{-3}$	$8.4526526284 \times 10^{-4}$	$6.9278694911 \times 10^{-4}$
6	$3.2985183310 \times 10^{-3}$	$1.1969571985 \times 10^{-3}$	$4.9185251316 \times 10^{-4}$	$4.2479149517 \times 10^{-4}$
7	$2.3750122009 \times 10^{-3}$	$8.3456517096 \times 10^{-4}$	$3.3594205748 \times 10^{-4}$	$2.7662648345 \times 10^{-4}$

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