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Exact solution of nonlinear time-fractional reaction-diffusion-convection equation via a new coupling method

Ali Khalouta

Laboratory of Fundamental and Numerical Mathematics, Department of Mathematics, Faculty of Sciences, Ferhat Abbas Sétif University 1, 19000 Sétif, Algeria

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Abstract

The main aim of this work is to find the exact solution in the form of the Mittag-Leller function for the nonlinear time-fractional reaction-diffusion-convection equation via a new coupling method namely, the Aboodh variational iteration method (AVIM). The proposed method is a coupling of the Aboodh transform method with the variational iteration method and the fractional derivative defined with the Liouville-Caputo operator. Three different numerical applications are given to demonstrate the validity and applicability of the proposed method and compare it to existing methods. The results shown through figures and tables demonstrate the accuracy of our method. It is concluded here that the proposed method is very efficient, simple and can be applied to other nonlinear problems arising in science and engineering.

Keywords: nonlinear time-fractional reaction-diffusion-convection equation, Liouville-Caputo fractional derivative, Aboodh transform, variational iteration method, Mittag-Leffler function, exact solution
2020 MSC: Primary 35R11, 26A33, Secondary 34K28, 35C05.

1 Introduction

Nonlinear fractional differential equations (NFDEs) have appeared in many physical and engineering processes. The wide applicability of these equations is the main reason why they have gained so much attention from many mathematicians and physicists. The study of analytical or numerical solutions of NFDEs plays an important role in mathematical physics, engineering and other sciences. Therefore, an effective and appropriate methods to solve them is very important.

Recently, the fractional natural decomposition method (FNDM) [12], Sumudu homotopy perturbation method (SHPM) [2], homotopy analysis transform method (HATM) [7], optimal homotopy asymptotic method (OHAM) [4], Elzaki projected differential transform method (EPDTM) [11], fractional residual pawer series method (FPSM) [9], fractional reduced differential transform method (FRDTM) [3] have been used for solving a wide range of NFDEs.

Email address: nadjibkh@yahoo.fr; ali.khalouta@univ-setif.dz (Ali Khalouta)

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In this paper we will suggest a new coupling method namely, the Aboodh variational iteration method (AVIM) to find the exact solution of nonlinear time-fractional reaction-diffusion-convection equations. The exact solution is given in terms Mittag-Leffler function.

We consider the nonlinear time-fractional reaction-diffusion-convection equation of the following type

$$D_t^{\alpha} u = (a(u)u_x)_x + b(u)u_x + c(u), \tag{1.1}$$

with the initial condition

$$u(x,0) = u_0(x), \tag{1.2}$$

where D_t^{α} is the Liouville-Caputo time-fractional derivative operator of order α with $0 < \alpha \leq 1$, $u = \{u(x,t), (x,t) \in \mathbb{R} \times \mathbb{R}^+\}$ is an unknown function and the arbitrary smooth functions $a(u), b(u)$ and $c(u)$ denote the diffusion term, the convection term and the reaction term respectively.

The rest of this paper is arranged as follows. The section 2, is devoted to some basic definitions and theorems of the fractional calculus theory and Aboodh transform. In section 3, we explain the variational iteration method (VIM). Section 4 is devoted to applying the proposed method for the nonlinear time-fractional reaction-diffusion-convection equation (1.1) with the initial conditions (1.2). Moreover, in section 5 the proposed method is applied to three different numerical applications. In section 6, we give the conclusion of this work. Finally, some research perspectives are given in section 7.

2 Basical definitions

This section introduces some basic definitions and theorems of the fractional calculus theory and Aboodh transform, which are useful for our method.

Definition 2.1. [6] A real function $f(t), t > 0$, is considered to be in the space C_{μ}, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, so that $f(t) = t^p h(t)$, where $h(t) \in C\left([0,\infty[\right)$, and it is said to be in the space C_{μ}^{n} if $f^{(n)} \in C_{\mu}$, $n \in \mathbb{N}$.

Definition 2.2. [6] The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of $f \in C_{\mu}, \mu \geq -1$ is defined as follows

$$I^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \int\limits_0^t (t-\xi)^{\alpha-1} f(\xi) d\xi, t > 0, \alpha > 0, \\ f(t), \qquad \alpha = 0, \end{cases} \tag{2.1}$$

where $\Gamma(.)$ is the well-known gamma function.

Definition 2.3. [6] The Liouville-Caputo fractional derivative of order $\alpha > 0$ of $f \in C_{-1}^{n}, n \in \mathbb{N}$ is defined as follows

$$D^{\alpha} f(t) = \begin{cases} I^{n-\alpha} f^{(n)}(t), t > 0, n-1 < \alpha < n, \\ f^{(n)}(t), \qquad \alpha = n. \end{cases} \tag{2.2}$$

Definition 2.4. [6] Let n to be the smallest integer that exceeds α. The Liouville-Caputo time-fractional derivative operator of order $\alpha \in \mathbb{R}^+$ is defined as follows

$$D_t^{\alpha} u(x,t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \int\limits_0^t (t-\xi)^{n-\alpha-1} u^{(n)}(x,\xi) d\xi, n-1 < \alpha < n, \\ u^{(n)}(x,t), \qquad \alpha = n. \end{cases} \tag{2.3}$$

Definition 2.5. [6] The Mittag-Leffler function is defined as follows

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)}, \alpha \in \mathbb{C}, Re(\alpha) > 0. \tag{2.4}$$

For $\alpha = 1$, $E_{\alpha}(z)$ reduces to e^z.

Definition 2.6. [1] The Aboodh transform of the function $f(t)$ of exponential order is defined over the set of functions

$$A = \left\{ f(t) \mid \exists M, k_1, k_2 > 0, |f(t)| < M \exp\left(\frac{|t|}{k_j}\right), \text{ if } t \in (-1)^j \times [0, \infty) \right\},$$

by the following integral

$$\mathcal{A}[f(t)] = K(v) = \frac{1}{v}\int_0^\infty f(t)e^{-vt}dt, t \geq 0, k_1 < v < k_2, \tag{2.5}$$

where v is the factor of the variable t.

Some basic properties of the Aboodh transform are given as follows:

Property 1: The Aboodh transform is a linear operator. That is, if λ and μ are non-zero constants, then

$$\mathcal{A}[\lambda f(t) \pm \mu g(t)] = \lambda \mathcal{A}[f(t)] \pm \mu \mathcal{A}[g(t)].$$

Property 2: If $f^{(n)}(t)$ is the $n-$th derivative of the function $f(t) \in A$ with respect to "t" then its Aboodh transform is given by

$$\mathcal{A}\left[f^{(n)}(t)\right] = v^n K(v) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{2-n+k}}.$$

Property 3: (Convolution property) Suppose $K(v)$ and $G(v)$ are the Aboodh transforms of $f(t)$ and $g(t)$, respectively, both defined in the set A. Then the Aboodh transform of their convolution is given by

$$\mathcal{A}[(f * g)(t)] = vK(v)G(v),$$

where the convolution of two functions is defined by

$$(f * g)(t) = \int_0^t f(\xi)g(t-\xi)d\xi = \int_0^t f(t-\xi)g(\xi)d\xi.$$

Property 4: Some special Aboodh transforms

$$\begin{aligned} \mathcal{A}(1) &= \frac{1}{v^2}, \\ \mathcal{A}(t) &= \frac{1}{v^3}, \\ \mathcal{A}(t^n) &= \frac{n!}{v^{n+2}}, n = 0, 1, 2, ... \end{aligned}$$

Property 5: The Aboodh transform of t^α is given by

$$\mathcal{A}[t^\alpha] = \frac{\Gamma(\alpha+1)}{v^{\alpha+2}}, \alpha \geq 0.$$

Theorem 2.7. If $K(v)$ is the Aboodh transform of $f(t)$, then the Aboodh transform of the Riemann-Liouville fractional integral for the function $f(t)$ of order α, is given by

$$\mathcal{A}[I^\alpha y(t)] = \frac{1}{v^\alpha}K(v). \tag{2.6}$$

Proof . The Riemann-Liouville fractional integral for the function $f(t)$, as in (2.1) , can be expressed as the convolution

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1} * f(t). \tag{2.7}$$

Applying the Aboodh transform in the Eq. (2.7) and using the properties (3) and (5), we have

$$\begin{aligned} \mathcal{A}[I^\alpha f(t)] &= \mathcal{A}\left[\frac{1}{\Gamma(\alpha)}t^{\alpha-1} * f(t)\right] = v\mathcal{A}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right]\mathcal{A}[f(t)] \\ &= v\frac{1}{v^{\alpha+1}}K(v) = \frac{1}{v^\alpha}K(v). \end{aligned}$$

The proof is complete. □

Theorem 2.8. Let $n \in \mathbb{N}^*$ and $\alpha > 0$ be such that $n-1 < \alpha \leq n$ and $K(v)$ be the Aboodh transform of the function $f(t)$, then the Aboodh transform of the Liouville-Caputo fractional derivative of $f(t)$ of order α, is given by

$$\mathcal{A}\left[D^{\alpha} f(t)\right] = v^{\alpha} K(v) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{2-\alpha+k}}. \tag{2.8}$$

Proof . Let

$$g(t) = f^{(n)}(t),$$

then by the Definition of the Liouville-Caputo fractional derivative 2.3, we obtain

$$\begin{aligned} D^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} g(\xi) d\xi \\ &= I^{n-\alpha} g(t). \end{aligned} \tag{2.9}$$

Applying the Aboodh transform on both sides of (2.9) using the Theorem 2.7, we get

$$\mathcal{A}\left[D^{\alpha} f(t)\right] = \mathcal{A}\left[I^{n-\alpha} g(t)\right] = \frac{1}{v^{n-\alpha}} G(v). \tag{2.10}$$

Also, we have from the properties (1) and (2)

$$\mathcal{A}\left[g(t)\right] = \mathcal{A}\left[f^{(n)}(t)\right],$$

and

$$G(v) = v^{n} K(v) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{2-n+k}}.$$

Hence, (2.10) becomes

$$\mathcal{A}\left[D^{\alpha} f(t)\right] = \frac{1}{v^{n-\alpha}} \left(v^{n} K(v) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{2-n+k}} \right) = v^{\alpha} K(v) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{2-\alpha+k}}.$$

The proof is complete. □

3 Variational iteration method

Idea of variational iteration method (VIM) depends on the general Lagrange's multiplier method. This method has a main feature, which is the solution of a mathematical problem with linearization assumption used as initial approximation or trial function. This approximation converges rapidly to an accurate solution.

To illustrate the basic concepts of the VIM, we consider the following nonlinear differential equation

$$Lu(x,t) + Nu(x,t) = f(x,t), \tag{3.1}$$

where L is a linear operator and N is a nonlinear operator, and $f(x,t)$ is an inhomogeneous term.

According to the VIM [10], we can construct a correction functional as follows

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi) \left[Lu(x,\xi) + N\widetilde{u}(x,\xi) - f(x,\xi)\right] d\xi, \tag{3.2}$$

wherel $\lambda(\xi)$ is a general Lagrangian multiplier, which can be identified optimally via the variational theory and integration by parts. The subscript n denotes the $n^{th}-$order approximation, $\widetilde{u}_n$ is considered as a restricted variation (i.e. $\delta\widetilde{u}_n = 0$).

So, we first determine the Lagrange multiplier $\lambda(\xi)$ that will be identified optimally via integration by parts. The successive approximations $u_{n+1}, n \geq 0$ of the solution $u(x,t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0 . Consequently, the Solution

$$u(x,t) = \lim_{n \longrightarrow \infty} u_n(x,t).$$

4 Description of the Aboodh variational iteration method (AVIM)

Theorem 4.1. Consider the nonlinear time-fractional reaction-diffusion-convection equation (1.1) with the initial condition (1.2). Then, by the AVIM the exact solution of Eqs. (1.1) and (1.2) is given as a limit of the successive approximations $\{u_n(x,t)\}_{n\geq 0}$, in other words

$$u(x,t) = \lim_{n \longrightarrow \infty} u_n(x,t).$$

Proof . To prove the above theorem, we consider the following nonlinear time-fractional reaction-diffusion-convection equation (1.1) with the initial condition (1.2).

First we define

$$Nu = (a(u)u_x)_x \,, Mu = b(u)u_x, Ku = c(u). \tag{4.1}$$

Eq. (1.1) is written in the form

$$D_t^{\alpha} u(x,t) = Nu(x,t) + Mu(x,t) + Ku(x,t). \tag{4.2}$$

Applying the Aboodh transform on both sides of (4.2) and using the Theorem 2.8, we get

$$\mathcal{A}\left[u(x,t)\right] = \frac{1}{v^2} u(x,0) + \frac{1}{v^{\alpha}} \mathcal{A}\left[Nu(x,t) + Mu(x,t) + Ku(x,t)\right]. \tag{4.3}$$

After that, let us take the inverse Aboodh transform on both sides of (4.3) we have

$$u(x,t) = u_0(x) + \mathcal{A}^{-1}\left(\frac{1}{v^{\alpha}} \mathcal{A}\left[Nu(x,t) + Mu(x,t) + Ku(x,t)\right]\right). \tag{4.4}$$

Take the first partial derivative with respect to t of Eq. (4.4) to obtain

$$0 = \frac{\partial}{\partial t} u(x,t) - \frac{\partial}{\partial t} u_0(x) - \mathcal{A}^{-1}\left(\frac{1}{v^{\alpha}} \mathcal{A}\left[Nu(x,t) + Mu(x,t) + Ku(x,t)\right]\right). \tag{4.5}$$

According to the variational iteration method, we can construct a correct functional as follows

$$\begin{aligned} u_{n+1}(x,t) \quad &= \quad u_n(x,t) - \int_0^t \left[\frac{\partial u_n(x,t)}{\partial \xi} - \frac{\partial}{\partial \xi} \mathcal{A}^{-1}\left(\frac{1}{v^{\alpha}} \mathcal{A}\left[Nu_n(x,t) + Mu_n(x,t)\right.\right.\right. \\ &\left.\left.\left. + Ku_n(x,t)\right]\right) - \frac{\partial}{\partial t} u_0(x)\right] d\xi. \end{aligned} \tag{4.6}$$

Or equivalently

$$u_{n+1}(x,t) = u_0(x) + \mathcal{A}^{-1}\left(\frac{1}{v^{\alpha}} \mathcal{A}\left[Nu_n(x,t) + Mu_n(x,t) + Ku_n(x,t)\right]\right). \tag{4.7}$$

The exact solution is given as a limit of the successive approximations $\{u_n(x,t)\}_{n\geq 0}$,in other words,

$$u(x,t) = \lim_{n \to \infty} u_n(x,t). \tag{4.8}$$

This completes the proof. □

5 Numerical Applications

This section demonstrates the accuracy and effectiveness of the proposed method by presenting three different types of nonlinear time-fractional reaction-diffusion-convection equation.

Example 5.1. Consider the following nonlinear time-fractional reaction-diffusion-convection equation

$$D_t^\alpha u = u_{xx} + uu_x + u - u^2, \tag{5.1}$$

with the initial condition

$$u(x,0) = 1 + \exp(x), \tag{5.2}$$

where D_t^α is the Liouville-Caputo time-fractional derivative operator of order α, $0 < \alpha \leq 1$ and u is a function of $(x,t) \in \mathbb{R} \times \mathbb{R}^+$.

According to (4.7), we can construct the iteration formula as follows

$$u_{n+1}(x,t) = 1 + \exp(x) + \mathcal{A}^{-1}\left(\frac{1}{v^\alpha}\mathcal{A}\left[u_{nxx} + u_n u_{nx} + u_n - u_n^2\right]\right). \tag{5.3}$$

Using the iteration formula (5.3), we obtain

$$\begin{aligned}
u_0(x,t) &= 1 + \exp(x), \\
u_1(x,t) &= 1 + \exp(x)\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)}\right), \\
u_2(x,t) &= 1 + \exp(x)\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right), \\
u_3(x,t) &= 1 + \exp(x)\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right), \\
&\vdots
\end{aligned}$$

and so on.

Then, the general term in successive approximation is given by

$$u_n(x,t) = 1 + \exp(x)\sum_{k=0}^{n}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}.$$

Consequently, the exact solution of Eqs. (5.1) and (5.2) in a closed form is given by

$$\begin{aligned}
u(x,t) &= \lim_{n\to\infty} u_n(x,t) = \left(1 + \exp(x)\sum_{k=0}^{\infty}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\right) \\
&= 1 + \exp(x)E_\alpha(t^\alpha).
\end{aligned}$$

where $E_\alpha(t^\alpha)$ is the Mittag-Leffler functions defined by Eq. (2.4).

For $\alpha = 1$, then

$$u(x,t) = 1 + \exp(x)E_1(t) = 1 + \exp(x+t).$$

The solution is the same as that obtained by the RPSM [5].

Example 5.2. Consider the following nonlinear time-fractional reaction-diffusion-convection equation

$$D_t^\alpha u = (uu_x)_x + 3uu_x + 2(u - u^2), \tag{5.4}$$

with the initial condition

$$u(x,0) = 2\sqrt{\exp(x) - \exp(-4x)}, \tag{5.5}$$

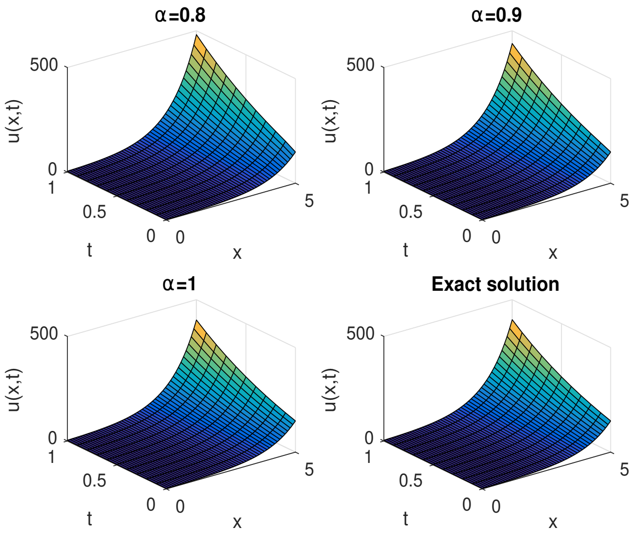

Figure 1: 3D plots graphs of the $4^{th}-$order approximate solution and exact solution for Example 5.1.

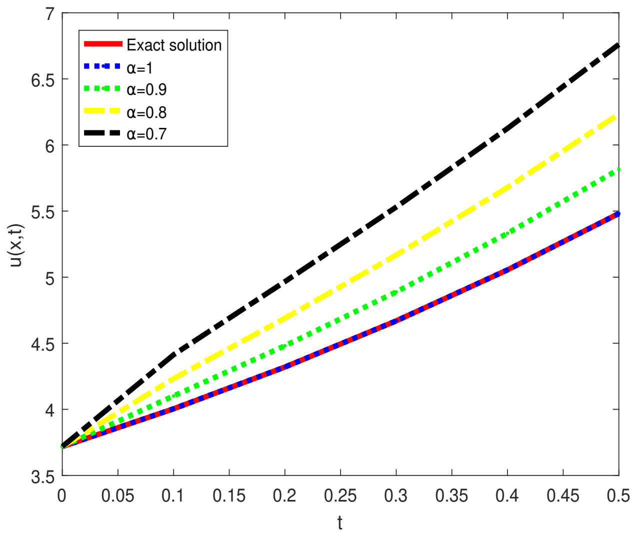

Figure 2: 2D plots graphs of the $4^{th}-$order approximate solution and exact solution for Example 5.1 when $x = 1$.

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	exact solution	$\lvert u_{exact} - u_{AVIM}\rvert$
0.01	3.8409	3.7928	3.7635	3.7456	3.7456	2.2690×10^{-12}
0.03	3.9922	3.9020	3.8417	3.8011	3.8011	5.5322×10^{-10}
0.05	4.1212	4.0005	3.9165	3.8577	3.8577	7.1383×10^{-9}
0.07	4.2412	4.0947	3.9904	3.9154	3.9154	3.8520×10^{-8}
0.09	4.3565	4.1868	4.0642	3.9743	3.9743	1.3579×10^{-7}

Table 1: The numerical values of the 4^{th} order approximpate solution by AVIM and exact solution for Example 5.1 when $x = 1$.

where D_t^{α} is the Liouville-Caputo time-fractional derivative operator of order α, $0 < \alpha \leq 1$ and u is a function of $(x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$.

According to (4.7), we can construct the iteration formula as follows

$$\begin{aligned} u_{n+1}(x,t) &= 2\sqrt{\exp(x)-\exp(-4x)} \\ &+\mathcal{A}^{-1}\left(\frac{1}{v^{\alpha}}\mathcal{A}\left[(u_n u_{nx})_x+3u_n u_{nx}+2(u_n-u_n^2)\right]\right). \end{aligned} \tag{5.6}$$

Using the iteration formula (5.6), we obtain

$$\begin{aligned} u_0(x,t) &= 2\sqrt{\exp(x)-\exp(-4x)}, \\ u_1(x,t) &= 2\sqrt{\exp(x)-\exp(-4x)}\left(1+\frac{2t^{\alpha}}{\Gamma(\alpha+1)}\right), \\ u_2(x,t) &= 2\sqrt{\exp(x)-\exp(-4x)}\left(1+\frac{2t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right), \\ u_3(x,t) &= 2\sqrt{\exp(x)-\exp(-4x)}\left(1+\frac{2t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2^2t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{2^3t^{3\alpha}}{\Gamma(3\alpha+1)}\right), \\ &\vdots \end{aligned}$$

and so on.

Then, the general term in successive approximation is given by

$$u_n(x,t) = 2\sqrt{\exp(x)-\exp(-4x)}\sum_{k=0}^{n}\frac{(2t^{\alpha})^k}{\Gamma(k\alpha+1)}.$$

Consequently, the exact solution of Eqs. (5.4) and (5.5) in a closed form is given by

$$\begin{aligned} u(x,t) &= \lim_{n\to\infty} u_n(x,t) \\ &= 2\sqrt{\exp(x)-\exp(-4x)}\sum_{n=0}^{\infty}\frac{t^{k\alpha}}{\Gamma(n\alpha+1)} \\ &= 2\sqrt{\exp(x)-\exp(-4x)}E_{\alpha}\left(2t^{\alpha}\right). \end{aligned}$$

where $E_\alpha(2t^\alpha)$ is the Mittag-Leffler functions defined by Eq. (2.4).

For $\alpha = 1$, then

$$u(x,t) = 2\sqrt{\exp(x)-\exp(-4x)}E_1(2t) = 2\exp(2t)\sqrt{\exp(x)-\exp(-4x)}.$$

The solution is the same as that obtained by the RPSM [5].

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	exact solution	$\lvert u_{exact} - u_{AVIM}\rvert$
0.01	3.5918	3.4695	3.3966	3.3527	3.3527	8.7928×10^{-11}
0.03	3.9939	3.7489	3.5922	3.4895	3.4895	2.1510×10^{-8}
0.05	4.3591	4.0120	3.7851	3.6319	3.6319	2.7849×10^{-7}
0.07	4.7179	4.5398	3.9809	3.7802	3.7802	1.5079×10^{-6}
0.09	5.0799	4.5398	4.1818	3.9344	3.9344	5.3341×10^{-6}

Table 2: The numerical values of the 4^{th} order approximpate solution by AVIM and exact solution for Example 5.2 when $x = 1$.

Example 5.3. Consider the following nonlinear time-fractional reaction-diffusion-convection equation

$$D_t^{\alpha} u = u_{xx} - u_x + uu_x + u - u^2, \tag{5.7}$$

with the initial condition

$$u(x,0) = \exp(x), \tag{5.8}$$

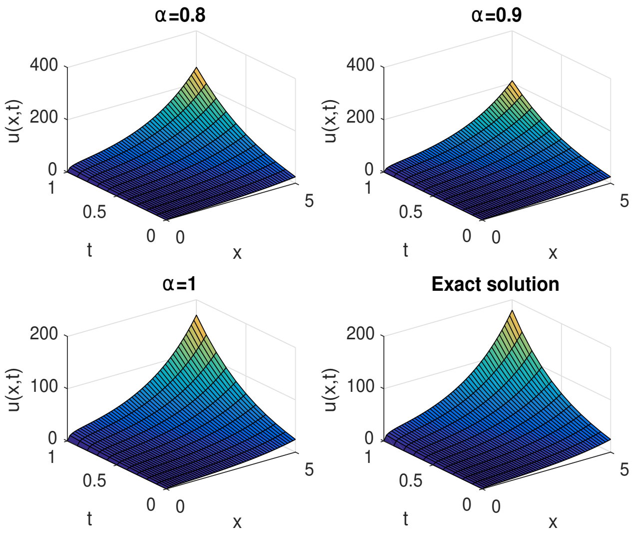

Figure 3: 3D plots graphs of the $4^{th}-$order approximate solution and exact solution for Example 5.2.

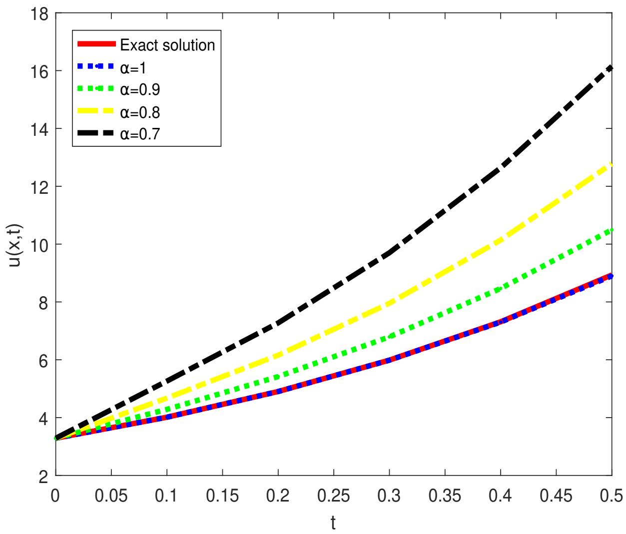

Figure 4: 2D plots graphs of the $4^{th}-$order approximate solution and exact solution for Example 5.2 when $x = 1$.

where D_t^{α} the Liouville-Caputo time-fractional derivative operator of order $\alpha, 0 < \alpha \leq 1$ and u is a function of $(x,t) \in \mathbb{R} \times \mathbb{R}^{+}$.

According to (4.7), we can construct the iteration formula as follows

$$u_{n+1}(x,t) = e^{x} + \mathcal{A}^{-1}\left(\frac{1}{v^{\alpha}}\mathcal{A}\left[u_{nxx} - u_{nx} + u_n u_{nx} + u_n - u_n^2\right]\right). \tag{5.9}$$

Using the iteration formula (5.9), we obtain

$$\begin{aligned} u_0(x,t) &= \exp(x), \\ u_1(x,t) &= \exp(x)\left(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right), \\ u_2(x,t) &= \exp(x)\left(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right), \\ u_3(x,t) &= \exp(x)\left(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right), \\ u_4(x,t) &= \exp(x)\left(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}\right), \\ &\vdots \end{aligned}$$

Then, the general term in successive approximation is given by

$$u_n(x,t) = \exp(x)\sum_{k=0}^{n}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}.$$

Consequently, the exact solution of Eqs. (5.7) and (5.8) in a closed form is given by

$$\begin{aligned} u(x,t) &= \lim_{n\to\infty} u_n(x,t) = \exp(x)\sum_{k=0}^{\infty}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)} \\ &= \exp(x)E_{\alpha}\left(t^{\alpha}\right). \end{aligned}$$

where $E_{\alpha}(t^{\alpha})$ is the Mittag-Leffler functions defined by Eq. (2.4).

For $\alpha = 1$, then

$$u(x,t) = \exp(x)E_1(t) = \exp(x+t).$$

The solution is the same as that obtained by the HPM [8].

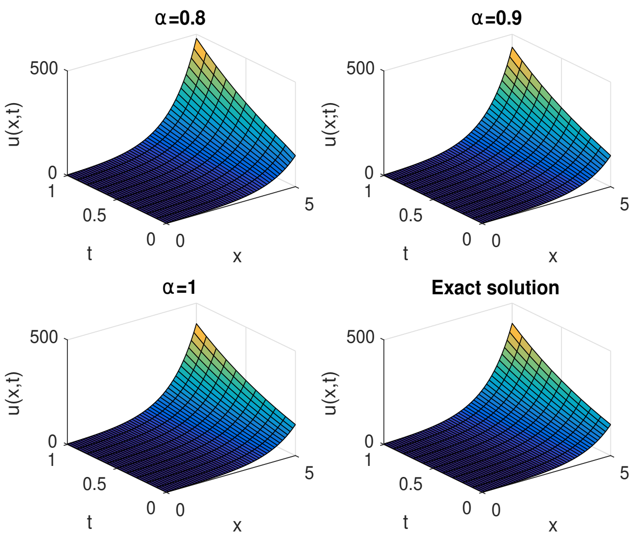

Figure 5: 3D plots graphs of the 4^{th}−order approximate solution and exact solution for Example 5.3.

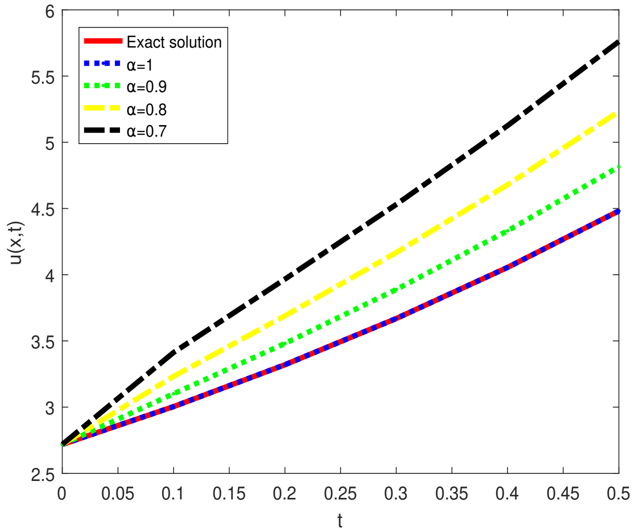

Figure 6: 2D plots graphs of the $4^{th}-$order approximate solution and exact solution for Example 5.3 when $x = 1$.

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	exact solution	$\lvert u_{exact} - u_{AVIM}\rvert$
0.01	2.8409	2.7928	2.7635	2.7456	2.7456	2.2690×10^{-12}
0.03	2.9922	2.9020	2.8417	2.8011	2.8011	5.5322×10^{-10}
0.05	3.1212	3.0004	2.9165	2.8577	2.8577	7.1383×10^{-9}
0.07	3.2412	3.0946	2.9904	2.9154	2.9154	3.8520×10^{-8}
0.09	3.3565	3.1868	3.0642	2.9743	2.9743	1.3579×10^{-7}

Table 3: The numerical values of the 4^{th} order approximpate solution by AVIM and exact solution for Example 5.3 when $x = 1$.

6 Conclusion

In this work, we proposed a new coupling method namely, the Aboodh variational iteration method (AVIM) for obtaining the exact solution of the nonlinear time-fractional reaction-diffusion-convection equation. The exact solution is given in the form of the Mittag-Leffler function. Three different examples are presented in order to validate and illustrate the effectiveness of the proposed method. We have noticed that the results obtained by the AVIM are in perfect agreement with the results obtained from the exisiting methods. Consequently and in conclusion, the AVIM can be considered as a promising method and can be applied to other nonlinear fractional partial differential equations which appear in various scientific fields and dynamical systems in the applied mathematics and engineering.

7 Open problems

In future works, we will extend the proposed method for solving a wide variety of nonlinear fractional partial differential equations that include higher-order fractional derivatives α, where $n-1 < \alpha \leq n$ and $n > 1$. In addition, we will try to apply this method in solving nonlinear fractional partial differential equations, but with different fractional derivative operators such as: Caputo-Fabrizio fractional derivative, Atangana-Baleanu-Caputo fractional derivative, Conformable fractional derivative, etc...

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