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Spectral method for the diffusion equation with a source term

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Abstract

The aim of this paper is to investigate the Legendre spectral method for solving the diffusion equation with a source term and mixed initial-boundary value problem in a finite rectangle Ω_2, we use some techniques to convert the problem to a system of ordinary differential equations and by an analysis matrical we find a general term defines all ordinary differential equations of this system, we solve this general term we get the desired approximate solution, we also present the error estimate.

Keywords: Diffusion equation, Spectral method, Orthogonal polynomials, Quadrature formula, Orthogonal matrix, Eigenvalues, Error estimate

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1 Introduction

The differential equations play a very important role in all fields of science like mathematics and Mathematical Physics and other Sciences, and a long time ago the scientists and researchers face difficulties in resolving many of these equations, for this we turned in recent years, especially after the emergence of the computer to search for approximate solution instead of the exact solution for these problems, these methods gave his fruit.

The main motivation in this paper is the numerical analysis of discretization of the in homogeneous mixed initial-boundary value problem using spectral element method, this method is associated with quadrature formulas which allow for a complete discretization of the right-hand side and of the linear form involved in the variational formulation, see also [4, 5, 8, 12, 25].

The concerned problem refers to the equation:

$$\begin{cases} \partial_t u - a\partial_x^2 u + cu = f & \text{in } \Omega_2 \\ u(x,t) = 0 & \text{on } \partial\Lambda \times I \\ u(x,0) = u_0(x) & \text{in } \Lambda \end{cases} \tag{1.1}$$

where Ω_2 is a regular finte rectangle defined by $\Omega_2 = \Lambda \times I = (-1,1) \times (0,T)$, also a and c are the positive constants, which $u(x,t)$ represents the temperature at point x and time t, the discretization consists therefore the space variable and the time variable.

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Then the problem **(1.1)** is a problem of one space variable, by using the orthogonal matrix we reduce this problem to a system of ordinary differential equations.

In this work we construct approximate solution to the boundary value problem **(1.1)** in the following form

$$u_N(x,t) = \sum_{n=0}^{N} b_n(t) l_n(x). \tag{1.2}$$

Where $l_n(x), 0 \leq n \leq N$, are the Lagrangian interpolates at the points $x_i \in \overline{\Lambda} = [-1,1]$, $0 \leq i \leq N$, these interpolates satisfy the property $l_n(x_j) = \delta_{nj}, 1 \leq n, j \leq N-1$, where δ_{nj} is the Kronecker delta and the points $x_j, 0 \leq j \leq N$ are the collocation points on the Gauss-Lobatto Legendre grid. The grid made by $x_j, 0 \leq j \leq N$, is denoted by S_{N+1}. The choice of the form **(1.2)** for the approximation solution, added to some techniques give a linear system which can be written in a matricial form as $\Gamma Db - Ab = \Gamma G$, where A is a square matrix and Γ is a diagonal invertible matrix and the operator $D = \frac{d}{dt}$. We write $b = Pv$ where P is an orthogonal matrix such that $P^{-1}\left(\Gamma^{-1}A\right)P = C$ is a diagonal matrix, then we obtain a system of $N-1$ ordinary differential equations, we can use the Lagrange method to solve for each component $v_i(t)$ of v, finally we conclude the expression of functions $b_n(t)$ and for which we obtain the desired approximate solution, see also [1, 2, 15, 26, 27].

2 Orthogonal polynomials

We work in the model domain Λ and we use the Legendre polynomials $L_n, n \geq 0$: each polynomial L_n has a degree n, it is orthogonal to the other ones in

$$L^2(\Lambda) = \left\{ \varphi : \Lambda \to \mathbb{R}, \text{measurable} / \int_{-1}^{1} \varphi^2(x)\, dx < +\infty \right\}, \tag{2.1}$$

and satisfies the following property

$$\int_{-1}^{1} L_n(x) L_m(x) dx = \frac{2}{2n+1} \delta_{nm}, \tag{2.2}$$

where δ_{nm} denotes the Kronecker symbol.

$$h_n'(x) = -n(n+1) L_n(x),\ h_n(x) = \left(1-x^2\right) L_n'(x),\ n \geq 0. \tag{2.3}$$

$$h_n(x) = \frac{n(n+1)}{2n+1}\left(L_{n-1}(x) - L_{n+1}(x)\right). \tag{2.4}$$

$$\|h_n(x)\|^2_{L^2(\Lambda)} = \frac{4\left[n(n+1)\right]^2}{(4n^2-1)(2n+3)}. \tag{2.5}$$

3 Variational Formulation

3.1 The spaces

The pivot space of the problem (1.1) is the space $L^2(\Lambda)$, and the variational space is the Sobolev space

$$H^1(\Lambda) = \left\{ v \in L^2(\Lambda)\ /\ \partial_x v \in L^2(\Lambda) \right\}, \tag{3.1}$$

and the corresponding norms are defiended respectively as

$$\|v\|^2_{L^2(\Lambda)} = \int_\Lambda v^2 dx, \tag{3.2}$$

$$\|v\|^2_{H^1(\Lambda)} = \int_\Lambda (v^2 + (\partial_x v)^2) dx. \tag{3.3}$$

3.2 The continuous problem

To introduce the variational formulation for the continuous problem (1.1), we need the subspace of the variational space with zero Dirichlet trace

$$H_0^1(\Lambda) = \left\{ v \in L^2(\Lambda) \ / \ v_x \in L^2(\Lambda),\ v(1) = v(-1) = 0 \right\}. \tag{3.4}$$

We introduce the product in $L^2(\Lambda)$

$$(f, v) = \int_\Lambda f(x,t)v(x,t)dx. \tag{3.5}$$

The continuous problem (1.1) admits the equivalent variational formulation:
Find u in $H_0^1(\Lambda)$, such that,

$$\forall v \in H_0^1(\Lambda), \theta(u,v) = \langle f, v \rangle, \tag{3.6}$$

where

$$\theta(u,v) = \int_\Lambda \left(\partial_t u - a\partial_x^2 u + cu\right) v dx, \tag{3.7}$$

integrating by parts leads to,

$$\theta(u,v) = \int_\Lambda \left(\partial_t uv + a\partial_x u \partial_x v + cuv\right) dx. \tag{3.8}$$

4 Discrete space and form

Let us denoted by N the parameter of discretization for the problem **(1.1)**, in spectral method N represents the degree of polynomials. The approximate space is essentially generated by the finite dimensional subspace of $L^2(\Lambda)$, $P_N^0(\Lambda)$ is the approximate space of the space $H_0^1(\Lambda)$, where

$$P_N^0(\Lambda) = \{p_n \in P_N(\Lambda) \ / \ p_n(1) = p_n(-1) = 0\}, \tag{4.1}$$

where $P_N(\Lambda)$ is the set of polynomials of degree less than or equal to N. We consider also the exact quadrature formula and introduce a bilinear form a N with approach to the form a and we approximate the scalar $(.,.)$ for $(.,.)_N$.

4.1 The Discrete problem

Firstly we observe that the Lagrange polynomials $l_n(x)$, $0 \le n \le N$, form a basis of $P_N^0(\Lambda)$, then the exact solution u of problem **(1.1)** is approached by the solution u_N^1 belonging to $P_N^0(\Lambda)$ with $\left(u_N^1 - u_0\right) \in P_N^0(\Lambda)$

$$\left\{ \begin{array}{c} \text{find } u_N^1 \in P_N^0(\Lambda), \text{ s.t} \\ \forall v_N \in P_N^0(\Lambda), \theta_N(u_N^1, v_N) = (f_N, v_N)_N \end{array} \right. \tag{4.2}$$

where

$$\theta_N(u_N^1, v_N) = \sum_{k=0}^{N} \left(\partial_t u_N^1 v_N + a\partial_x u_N^1 \partial_x v_N + cu_N^1 v_N\right)(x_k, t)\rho_k. \tag{4.3}$$

where $x_k, \rho_k, 0 \le k \le N$, are defined in proposition **4.1**, $u_N^1 = u_N + u_{N0}, u_N \in P_N^0(\Lambda)$, and the problem (4.2) is equivalent to the following problem: Find u_N^1 in $P_N^0(\Lambda)$ with $u_N = u_N^1 - u_{N0}$ in $P_N^0(\Lambda)$ such that, $\forall v_N \in P_N^0(\Lambda)$

$$\theta_N(u_N, v_N) = \Delta_N(u_{N0}, v_N), \tag{4.4}$$

where

$$\Delta_N(u_{N0}, v_N) = (f_N, v_N)_N - \theta_N(u_{N0}, v_N). \tag{4.5}$$

4.2 Existence and uniqueness of solution

4.2.1 Quadrature formula

Proposition 4.1. There exists a unique set of $N-1$ nodes x_j, $1 \leq j \leq N-1$ in Λ and with the condition $x_0 = -1, x_N = 1$, there exists $N+1$ positive weights $\rho_j, 0 \leq j \leq N$, such that the following exactness property holds:

$$\forall \varphi \in P_{2N-1}(\Lambda), \int_{-1}^{1} \varphi(x)\, dx = \sum_{j=0}^{N} \varphi(x_j)\, \rho_j. \tag{4.6}$$

where x_j, $1 \leq j \leq N-1$, are the roots of the polynomial $L_N^{'}$. and the weights ρ_j are given by:

$$\begin{cases} \rho_0 = \rho_N = \frac{2}{N(N+1)} \\ \rho_j = \frac{\rho_0}{L_N^2(x_j)}, 1 \leq j \leq N-1 \end{cases} \tag{4.7}$$

Proof . See [5, 6]. □

Definition 4.2. We define the discrete product for all polynomials v_N, u_N in $P_N^0(\Lambda)$ as:

$$(u_N, v_N)_N = \sum_{k=0}^{N} u_N(x_k, t) v_N(x_k, t) \rho_k. \tag{4.8}$$

Lemma 4.3. The polynomial $h_{N-1} \in P_N^0(\Lambda)$ verifies the double inequality:

$$\|h_{N-1}\|^2_{L^2(\Lambda)} \leq (h_{N-1}, h_{N-1})_N \leq \frac{3}{2} \|h_{N-1}\|^2_{L^2(\Lambda)}. \tag{4.9}$$

Proof . See [1]. □

Proposition 4.4. For all polynomial $h_n \in P_n^0(\Lambda)$ we have

$$n \|h_n\|_{L^2(\Lambda)} \leq \left\| h_n^{'} \right\|_{L^2(\Lambda)} \leq 3n \|h_n\|_{L^2(\Lambda)}. \tag{4.10}$$

Proof . See [1]. □

Also the lagrange's polynomials $l_j(x)$, $j = \overline{1, N-1}$ can be written in the following form

$$l_j(x) = \sum_{k=0}^{N-1} \gamma_{kj} h_k(x), \tag{4.11}$$

using **(2.3),** then we get

$$l_j(x) = \sum_{k=0}^{N-1} \lambda_{kj} L_k(x). \tag{4.12}$$

Proposition 4.5. The set of polynomials $\{L_n(\zeta)\}, n = 0..N$ forms a basis to the polynomial space $P_N(\Lambda)$, then any polynomial $\varphi_N \in P_N(\Lambda)$ can be written as $\varphi_N(\zeta) = \sum\limits_{n=0}^{N} \alpha_n L_n(\zeta)$ and we have the following inequality:

$$c_1 \log(2N+1) \leq \|\varphi\|^2_{L^2(\Lambda)} \leq c_2 \log(\exp(2)(2N+1)), \tag{4.13}$$

where $(c_1, c_2) = \left(\min(\alpha_n^2), \max(\alpha_n^2)\right)$.

Proof . See [1]. □

Proposition 4.6. For a positive integer m the Sobolev space $H^m(\Lambda)$ is defined by:

$$H^m(\Lambda) = \left\{ \varphi \in L^2(\Lambda) : 1 \leq k \leq m, \frac{d^k}{dx^k} \varphi \in L^2(\Lambda) \right\}, \tag{4.14}$$

with the norm:

$$\|\varphi\|^2_{H^m(\Lambda)} = \int_\Lambda \sum_{k=0}^{m} \left(\frac{d^k}{dx^k}\varphi \right)^2 (x)\, dx. \tag{4.15}$$

Proposition 4.7. The bilinear form $\theta_N(.,.)$ in **(4.4)** satisfies the following properties of continuity:

$$\forall u_N \in P_N^0(\Lambda),\quad \forall v_N \in P_N^0(\Lambda),\quad |\theta_N(u_N,v_N)| \leq \max\left(a, \frac{3}{2}\,(c + C_4)\right) \left(||u_N||_{H_0^1(\Lambda)} . ||v_N||_{H_0^1(\Lambda)} \right), \tag{4.16}$$

and ellipticity:

$$\forall u_N \in P_N^0(\Lambda),\quad |\theta_N(u_N,u_N)| \geq \min(a, c + C_3) \left(||u_N||^2_{H_0^1(\Lambda)} \right). \tag{4.17}$$

Proof . Continuity:

$$\theta_N(u_N,v_N) = \sum_{k=0}^{N} \partial_t u_N(x_k,t) v_N(x_k,t)\rho_k + a\sum_{k=0}^{N} \partial_x u_N(x_k,t)\partial_x v_N(x_k,t)\rho_k + c\sum_{k=0}^{N} u_N(x_k,t)v_N(x_k,t)\rho_k.$$

We consider the solution and its derivatives are bounded then there exists two real positive constants C_3 and C_4 such that

$$C_3\,|u_N(x_k,t)| \leq |\partial_t u_N(x_k,t)| \leq C_4\,|u_N(x_k,t)|\,. \tag{4.18}$$

Using lemmas **(4.3)**, the exact quadrature formula and the Schwarz inequality then we obtain the desired results also and ellipticity:

$$\theta_N(u_N,u_N) = \sum_{k=0}^{N} \partial_t u_N(x_k,t) u_N(x_k,t)\rho_k + a\sum_{k=0}^{N} \partial_x u_N(x_k,t)\partial_x u_N(x_k,t)\rho_k + c\sum_{k=0}^{N} u_N(x_k,t)u_N(x_k,t)\rho_k.$$

using the exactingness quadrature formula we can write,

$$\theta_N(u_N,u_N) = \sum_{k=0}^{N} \partial_t u_N(x_k,t) u_N(x_k,t)\rho_k + a\int_{-1}^{1} \partial_x u_N(x,t)\partial_x u_N(x,t)dx + c\sum_{k=0}^{N} u_N(x_k,t)u_N(x_k,t)\rho_k, \tag{4.19}$$

then from **(4.18)** we can write:

$$|\theta_N(u_N,u_N)| \geq C_3\sum_{k=0}^{N} u_N(x_k,t)u_N(x_k,t)\rho_k + a\int_{-1}^{1} \partial_x u_N(x,t)\partial_x u_N(x,t)dx + c\sum_{k=0}^{N} u_N(x_k,t)u_N(x_k,t)\rho_k.$$

Using **(4.9)** we can write:

$$|\theta_N(u_N,u_N)| \geq \min(a, c + C_3) \left(||u_N||^2_{H_0^1(\Lambda)} \right).$$

then for this inequality yields the desired result. □

Proposition 4.8. (The inequality of stability) For any continuous function $g = u_0$ on Λ, the problem **(4.4)** has a unique solution u_N in $P_N^0(\Lambda)$, and this solution verifies the inequality of stability:

$$\|u_N(x,t)\|_{H_0^1(\Lambda)} \leq \gamma \left(\|f_N\,(x,t)\|_{L^2(\Lambda)} + \|g_N\,(x)\|_{L^2(\Lambda)} \right). \tag{4.20}$$

Proof . Using **(4.4)** we can write:

$$\theta_N(u_N,u_N) = (f_N,u_N)_N - \theta_N(g_N,u_N) \leq |(f_N,u_N)_N| + |\theta_N(g_N,u_N)|\,, \tag{4.21}$$

using Schwarz inequality we can write

$$\begin{aligned} |(f_N,u_N)_N| + |\theta_N(g_N,u_N)| \leq\ & \tfrac{3}{2}\,\|f_N\,(x,t)\|_{L^2(\Lambda)} . \|u_N(x,t)\|_{L^2(\Lambda)} \\ & + a\,\|\partial_x g_N(x,t)\|_{L^2(\Lambda)} . \|\partial_x u_N(x,t)\|_{L^2(\Lambda)} \\ & + \tfrac{3c}{2}\,\|g_N(x,t)\|_{L^2(\Lambda)} . \|u_N(x,t)\|_{L^2(\Lambda)}\,, \end{aligned}$$

the quantities $\|\partial_x g_N(x,t)\|_{L^2(\Lambda)}$, and $\|\partial_x u_N(x,t)\|_{L^2(\Lambda)}$ are bounded then there exists a positive number γ such that,

$$\theta_N(u_N,u_N) \leq |(f_N,u_N)_N| + |\theta_N(g_N,u_N)| \leq \gamma \left(\|f_N\,(x,t)\|_{L^2(\Lambda)} + \|g_N\,(x)\|_{L^2(\Lambda)} \right) \|u_N(x,t)\|_{H_0^1(\Lambda)}\,,$$

using **(4.17),** yields the desired result. □

5 Numerical experiment

At the points x_k, $1 \leq k \leq N-1$ the problem **(1.1)** is equivalent to,

$$\begin{cases} \sum\limits_{n=1}^{N-1} l_n(x_k)b_n'(t) + [cl_n(x_k) - al_n'(x_k)]\, b_n(t) = \sum\limits_{n=1}^{N-1} f_n(t)l_n(x_k) + au_0''(x_k) - cu_0(x_k) \text{ in } \Lambda \cap S_{N+1} \\ u_N\,(x_k,t) = 0, \qquad \text{on } \partial\Lambda \cap S_{N+1} \\ u_N(x,0) = u_{N0}(x) \\ f(x,t) = \sum\limits_{n=1}^{N-1} f_n(t) l_n\,(x)\,, \quad f_n(t) = \sum\limits_{j=1}^{N-1} f_{jn} l_j\,(t)\,,\ f_{jn} = f\,(x_j, t_n) \end{cases} \tag{5.1}$$

Since the functions

$$cl_n(x) - al_n''(x),\ 1 \leq n \leq N-1$$

are polynomials with degree N, we multiply both sides by $l_m(x_k)\rho_k$ and applying the sum, by using the quadrature formula, when m varies from 1 to $N-1$, we obtain a linear system, then we can write this system in a matricial form:

$$\Gamma Db - Ab = \Gamma F \tag{5.2}$$

where A is a square symmetric define positive matrix with order $N-1$, its elements have the form:

$$\alpha_{mn} = (-cl_n(x_k)l_m(x_k) - a(l_n'(x_k)l_m'(x_k))\rho_k,\ n = \overline{1, N-1}\,\},m = \overline{1, N-1}$$

Γ is a diagonal invertible matrix its elements are defined as:

$$\gamma_{mn} = \begin{cases} \rho_m \ , n = m \\ 0, \quad n \neq m \end{cases} \quad ,\ m, n = \overline{1, N-1}$$

F is a known vector where:

$$\alpha_{mn} = (-cl_n(x_k)l_m(x_k) - a(l_n'(x_k)l_m'(x_k))\rho_k,\ n = \overline{1, N-1}\,\},m = \overline{1, N-1}$$

and the vector b is an unknown vector where

$$b(t) = (b_1(t), b_2(t), b_3(t), \ldots.., b_{N-2}(t), b_{N-1}(t))^t$$

the operator,

$$D = \frac{d}{dt}$$

multiplying **(5.2)** by the invertible matrix Γ^{-1} of Γ then we find

$$Db - \Gamma^{-1}Ab = F \tag{5.3}$$

the matrix $\Gamma^{-1}A$ has positive eigenvalues and there exists an orthogonal invertible matrix P such that,

$$P^{-1}\left(\Gamma^{-1}A\right)P = C$$

where C is a diagonal matrix, the elements of the diagonal are the eigenvalues $\lambda_i = \alpha_{ii}, i = \overline{1, N-1}$ of the matrix $\Gamma^{-1}A$, if we consider the vector v such that

$$b = Pv$$

then the system **(5.3)** becomes

$$PDv - (\Gamma^{-1}A)Pv = F \tag{5.4}$$

multiplying multiplying **(5.4)** by the matrix P^{-1} we obtain,

$$Dv - Cv = P^{-1}F \tag{5.5}$$

The matricial form **(5.5)** has $N-1$ linear ordinary differential equations defined as

$$v_k'(t) - \lambda_k v_k(t) = h_k(t) \tag{5.6}$$

$$\text{where} \quad h_k(t) = \sum_{j=1}^{N-1} p^{-1}(k,j)\left(f_j(t) + a u_0''(x_k) - c u_0(x_k)\right),\ 1 \leq k \leq N-1 \tag{5.7}$$

$p^{-1}(k,j)$ are the elements of the inverse matrix P^{-1}. To solve the equations **(5.6)** we use Lagrange's method [27], we may write the solution in the closed form:

$$v_k(t) = e^{\lambda_k t}\left(\int_0^t e^{-\lambda_k s} h_k(s)ds + d_k\right) \tag{5.8}$$

where d_k is constant to be determined, using the boundary conditions then **(5.8)** may be written in the following form:

$$v_k(t) = e^{\lambda_k t}\left(\int_0^t e^{-\lambda_k s} h_k(s)ds + \sum_{j=1}^{N-1} p_{kj}^{-1} u_0(x_k)\right) \tag{5.9}$$

Finally we obtain the functions,

$$b_n(t) = \sum_{j=1}^{N-1} p_{nj} v_j(t) \tag{5.10}$$

where p_{nj}, $1 \leq n, j \leq N-1$ are the elements of the matrix P, and the approximation solution is:

$$u(x,t) = \sum_{n=1}^{N-1}\sum_{j=1}^{N-1} p_{nj}\left(\int_0^t e^{-\lambda_k s} h_k(s)ds + \sum_{j=1}^{N-1} p_{kj}^{-1} u_0(x_k)\right) e^{\lambda_k t} l_n(x).$$

If the time t defined in the interval $I = [0, T]$, we can consider the solution in the form

$$u(x,t) = \sum_{n=1}^{N-1}\sum_{j=1}^{N-1} u_{nj} l_n(x)\, l_j(t), \quad b_n(t) = \sum_{j=1}^{N-1} u_{nj} l_j(t) \tag{5.11}$$

using **(5.10)** and **(5.11)** then we determine the coefficients

$$u_{nj} = \sum_{j=1}^{N-1} p_{nj}\left(\int_0^{t_j} e^{-\lambda_k s} h_k(s)ds + \sum_{j=1}^{N-1} p_{kj}^{-1} u_0(x_k)\right) e^{\lambda_k t_j}$$

and the approximate solution is

$$u_N(x,t) = \sum_{n=1}^{N-1}\sum_{m=1}^{N-1}\left(\sum_{j=1}^{N-1} p_{nj}\left(\int_0^{t_j} e^{-\lambda_k s} h_k(s)ds + \sum_{j=1}^{N-1} p_{kj}^{-1} u_0(x_k)\right) e^{\lambda_k t_j}\right) l_n(x) l_m(t) + \Phi(x)$$

$\Phi(x) = \sum\limits_{n=1}^{N-1} u_0(x_n) l_n(x)$

and using **(5.7)** we get

$$\begin{aligned} u_N(x,t) \quad = \quad & \sum_{n=1}^{N-1}\sum_{m=1}^{N-1}\left(\sum_{j=1}^{N-1} p_{nj}\left(\int_0^{t_j} e^{-\lambda_k (s-t_j)} \sum_{j=1}^{N-1} p^{-1}(k,j)\left(f_j(s) + a u_0''(x_k) - c u_0(x_k)\right)\right) ds \right.\\ & \left. + \left(\sum_{j=1}^{N-1} p_{kj}^{-1} u_0(x_k)\right) e^{\lambda_k t_j}\right) l_n(x) l_m(t) + \Phi(x) \end{aligned}$$

5.1 Numerical integration

The function

$$q_k(s) = e^{-\lambda_k(s-t)} h_k(s) \tag{5.12}$$

is explicit but we can not always calculate its primitive explicitly, in this case we use the polynomial interpolation and seek numerical approximation of the integral. Then the Lagrange polynomial interpolation is

$$q_{Nj}(s) = \sum_{n=0}^{N} q_j(t_n) l_j(s)$$

where t_n defined by $t_n = \frac{T}{2}(x_n + 1)$, $n = \overline{0, N}$, and x_n, $0 \leq n \leq N$, are the collocation points on the Gauss-Lobatto Legendre grid, then the approximation of the integral **(5.9)**

$$v_{Nj}(t) = \int_0^t q_{Nj}(s)ds + \left(\sum_{j=1}^{N-1} p_{kj}^{-1} u_0(x_k) \right) e^{\lambda_k t}$$

then we obtain

$$b_n(t) = \sum_{j=1}^{N-1} p_{nj}(t_n) v_{Nj}(t)$$

where p_{nj}, $1 \leq n, j \leq N-1$ are the entries of the matrix P, using **(1.2)** we get the approximate solution

$$u_N(x,t) = \sum_{n=1}^{N-1} \sum_{j=1}^{N-1} p_{nj} v_{Nj}(t) l_n(x)$$

5.2 Error estimation

Definition 5.1. The polynomial space $P_N^0(\Lambda)$ is dense in the space of continuous functions on Λ hence in $H_0^1(\Lambda)$ then any function $u \in H_0^1(\Lambda)$ admits the expansion

$$u(x,t) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \alpha(k,l) h_k(x) t_l(t) \tag{5.13}$$

We know

$$t_n(t) = \frac{n(n+1)}{2(2n+1)} \left(p_{n-1}(t) - p_{n+1}(t) \right) \tag{5.14}$$

where

$$p_n(t) = L_n\left(\frac{2}{T}t - 1\right), n \geq 0 \tag{5.15}$$

and using **(5.14)** then

$$u(x,t) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \gamma(k,l) h_k(x) p_l(t) \tag{5.16}$$

Proposition 5.2. The following estimate holds between the exact solution u in $H_0^1(\Lambda)$ and the approximate solution $u_N \in P_N^0(\Lambda)$ verify,

$$\|u - u_N\|_{L^2(\Lambda)} \leq 3CN^{-1} \left(\|(u_0 - u_{N0})\|_{L^2(\Lambda)} + \|f - f_N\|_{L^2(\Lambda)} \right) \tag{5.17}$$

Proof . Using the ellipticity condition **(4.17)** and **(4.10)** we can write,

$$\begin{aligned} N^2 \|u - u_N\|_{L^2(\Lambda)}^2 &\leq \theta(u - u_N, u - u_N) = (f - f_N, u - u_N)_N - \theta(u_0 - u_{N0}, u - u_N) \\ &\leq C \left(\left| \int_\Lambda (f - f_N)(u - u_N)\, dx \right| + |\theta(u_0 - u_{N0}, u - u_N)| \right) \end{aligned} \tag{5.18}$$

$$\left|\int_{\Lambda}(f-f_N)(u-u_N)\,dx\right| \leq \|f-f_N\|_{L^2(\Lambda)}\|u-u_N\|_{L^2(\Lambda)} \tag{5.19}$$

$$\begin{aligned}|\theta(u_0-u_{N0},u-u_N)| &\leq \left|a\int_{\Lambda}\partial_x(u_0-u_{N0})\,\partial_x(u-u_N)\,dx\right| + \left|\int_{\Lambda}\partial_t(u_0-u_{N0})(u-u_N)\,dx\right| \\ &\quad + \left|c\int_{\Lambda}(u_0-u_{N0})(u-u_N)\,dx\right|\end{aligned}$$

the function u_0 is independent of the variable t then

$$\int_{\Lambda}\partial_t(u_0-u_{N0})(u-u_N)\,dx = 0$$

and

$$\left|c\int_{\Lambda}(u_0-u_{N0})(u-u_N)\,dx\right| \leq c\,\|(u_0-u_{N0})\|_{L^2(\Lambda)}\|(u-u_N)\|_{L^2(\Lambda)}$$

and

$$\left|a\int_{\Lambda}\partial_x(u_0-u_{N0})\,\partial_x(u-u_N)\,dx\right| \leq a\,\|\partial_x(u_0-u_{N0})\|_{L^2(\Lambda)}\|\partial_x(u-u_N)\|_{L^2(\Lambda)} \tag{5.20}$$

using **(5.19), (5.20)** and **(4.10)** we get

$$N^2\|u-u_N\|^2_{L^2(\Lambda)} \leq 3CN\left(\|(u_0-u_{N0})\|_{L^2(\Lambda)} + \|f-f_N\|_{L^2(\Lambda)}\right)\|(u-u_N)\|_{L^2(\Lambda)}$$

finally we find the desired results □

5.3 Condition number

Definition 5.3. The condition number of a $n\times n$ non-singular matrix A is defined by:

$$k_P(A) = \|A\|_P\left\|A^{-1}\right\|_P \tag{5.21}$$

where $\|A\|_P$ is the spectral norm defined by $\rho = (A^tA)^{\frac{1}{2}}$.

Remark 5.4. The condition number of a matrix A gives a measure of how sensitive systems of equations, with coefficients matrix A, are to small perturbations such as those caused by rounding. Then if the condition number of a matrix is large, the effect of rounding error in the solution process may be serious [27].

To compute the condition number of different order of these matrix we use the spectral norm, and all operations are made by the Maple 12 (Maple 2008), using [11] (Richards 2002).

5.4 Figure illustration

We consider the true explicit solution: $u(x,t) = \exp(-0.05\pi^2t)\sin(\pi x)$, $u(x,0) = u_0(x) = \sin(\pi x)$ and $f(x,t) = \left(((-1.05)\,\pi^2+1)\sin(\pi x)\right)\exp(-0.05\pi^2t) - (-\pi^2+1)\sin(\pi x)$.

The figures 1 and 2 present the behavior of the condition number and the error, N vary from 3 to 12 we plot $(N, \log(k_P(A)))$. In Figure 3, we present the behavior of the functions $b_n(t)$, when n vary from 3 to 10, and the figures 4 and 5, presents the true and the approximate solutions u and u_N respectively, these plots occurs when $N = 9$.

Remark 5.5. This figure shows that the error decreases rapidly when N increass. Here we plot $\left(N, \|u-u_N\|_{L^2(\Lambda)}\right)$.

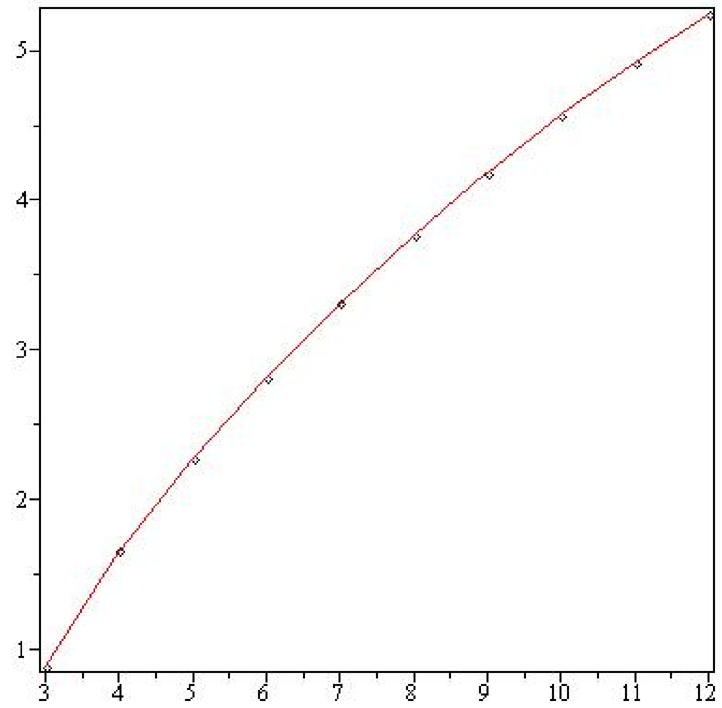

Figure 1: The behavior of the condition number when N vary from 3 to 12

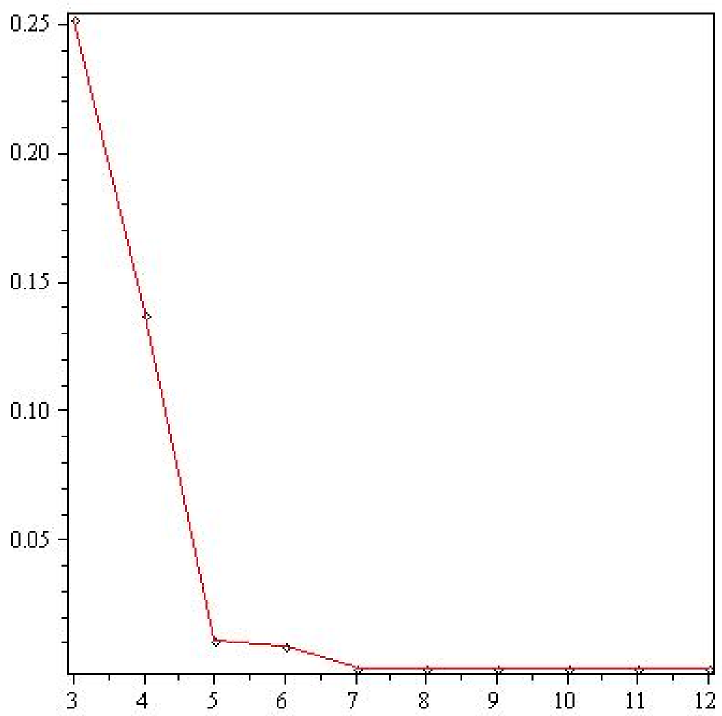

Figure 2: The behavior of the error when N vary from 3 to 12

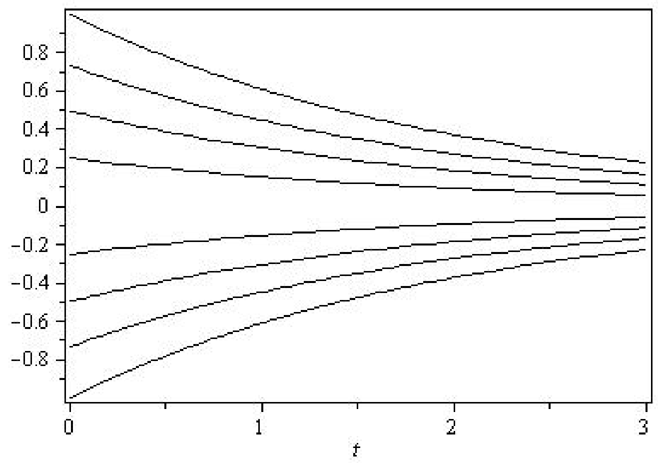

Figure 3: Plots of the functions $b_n(t)$, n vary from 3 to 10

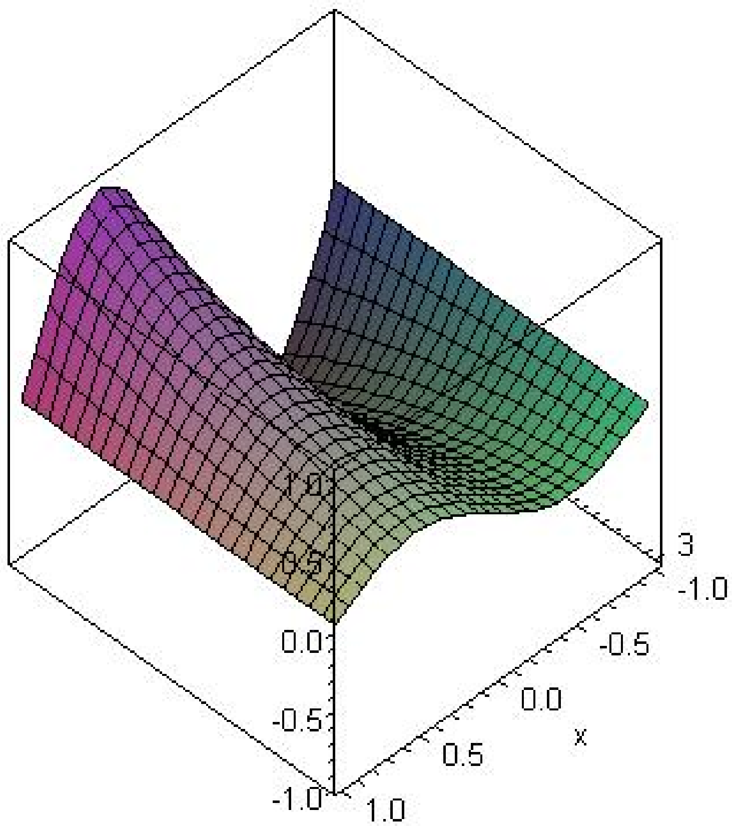

Figure 4: The true solution when $N = 9$

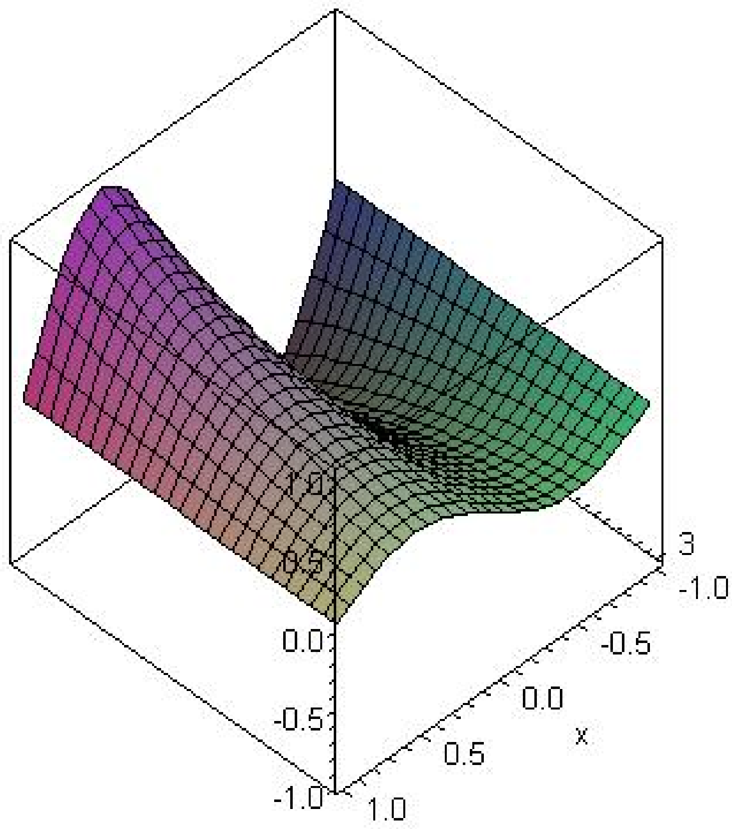

Figure 5: The approximation solution when $N = 9$

6 Conclusion

We know that many ordinary or partial differential equations do not admit exact solution, so we seek the approximate solution, in this article I have described a numerical method converges quickly to the solution of the problem, this method based on the properties of orthogonal polynomials and matrix analysis.

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